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SEPARATION PROPERTIES IN X AND 2^X MEASURABLE MULTIFUNCTIONS AND GRAPHS

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ABSTRACT. This paper investigates, in a general framework, relations among some separation properties of a space X, the countable separation of the hyperspace 2^X, endowed with a suitable σ-algebra, and the measurability of graphs of measurable multifunctions. Several results can be deduced which apply to properties of separation and measurability of different kinds.

Introduction

It is known that a countably R-separated topological space (X, τ_X) satisfies the following property (see [6], theorem 3.1):

(i) *for each measurable space* (T, σ_T) *and each cosed-valued measurable multifunction* $F\colon T \to X$, *the graph of* F *lies in* $\sigma_T \times \sigma(\tau_X)$.

On the other hand, if X is a second countable space, then the following property holds (see [9], theorem 4.2 (b)):

(ii) *for each measurable space* (T, σ_T) *and each closed-valued weakly measurable multifunction* $F\colon T \to X$, *the graph of* F *lies in* $\sigma_T \times \sigma(\tau_X)$.

In a previous paper ([2]), relationships have been studied among the countable R-separation of a topological space (X, τ_X), the measurability of some sets constructed by means of measurable multifunctions and the countable separation of 2^X, endowed with the σ-algebra $\sigma(\mathfrak{B}^+)$ generated by the family $\{\langle U\rangle\colon U \in \tau_X\}$ ($\langle U\rangle$ denotes the class of all non-empty closed substs of U). In particular, the property (i) implies the countable separation of 2^X; moreover, each of this two conditions is equivalent to saying that X is countably R-separated provided that X is R_0.

In this paper we study, in a general framework, the relations among some separation properties of a space X, equipped with a topology and a σ-algebra,

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the countable separation of 2^X, endowed with a suitable σ-algebra, and the measurability of graphs of measurable multifunctions, replacing the R_0-axiom with a weaker requirement on the multifunction $x \mapsto \overline{\{x\}}$.

Several results can be deduced which apply to the properties of separation and measurability of different kinds. For example, the property (ii), the second countability of X and the countable separation of 2^X, endowed with the σ-algebra $\sigma(\mathfrak{B}^-)$ generated by the family $\{\langle X, U\rangle : U \in \tau_X\}$ ($\langle X, U\rangle$ denotes the class of all non-empty closed sets which intercept U), are equivalent.

Also, a certain connection with the problem of the measurability of functions of two variables is stated, since any property of (strong) countable R-separation introduced here ensures the existence of a regular $\mathscr{P}$-system.

1. Preliminaries

Let T, X, Y be sets, F: $T \to X$ a multifunction, i.e. a function from T to the family $\mathscr{P}(X)$ of all subsets of X, and g: $Y \to X$ a function.

We denote by $Gr(F)$ the graph of F, i.e. the set $\{(t, x) \in T \times X : x \in F(t)\}$ and by $Gr(F, g)$ the set $\{(t, y) \in T \times Y : g(y) \in F(t)\}$.

If $U \subset X$, we put $F^+(U) = \{t \in T : F(t) \subset U\}$ and $F^-(U) = \{t \in T : F(t) \cap \cap U \neq \emptyset\}$. We have the fundamental relation $F^+(U) = T - F^-(X - U)$.

If (T, σ_T) and (X, σ_X) are measurable spaces (we mean that σ_T and σ_X are σ-algebras) and $\mathscr{U} \subset \sigma_X$, F is said to be $\mathscr{U}$-measurable if for each $U \in \mathscr{U}$ we have $F^+(U) \in \sigma_T$. Moreover, if (Y, σ_Y) is a measurable space, g is said to be measurable if $g^{-1}(M) \in \sigma_Y$ for each $M \in \sigma_X$.

If Z is a set and $\mathscr{B}$ a family of subsets of Z, $\sigma(\mathscr{B})$ denotes the σ-algebra on Z generated by $\mathscr{B}$. $(Z, \sigma(\mathscr{B}))$ is said to be countably separated if there exists a countable family $\mathscr{B}' \subset \mathscr{B}$ which T_0-separates Z.

If (X, τ_X) is a topological space, we denote by $\mathscr{F}(X)$ the family of all closed subsets of X, by 2^X the family of all non-empty closed subsets of X and by Ω the set $\{(C, x) : C \in 2^X, x \in C\}$.

