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WITT EQUIVALENCE OF QUADRATIC EXTENSIONS OF GLOBAL FIELDS

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ABSTRACT. In the paper we show that the Witt equivalence class of a global field F determines, and is completely determined by the finite set of Witt equivalence classes of all quadratic extensions of the field F.

Introduction

The classical theory of quadratic forms over global fields has been supplemented lately with a solution of the problem of the *Witt equivalence of global fields*. Here fields are meant to be Witt equivalent if their Witt rings of non-degenerate symmetric bilinear forms are isomorphic. For the Witt equivalence of global fields a *Hasse Principle* has been established in [6] and [1] (cf. [9]) that fits perfectly the classical theory and enables to decide if two given global fields are Witt equivalent.

In the case of quadratic number fields the classification up to the Witt equivalence has been obtained earlier by combining the results in [2], [3] and [10]. It turns out that there are exactly 7 Witt equivalence classes of quadratic number fields, represented by the fields $\mathbb{Q}(\sqrt{d})$, $d = -1, \pm 2, \pm 7, \pm 17$.

In this note we use the Hasse Principle for the Witt equivalence of global fields to study the Witt equivalence of quadratic extensions of an arbitrary global field F. We show that the Witt equivalence class of a global field F determines, and is completely determined by the finite set of Witt equivalence classes of all quadratic extensions of the field F.

Although our proof depends largely on the special properties of global fields, it is still meaningful to ask if the result carries over to more general classes of fields. No explicit counter-examples seem to be known, while there are several instances, where it is known that the Witt equivalence class of a field is preserved under suitable quadratic extensions.

Our notation is standard. For a field F, we write $F^{\cdot}$ for the multiplicative group of non-zero elements of F and $F^{\cdot 2}$ for the subgroup of squares. The level

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$s(F)$ is the smallest number s of summands in representation of -1 as a sum of squares over F. If F is a *global field*, i.e., a number field or a function field in one variable over a finite field of constants, and if P is a prime of F, then F_P denotes *the completion of F at P*. If F and E are arbitrary global fields of a characteristic 2, then F and E are Witt equivalent (cf. [9], Theorem(1.1)), hence so are quadratic extensions of F and E. Thus our problem becomes trivial in the characteristic 2 and so we will assume that our global fields have the characteristic $\neq 2$.

1. Going -up

Let $\mathcal{C}$ be a class of fields closed with respect to quadratic extensions. Given a field F in $\mathcal{C}$ we say that *the quadratic extensions preserve the Witt equivalence class of F in $\mathcal{C}$ if for every field E in $\mathcal{C}$ which is Witt equivalent with F and for every quadratic extension $F(\sqrt{a})$ of F, there is a quadratic extension $E(\sqrt{b})$ of E such that $F(\sqrt{a})$ and $E(\sqrt{b})$ are Witt equivalent.*

In this section we observe that quadratic extensions preserve the Witt equivalence class of any global field F (in the class of global fields). This follows from the following slightly stronger result due to J. Carpenter [1].

Theorem 1. *Let F and E be two Witt equivalent global fields. Suppose $i\colon WF \to WE$ is a Witt ring isomorphism mapping the 1-dimensional form $\langle a\rangle \in WF$ onto the 1-dimensional form $\langle b\rangle \in WE$. Then the quadratic extensions $F(\sqrt{a})$ and $E(\sqrt{b})$. are Witt equivalent.*

Proof. If F is a number field and a global field E is Witt equivalent with F, then E is also a number field (cf. [9], Thm (1.5)). In this case it has been proved in [1], Cor. 4.2, p. 50 that $F(\sqrt{a})$ and $E(\sqrt{b})$ are reciprocity equivalent, hence Witt equivalent by [10] Prop. (1.3) (see also [6]).