Moreover, if $U_1, \ldots, U_n \in \mathscr{P}(X)$, we denote by $\langle U_1, \ldots, U_n\rangle$ the family $\left\{C \in 2^X : C \subset \bigcup_{i=1}^{n} U_i \text{ and } C \cap U_i \neq \emptyset \ \forall i = 1, \ldots, n\right\}$. If $\mathscr{U} \subset \mathscr{P}(X)$, we put $\mathfrak{U} = \{\langle U_1, \ldots, U_n\rangle : U_1, \ldots, U_n \in \mathscr{U}, n \in \mathbf{N}\}$, $\mathfrak{U}^+ = \{\langle U\rangle : U \in \mathscr{U}\}$ and $\mathfrak{U}^- = \{\langle X, U\rangle : U \in \mathscr{U}\}$.

In the particular case in which $\mathscr{U} = \tau_X$, we obtain, respectively, a base for the finite topology on 2^X, a base for the upper semifinite topology on 2^X and a subbase for the lower semifinite topology on 2^X (see [7]). To emphasize this case, we use the notation $\mathfrak{B}$, $\mathfrak{B}^+$ and $\mathfrak{B}^-$.

2. Separation. Measurability. Graphs

Let (X, τ_X) be a topological space and (X, σ_X) a measurable space.

Definition 2.1. *We say that a family $\mathscr{U} \subset \sigma_X$ separates (resp. weakly separates) closed sets and points of X if for each closed set C and each point $x \notin C$ there exists $U \in \mathscr{U}$ such that $C \subset U$ and $x \notin U$ (resp. $\overline{\{x\}} \not\subset U$).*

Definition 2.2. *X is said to be countably R-separated (resp. weakly countably R-separated) by a family $\mathscr{U} \subset \sigma_X$, with $\emptyset \in \mathscr{U}$, if there exists a contable subfamily $\mathscr{W}$ of $\mathscr{U}$ which separates (resp. weakly separates) closed sets and points of X.*

Remark 2.1. If $\tau_X \subset \sigma_X$ [(1)], several properties on X can be obtained, specifying the family $\mathscr{U}$ in definition 2.2. We point out the following ones:

The cR-separation (resp. wcR-separation), obtained if $\mathscr{U} = \tau_X$ (see [2], definitions 3.2 and 3.1).

The second axiom of countability, obtained if $\mathscr{U} = \mathscr{F}(X)$ (given a countable family $\mathscr{W} \subset \mathscr{F}(X)$, separates (weakly separates) closed sets and points of X iff $\mathscr{B} = \{X - U : U \in \mathscr{W}\}$ is a base for τ_X). In this case, the two properties of countable R-separation coincide.

— The cT_0R-separation (resp. wcT_0R-separation), obtained if $\mathscr{U} = \tau_X \cup \cup \mathscr{F}(X)$. Indeed, if $\mathscr{W} \subset \tau_X \cup \mathscr{F}(X)$ is a countable family which separates (resp. weakly separates) closed sets and points of X, then the countable family $\mathscr{A} = \{U: U \in \mathscr{W} \cap \tau_X\} \cup \{X - U : U \in \mathscr{W} \cap \mathscr{F}(X)\}$ T_0R-separates (resp. wT_0R-separates) X, namely for each closed set C and any $x \notin C$ there exists $A \in \mathscr{A}$ such that ($x \in A$ and $A \cap C = \emptyset$) or ($C \subset A$ and $x \notin A$ (resp. $\overline{\{x\}} \not\subset A$)); conversely, if $\mathscr{A} \subset \tau_X$ is a countable family by which T_0R-separates (resp. wT_0R-separates) X, then the countable family $\mathscr{W} = \mathscr{A} \cup \{X - A : A \in \mathscr{A}\}$ separates (resp. weakly separates) closed sets and points of X.

The properties of separation which we can obtain by specifying the family $\mathscr{U}$ in definition 2.2 are as "strong" as $\mathscr{U}$ is "small". This leads us to consider the "minimal" property of countable R-separation byσ_X. This property is interesting because, as we can see in the following theorem, it gives a characterization of spaces in which a regular $\mathscr{P}$-system exists. The concept of $\mathscr{P}$-system has been introduced in [4]; it is very useful in issues concerning the measurability of functions of two variables (see [1], [4]) and the measurability of certain multifunctions defined by means of functions of two variables (see [8]).

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(1) We notice that if X is countably R-separated by $\mathscr{U}$, then $\tau_X \subset \sigma_X$. In fact, for each $A \in \tau_X$ we have $A = \bigcup_{\substack{U \in \mathscr{W} \\ X - U \subset A}} (X - U)$, $\mathscr{W}$ being a countable subfamily of $\mathscr{U}$ which separates closed sets and points of X.

First, we recall the definition of the $\mathscr{P}$- system and that of the regularity of a $\mathscr{P}$-system with respect to a topology.

Definition 2.3. ([4], definition 4.1) *Let* (X, σ_X) *be a measurable space and* $\mathscr{P} = \{P_n^k: \emptyset \neq P_n^k \in \sigma_X, k = 1, 2, \ldots, n \in N_k\}$, *where* N_k *is the set of all positive integers or a set* $\{1, \ldots, n_k\}$. $\mathscr{P}$ *is called a* $\mathscr{P}$*-system on* (X, σ_X) if $X = \bigcup_{n \in N_k} P_n^k$ *for each* $k = 1, 2, \ldots$.

Definition 2.4. ([4] definition 4.3) *Let* (X, τ_X) *be a topological space and* (X, σ_X) *a measurable space. A* $\mathscr{P}$*-system* $\mathscr{P}$ *on* (X, σ_X) *is* τ_X*-regular if for each* $A \in \tau_X$ *and each* $x \in A$ *there is* k_0 *such that, for* $k > k_0$, $x \in P_n^k \Rightarrow P_n^k \subset A$.

Theorem 2.1. *Let* (X, τ_X) *be a topological space and* (X, σ_X) *a measurable space. Then the following statements are equivalent*:

1) *X is countably R-separated by* σ_X;
2) *a* τ_X*-regular* $\mathscr{P}$*-system exists on* (X, σ_X);
3) *a second countable topology* τ'_X *exists on X, such that* $\tau_X \subset \tau'_X \subset \sigma_X$.

Proof. 1) $\Rightarrow$ 2): let $\mathscr{E} = \{E_k: k = 1, 2, \ldots\} \subset \sigma_X$ be a countable family such that $\mathscr{W} = \{X - E_k: k = 1, 2, \ldots\}$ separates closed sets and points of X. We shall construct, by induction, a τ_X-regular $\mathscr{P}$-system on (X, σ_X).

$\mathscr{A}_0 = \left\{E_k - \bigcup_{n<k} E_n\right\}_{k=1,2,\ldots}$ is a family of pairwise disjoint measurable sets such that $\bigcup_{A \in \mathscr{A}_0} A = X$. Let $\{P_n^1 : n \in N_1\} = \{A \in \mathscr{A}_0 : A \neq \emptyset\}$.

Now, suppose that $\{P_n^k: n \in N_k\}$ has been constructed for $k = 2, \ldots, p$, such that $P_n^k \neq \emptyset$ for all $n \in N_k$, $\bigcup_{n \in N_k} P_n^k = X$, $P_n^k \cap P_m^k = \emptyset$ if $n \neq m$, $x \in P_n^k \cap P_m^{k-1} \Rightarrow P_n^k \subset P_m^{k-1}$ and $x \in P_n^k \cap E_k \Rightarrow P_n^k \subset E_k$.

Define $\mathscr{A}_p = \{E_{p+1} \cap P_n^p : n \in N_p\}$ and $\mathscr{B}_p = \{P_n^p - E_{p+1} : n \in N_p\}$; by enumerating all non-empty sets in $\mathscr{A}_p \cup \mathscr{B}_p$ we obtain a countable family $\{P_n^{p+1} : n \in N_{p+1}\} \subset \sigma_X$ of pairwise disjoint sets such that $\bigcup_{n \in N_{p+1}} P_n^{p+1} = \bigcup_{n \in N_p} P_n^p = X$, $x \in P_n^{p+1} \cap P_m^p \Rightarrow P_n^{p+1} \subset P_m^p$ and $x \in P_n^{p+1} \cap E_{p+1} \Rightarrow P_n^{p+1} \subset E_{p+1}$.

It follows that $\mathscr{P} = \{P_n^k: k = 1, 2, \ldots, n \in N_k\}$ is a (disjoint) $\mathscr{P}$-system on (X, σ_X).