Now assume F and E to be global function fields of characteristic $\neq 2$. According to [9], theorem (1.3), F and E are Witt equivalent if and only if $s(F) = s(E)$. Thus if $s(F) = s(E) = 1$, then $s(F(\sqrt{a})) = s(E(\sqrt{b})) = 1$ and $F(\sqrt{a})$, $E(\sqrt{b})$ are Witt equivalent. If $s(F) = s(E) = 2$, then $s(F(\sqrt{a})) = 1$ iff $-a \in F^{\cdot 2}$ iff $\langle -a\rangle = \langle 1\rangle$ in WF iff $i\langle -a\rangle = \langle 1\rangle$ in WE iff $\langle -b\rangle = \langle 1\rangle$ in WE iff $s(E(\sqrt{b})) = 1$. This proves that $s(F(\sqrt{a})) = s(E(\sqrt{b}))$, hence $F(\sqrt{a})$ and $E(\sqrt{b})$ are Witt equivalent, as desired. ∎

Remarks. (1.2) While the quadratic extensions preserve the Witt equivalence of global fields, we notice that very little can be concluded about the ground fields of the Witt equivalent quadratic extension. Thus, for instance, if $F = \mathbb{Q}(\sqrt{-1})$ and $E = \mathbb{Q}(\sqrt{2})$, then the quadratic extensions $F(\sqrt{2})$ and

$E(\sqrt{-1})$ are not only Witt equivalent, but, in fact, they are identical fields. Nonetheless, F and E are not Witt equivalent (E is real, while F is not). Even if we consider two Witt equivalent quadratic extensions of a fixed number field F, then, in general, one cannot expect the existence of a Witt ring automorphism $i\colon WF \to WE$ mapping $\langle a\rangle$ onto $\langle b\rangle$. Thus for $F = \mathbb{Q}$, the rational numbers, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are Witt equivalent but there does not exist ring automorphism $i\colon W\mathbb{Q} \to W\mathbb{Q}$ such that $i(\langle 2\rangle) = \langle 3\rangle$. For otherwise, $i(\langle -1,2\rangle) = \langle -1,3\rangle$, and by *Harrison's Criterion* [4], p.21 (cf. [6] or [10]) we may assume that there is an automorphism of the group of square classes of $\mathbb{Q}$ preserving the value sets of forms corresponding under the map i. Now this is impossible since $\langle -1,2\rangle$ represents 1 over $\mathbb{Q}$, while $\langle -1,3\rangle$ does not. In section 2 we show how to read off the information on the Witt equivalence of ground fields from the behavior of the families of quadratic extensions under the Witt equivalence.

(1.3) There are some other known results about the preservation of the Witt equivalence under quadratic extensions. L. Szczepanik [7] proved such a result for general fields assuming that the quadratic extensions have the u-invariant at most 2. A major result in this respect is due to B. Jacob and R. Ware [5]. They prove that quadratic extensions preserve the Witt equivalence for all fields in the class of fields with so-called elementary type Witt ring. This class of fields is disjoint from the class of the global fields.

(1.4) Theorem 1 and the result mentioned in (1.3) answer, in part, Problem 6 in [8], pp. 60–61. We recall that the 10 problems proposed in [8] were first presented at the Summer School on Number Theory held in Kočovce in 1975.

2. Going—down

In this section we study the extent to which the family of all quadratic extensions of a field determines the Witt equivalence class of the ground field. When the ground field is a global field, we get the following result.

Theorem 2. *Let F and E be global fields. Suppose that there is a bijective map between the families of quadratic extensions of F and E such that the corresponding quadratic extensions of F and E are Witt equivalent. Then F and E are Witt equivalent.*

Proof. We consider first the simpler case when F and E are global function fields (of characteristic $\neq 2$). It suffices to show that $s(F) = s(E)$. Contrary to this, suppose that $s(F) = 1$ and $s(E) = 2$. Then any extension of F has level 1, while $s(E(\sqrt{a})) = 2$ whenever $-a \notin E^{\cdot 2}$. Thus $E(\sqrt{a})$ is not Witt equivalent with any quadratic extension of F, a contradiction.