Now we show that $\mathscr{P}$ is τ_X-regular. Let $A \in \tau_X$ and $x \in A$. Since $\mathscr{W}$ separates closed sets and points of X, there exists h such that $x \in E_h \subset A$. Moreover, there exists $m \in N_h$ such that $x \in P_m^h \subset E_h$. As for $k > h$, $x \in P_n^k \Rightarrow P_n^k \subset P_m^h$, the desired regularity is proved.

2) $\Rightarrow$ 3): (2) let $\mathscr{P} = \{P_n^k: k = 1, 2, \ldots, n \in N_k\}$ be a τ_X-regular $\mathscr{P}$-system on (X, σ_X). The (unique) topology on X which has $\mathscr{P}$ as a subbase is the required topology τ'_X.

In fact, τ'_X is, obviously, second countable and $\tau'_X \subset \sigma_X$; moreover, by the τ_X-regularity of $\mathscr{P}$ it follows, in particular, that $A = \bigcup_{P_n^k \subset A} P_n^k$ for every $A \in \tau_X$; hence, taking into account that $\mathscr{P} \subset \tau'_X$, $\tau_X \subset \tau'_X$.

3) $\Rightarrow$ 1): if $\mathscr{B}$ is a countable base for τ'_X, the family $\mathscr{W} = \{X - B : B \in \mathscr{B}\}$ separates closed sets and points of X. □

In general, the countable R-separation by $\mathscr{U}$ implies the weak countable R-separation by $\mathscr{U}$. The inverse implication holds if $\mathscr{U} = \mathscr{F}(X)$ or , more generally, if $\mathscr{U}$ satisfies the following

Property 2.1. *The multifunction $\bar{I}$: $X \to X$, defined by $\bar{I}(x) = \overline{\{x\}}$ for each $x \in X$, is such that for each $U \in \mathscr{U}$ we have $\bar{I}^+(U) \in \mathscr{U}$.*

Indeed, the following lemma holds:

Lemma 2.1. *If X is weakly countably R-separated by $\mathscr{U}$ and $\mathscr{U}$ satisfies the property* 2.1 *then X is countably R-separated by $\mathscr{U}$.*

Proof. Let $\mathscr{W} \subset \mathscr{U}$ be a countable family which weakly separates closed sets and points of X. Put $\mathscr{W}' = \{\bar{I}^+(W) : W \in \mathscr{W}\}$. By property 2.1, we have $\mathscr{W}' \subset \mathscr{U}$. Moreover, for each closed set C and each $x \notin C$ let $W \in \mathscr{W}$ be such that $C \subset W$ and $\overline{\{x\}} \not\subset W$; for each $y \in C$ we have $\overline{\{y\}} \subset C \subset W$, whence $C \subset \bar{I}^+(W)$, while $\overline{\{x\}} \not\subset W$ implies that $x \notin \bar{I}^+(W)$.

Thus the family $\mathscr{W}'$ separates closed sets and points of X. □

Remark 2.2. If $\tau_X \subset \mathscr{U} \subset \tau_X \cup \mathscr{F}(X)$, the property 2.1 is weaker then the R_0-axiom (see [3]). Indeed, the R_0-axiom on (X, τ_X) is equivalent to saying that $\bar{I}^+(U) = U$ for each $U \in \tau_X$, while $\mathscr{F}(X)$ always satisfies the property 2.1. Simple examples show that this relation is strict.

We do not know whether the countable R-separation by $\mathscr{U}$ implies that $\mathscr{U}$ satisfies the property 2.1. However, this is true if $\mathscr{U} = \tau_X$, i.e. if X is cR-separated; in this case, in fact, even the R_0-axiom holds.

Theorem 2.2. *Let (X, τ_X) be a topological space, (X, σ_X) a measurable space and $\mathscr{U} \subset \sigma_X$, with $\emptyset \in \mathscr{U}$. The following statements are equivalent:*

1) *X is weakly countably R-separated by the family $\mathscr{U}$.*

2) *$(2^X, \sigma(\mathfrak{U}^+))$ is countably separated.*

Proof. 1) $\Rightarrow$ 2): if $\mathscr{W} \subset \mathscr{U}$ is a countable family which weakly separates closed sets and points of X, then $\mathfrak{W}^+ = \{\langle W \rangle: W \in \mathscr{W}\}$ is a countable family, with $\mathfrak{W}^+ \subset \mathfrak{U}^+$, which T_0-separates elements of 2^X.