Now we assume that F and E are number fields. Then, according to [1], F and E are Witt equivalent if and only if the following three conditions hold:

(A) $r_1(F) = r_1(E)$, i.e., the numbers of real embeddings are equal.

(B) $-1 \in F^{\cdot 2}$ iff $-1 \in E^{\cdot 2}$.

(C) There is a bijective map $T\colon \Omega_2(F) \to \Omega_2(E)$ between the sets of dyadic primes of F and E preserving local degrees and local levels.

The requirement in (C) says that, whenever $P \in \Omega_2(F)$ and $Q \in \Omega_2(E)$ correspond to each other under the bijection T, then $[F_P : \mathbb{Q}_2] = [E_P : \mathbb{Q}_2]$ and $-1 \in F_P^{\cdot 2} \iff -1 \in E_Q^{\cdot 2}$. Thus we will prove that under the hypothesis of the Theorem 2, the fields F and E satisfy (A), (B), (C).

(A) If $r_1(F) = r_1(E) = 0$, the proof is clear. Thus assume $r_1(F) > 0$. Choose a totally positive element $a \in F^{\cdot} \setminus F^{\cdot 2}$. Then $r_1(F(\sqrt{a})) = 2 \cdot r_1(F)$. By hypothesis, there is $b \in E^{\cdot} \setminus E^{\cdot 2}$ such that $r_1(F(\sqrt{a})) = r_1(E(\sqrt{b}))$. On the other hand, clearly $r_1(E(\sqrt{b})) \leq 2 \cdot r_1(E)$. Hence $r_1(F) \leq r_1(E)$, and by symmetry, we get (A).

(B) Suppose $-1 \in F^{\cdot 2}$. Then -1 is a square in every quadratic extension of F, hence, by hypothesis, -1 is a square in every quadratic extension of E. It follows that $-1 \in E^{\cdot 2}$.

(C) Write $g_2(F)$ for $|\Omega_2(F)|$, the number of dyadic primes of F. First we show that $g_2(F) = g_2(E)$. Choose $a \in F^{\cdot} \setminus F^{\cdot 2}$, which is a square in every dyadic completion of F. Then $g_2(F(\sqrt{a})) = 2 \cdot g_2(F)$. On the other hand, $g_2(E(\sqrt{b})) \leq 2 \cdot g_2(E(\sqrt{b}))$ for any $b \in E^{\cdot}$. Since for a certain $b \in E^{\cdot} \setminus E^{\cdot 2}$ the fields $F(\sqrt{a})$ and $E(\sqrt{b})$ are Witt equivalent, we have $g_2(F(\sqrt{a})) = g_2(E(\sqrt{b}))$, by (C). It follows that $g_2(F) \leq g_2(E)$, and by symmetry we have $g_2(F) = g_2(E)$. Now let $\Omega_2(F) = \{P_1, \ldots, P_g\}$, and $\Omega_2(E) = \{Q_1, \ldots, Q_g\}$, where $g = g_2(F) = g_2(E)$. With the choice of a and b as above, we have $g_2(F(\sqrt{a})) = 2g = g_2(E(\sqrt{b}))$. Thus each P_i and each Q_j splits in $F(\sqrt{a})$ and $E(\sqrt{b})$, respectively. Assume $P_i = P_i' \cdot P_i''$ and $Q_j = Q_j' \cdot Q_j''$ so that

$$\Omega_2(F(\sqrt{a})) = \{P_1', P_1'', \ldots, P_g', P_g''\},$$
$$\Omega_2(E(\sqrt{b})) = \{Q_1', Q_1'', \ldots, Q_g', Q_g''\}.$$