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(2) This is proved also in lemma 3.1 of [1]. We give here a shorter proof.

2) $\Rightarrow$ 1): if X is not weakly countably R-separated by $\mathscr{U}$, for each countable family $\mathfrak{W} \subset \mathfrak{U}^+$ the family $\mathscr{W} = \{W: \langle W \rangle \in \mathfrak{W}\} \cup \{\emptyset\}$ does not weakly separate closed sets and points of X, i.e. there exist a non-empty closed set C^* and $x^* \notin C^*$ such that $C^* \subset W \Rightarrow \overline{\{x^*\}} \subset W$ or, equivalently, $C^* \in \langle W \rangle \Rightarrow C^* \cup \cup \overline{\{x^*\}} \in \langle W \rangle$. Since, on the other hand, $C^* \cup \overline{\{x^*\}} \in \langle W \rangle \Rightarrow C^* \in \langle W \rangle$, it follows that $\mathfrak{W}$ does not T_0-separate elements of 2^X. □

Remark 2.3. If $\mathscr{U} = \mathscr{F}(X)$, then $\sigma(\mathfrak{U}^+) = \sigma(\mathfrak{V}^-)$; while if $\mathscr{U} = \tau_X \cup \cup \mathscr{F}(X)$, then $\sigma(\mathfrak{U}^+) = \sigma(\mathfrak{V})$. These two equalities are easily shown, taking into account that for all subsets U, U_1, ..., U_n of X we have $\langle X, U \rangle = 2^X - \langle X - U \rangle$ and $\langle U_1, \ldots, U_n \rangle = \left[\bigcap_{i=1}^{n} (2^X - \langle X - U_i \rangle) \right] \cap \left\langle \bigcup_{i=1}^{n} U_i \right\rangle$.

We point out the following special formulations of theorem 2.2:

Corollary 2.1. ([2], theorem 3.1) *The following conditions are equivalent for a topological space* (X, τ_X):

1) *X is wcR-scparated;*
2) $(2^X, \sigma(\mathfrak{V}^+))$ *is countably separated.*

Corollary 2.2. *The following conditions are equivalent for a topological space* (X, τ_X):

1) *X is second countable;*
2) $(2^X, \sigma(\mathfrak{V}^-))$ *is countably separated.*

Proof. It follows by theorem 2.2 for $\mathscr{U} = \mathscr{F}(X)$, taking into account remark 2.1 and remark 2.3. □

Corollary 2.3. *The following conditions are equivalent for a topological space* (X, τ_X):

1) *X is wcT_0R-separated;*
2) $(2^X, \sigma(\mathfrak{V}))$ *is countably separated.*

Proof. It follows by theorem 2.2 for $\mathscr{U} = \tau_X \cup \mathscr{F}(X)$, taking into account remark 2.1 and remark 2.3. □

Remark 2.4. If in the following theorem 2.3, corollary 2.4 and theorem 2.4, $\mathscr{U}$ is τ_X, $\mathscr{F}(X)$, $\tau_X \cup \mathscr{F}(X)$ or $\sigma(\tau_X)$, then the $\mathscr{U}$-measurability of F is measurability, weak measurability, simultaneous measurability and weak measurbility, or $\mathscr{B}$-measurability, respectively.

Theorem 2.3. *Let X be countably R-separated by $\mathscr{U}$, F: $T \to X$ closed-valued and $\mathscr{U}$-measurable, and g: $Y \to X$ measurable. Then* $Gr(F, g) \in \sigma_T \times \sigma_Y$.

Proof. Let $\mathscr{W} \subset \mathscr{U}$ be countable family which separates closed sets and points of X. Then:

$$(t, y) \in (T \times Y) - Gr(F, g) \Leftrightarrow \exists W \in \mathscr{W}: g(y) \notin W \text{ and } F(t) \subset W,$$
$$\text{i.e. } y \notin g^{-1}(W) \text{ and } t \in F^+(W).$$

Therefore $(T \times Y) - Gr(F, g) = \bigcup_{W \in \mathscr{W}} [F^{+}(W) \times g^{-1}(X - W)] \in \sigma_T \times \sigma_Y$. □

Corollary 2.4. *If X and F are as in theorem 2.3, then $Gr(F) \in \sigma_T \times \sigma_X$.*

Theorem 2.4. *Let (X, τ_X) be a topological space, (X, σ_X) a measurable space and $\mathscr{U} \subset \sigma_X$, with $\emptyset \in \mathscr{U}$. Then the following properties are equivalent:*

1) *given two measurable spaces (T, σ_T) and (Y, σ_Y), a $\mathscr{U}$-measurable multifunction F: $T \to X$, with (non-empty and) closed values, and a measurable function g: $Y \to X$, then $Gr(F, g) \in \sigma_T \times \sigma_Y$;*

2) *for each measurable space (T, σ_T) and each $\mathscr{U}$-mesurable multifunction F: $T \to X$, with (non-empty and) closed values, we have $Gr(F) \in \sigma_T \times \sigma_X$;*

3) *$\Omega \in \sigma(\mathfrak{U}^{+}) \times \sigma_X$.*

Proof. Obviously, 1) ⇒ 2). We shall prove that 2) ⇒ 3) and 3) ⇒ 1)

2) ⇒ 3): put $(T, \sigma_T) = (2^X, \sigma(\mathfrak{U}^{+}))$. Then the multifunction I: $2^X \to X$, defined by $I(C) = C$ for each $C \in 2^X$, is $\mathscr{U}$-measurable. Hence $\Omega = Gr(I) \in \sigma(\mathfrak{U}^{+}) \times$
$\times\ \sigma_X$.

3) ⇒ 1): if F: $T \to X$ is $\mathscr{U}$-measurable, the function f: $T \to 2^X$, defined for each $t \in T$ by:

$$f(t) = \begin{cases} F(t) & \text{if } F(t) \neq \emptyset \\ X & \text{if } F(t) = \emptyset, \end{cases}$$

is measurable (2^X equipped with the σ-algebra $\sigma(\mathfrak{U}^{+})$).

In fact:

$$f^{-1}(\langle U \rangle) = \begin{cases} T & \text{if } U = X \\ F^{+}(U) - F^{+}(\emptyset) & \text{if } U \in \mathscr{U},\ U \neq X. \end{cases}$$

Therefore the function (f, g): $T \times Y \to 2^X \times X$, defined by $(f, g)(t, y) = (f(t), g(y))$ for all $(t, y) \in T \times Y$, is measurable and hence $Gr(F, g) = (f, g)^{-1}(\Omega) - (F^{+}(\emptyset) \times Y) \in \sigma_T \times \sigma_Y$. □

3. Further relations between measurability of graphs and separation

Definition 3.1. *If (X, τ_X) is a topological space, we say that the family $\mathfrak{R} \subset \mathscr{P}(2^X)$ separates closed sets and points of X if for each closed set C and each $x \notin C$ there exists $\mathscr{A} \in \mathfrak{R}$ such that:*

$$(C \in \mathscr{A} \text{ and } C \cup \overline{\{x\}} \notin \mathscr{A}) \text{ or } (C \cup \overline{\{x\}} \in \mathscr{A} \text{ and } C \notin \mathscr{A}).$$

Theorem 3.1. *Let (X, τ_X) be a topological space and $\mathfrak{B} \subset \mathscr{P}(2^X)$. If $\Omega \in \sigma(\mathfrak{B}) \times \sigma_X$, then there exists a countable family $\mathfrak{R} \subset \mathfrak{B}$ which separates closed sets and points of X.*

Proof. Define:

$\mathfrak{K} = \{K \subset 2^X \times X$: $\exists$ *a contable family* $\mathfrak{N} \subset \mathfrak{B}$ *such that* $K \in \sigma(\mathfrak{N}) \times \sigma_X\}$.

We claim:

1) $\mathfrak{K}$ is a σ-algebra.

In fact: 1) $\emptyset$ and $2^X \times X$, obviously, lie in $\mathfrak{K}$.

2) $K_n \in \mathfrak{K}\, \forall n \in \mathbf{N} \Rightarrow \forall n \exists$ a countable family $\mathfrak{N}_n \subset \mathfrak{B}$ such that $K_n \in \sigma(\mathfrak{N}_n) \times \sigma_X$. Hence $\bigcup_n K_n \in \sigma\left(\bigcup_n \mathfrak{N}_n\right) \times \sigma_X$ and, $\bigcup_n \mathfrak{N}_n$ being a countable family contained in $\mathfrak{B}$, $\bigcup_n K_n \in \mathfrak{K}$.