Since $F(\sqrt{a})$ and $E(\sqrt{b})$ are Witt equivalent, by (C) there is a bijection between $\Omega_2(F(\sqrt{a}))$ and $\Omega_2(E(\sqrt{b}))$ preserving local degrees and local levels. After renumbering the primes, we may, if necessary, assume that $P_i' \mapsto Q_i'$ under this bijection. Then for any i, $1 \leq i \leq g$, we have $-1 \in F_{P_i}^{\cdot 2} \iff -1 \in E_{Q_i}^{\cdot 2}$ and $[F_{P_i} : \mathbb{Q}_2] = \left[F(\sqrt{a})_{P_i'} : \mathbb{Q}_2\right] = \left[E(\sqrt{b})_{Q_i'} : \mathbb{Q}_2\right] = [E_{Q_i} : \mathbb{Q}_2]$, since

$F(\sqrt{a})_{P_i'} = F_{P_i}$ and $E(\sqrt{b})_{Q_i'} = E_{Q_i}$, by the splitting behaviour of P_i and Q_i. This proves (C) for F and E, and finishes the proof of Theorem 2.

As mentioned in the Introduction, there are exactly 7 Witt equivalence classes for quadratic number fields. The problem of determining the number of the Witt equivalence classes for number fields of a given degree n has been solved in [9]. It is proved there that for the cubic fields ($n = 3$) there are exactly 8 classes and for the quartic fields ($n = 4$) there are exactly 29 Witt equivalence classes. While for $n = 2$ and $n = 3$ a complete classification with respect to the Witt equivalence is available, for $n = 4$ we do not known even how to represent the 29 classes. Theorem 2 implies that each of the seven fields $Q(\sqrt{d})$, $d = -1, \pm 2, \pm 7, \pm 17$, which are pairwise Witt inequivalent, has a quadratic extension with a separate Witt equivalence class. Thus, among 29 Witt equivalence classes of quartic fields at least 7 can be represented by the fields $\mathbb{Q}(\sqrt{a}, \sqrt{b})$, $a \in \mathbb{Q}$, $b \in \mathbb{Q}(\sqrt{a})$. We do not know at the moment the exact number of such classes, nor we do know how many classes of quartic fields can be represented by the biquadratic fields $\mathbb{Q}(\sqrt{a}, \sqrt{b})$, $a, b \in \mathbb{Q}$.

Added in proof. In a forthcoming paper S. Jakubec and F. Marko proved that out of 29 Witt equivalence classes of quartic fields, exactly 26 can be represented by quadratic extensions of quadratic number fields. [Witt equivalence classes of quartic number fields. Math. Comp. (To appear).]

REFERENCES

[1] CARPENTER, J.: Finiteness Theorems for Forms over Number Fields. Dissertation, LSU Baton Rouge, La., 1989.

[2] CZOGALA, A.: Witt Rings of Algebraic Number Fields (Polish). Dissertation, Silesian University, Katowice, 1987.

[3] CZOGALA, A.: On reciprocity equivalence of quadratic number fields. Acta Arith. 58 (To appear.).

[4] HARRISON, D. K.: Witt Rings. Univ. of Kentucky, 1970.

[5] JACOB, B.—WARE, R.: Realizing dyadic factors of elementary type Witt rings and pro-2 Galois groups. Preprint (1989).

[6] PERLIS, R.—SZYMICZEK, K.—CONNER, P. E.—LITHERLAND, R.: Matching Witts with global fields. Preprint (1989).

[7] SZCZEPANIK, L.: On WR-equivalence of quadratic extensions of WR-equivalent fields. Univ. Slląnski w Katowicach, Prace mat. 10 (1980), p. 17-22.

[8] SZYMICZEK, K.: Quadratic forms over fields. Dissert. Math. 152 (1977), 1–63.

[9] SZYMICZEK, K.: Witt equivalence of global fields. Commun. Algebra 19 (1991), 1125–1149.

[10] SZYMICZEK, K.: Matching Witts locally and globally. Math. Slovaca 41 (1991), 315 330.

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