3) $K \in \mathfrak{K} \Leftrightarrow (2^X \times X) - K \in \mathfrak{K}$ folows from $K \in \sigma(\mathfrak{N}) \times \sigma_X \Leftrightarrow \Leftrightarrow (2^X \times X) - K \in \sigma(\mathfrak{N}) \times \sigma_X$.

2) $\sigma(\mathfrak{B}) \times \sigma_X \subset \mathfrak{K}$.

In fact, for each measurable rectangle, $\mathscr{A} \times B$ ($\mathscr{A} \in \sigma(\mathfrak{B})$, $B \in \sigma_X$), there is a countable family $\mathfrak{N} \subset \mathfrak{B}$ such that $\mathscr{A} \in \sigma(\mathfrak{N})$ (see [5], theorem D pg. 24). Hence $\mathscr{A} \times B \in \mathfrak{K}$. It follows that $\sigma(\mathfrak{B}) \times \times \sigma_X \subset \mathfrak{K}$, by definition of the product of σ-algebra and the preceding step.

3) $\Omega \in \mathfrak{K}$.

It follows, obviously, by hypothesis and step 2).

Now, let $\mathfrak{N} \subset \mathfrak{B}$ be a countable family such that $\Omega \in \sigma(\mathfrak{N}) \times \sigma_X$.

4) $\sigma(\mathfrak{N})$ separates closed sets and points of X:

Assume the contrary. Then there exist $C^* \in 2^X$ and $x^* \notin C^*$ such that:

$$(*)\ \forall \mathscr{A} \in \sigma(\mathfrak{N}),\ C^* \in \mathscr{A} \Leftrightarrow C^* \cup \{x^*\} \in \mathscr{A}.$$

Consider the σ-algebra:

$$\mathscr{L} = \{L \subset 2^X \times X : (C^*, x^*) \in L \Leftrightarrow (C^* \cup \overline{\{x^*\}}, x^*) \in L\}.$$

$(*)$ implies that $\sigma(\mathfrak{N}) \times \sigma_X \subset \mathscr{L}$ whence, in particular, $\Omega \in \mathscr{L}$, a contradiction.

Finally:

5) $\mathfrak{N}$ separates closed sets and points of X.

Suppose, on the contrary, that there exist $C^* \in 2^X$ and $x^* \notin C^*$ such that, for each $\mathscr{A} \in \mathfrak{N}$, $C^* \in \mathscr{A} \Leftrightarrow C^* \cup \overline{\{x^*\}} \in \mathscr{A}$; $\mathfrak{M} = \{\mathscr{M} \subset 2^X : : C^* \in \mathscr{M} \Leftrightarrow C^* \cup \overline{\{x^*\}} \in \mathscr{M}\}$ is a σ-algebra containing $\mathfrak{N}$ and, hence, $\sigma(\mathfrak{N})$; but this is in contrast with step 4). □

The following theorem shows that the conditions of theorem 2.4 imply those of theorem 2.2.

Theorem 3.2. *Let* (X, τ_X) *be a topological space*, (X, σ_X) *a measurable space and* $\mathscr{U} \subset \sigma_X$. *If* $\Omega \in \sigma(\mathfrak{U}^+) \times \sigma_X$, *then* X *is weakly countably R-separated by* $\mathscr{U} \cup \{\emptyset\}$.

Proof. Put $\mathfrak{B} = \mathfrak{U}^+$. Theorem 3.1 implies that there is $\mathfrak{N} \subset \mathfrak{B}$ ($\mathfrak{N}$ countable) which separates closed sets and points of X. This means that for each $C \in 2^X$ and each $x \notin C$ there is $\langle U \rangle \in \mathfrak{N}$ such that:

α) $$C \in \langle U \rangle \text{ and } C \cup \overline{\{x\}} \notin \langle U \rangle$$

or

β) $$C \cup \overline{\{x\}} \in \langle U \rangle \text{ and } C \notin \langle U \rangle .$$

But α) is the same as saying that $C \subset U$ and $\overline{\{x\}} \not\subset U$, while β) is not possible. It follows that the countable family $\{U: \langle U \rangle \in \mathfrak{N}\} \cup \{\emptyset\}$ weakly separates closed sets and points of X. □

The following theorem sumarizes the results obtained so far.

Some interesting special formulations of it can be obtained, according to remarks 2.1, 2.2, 2.3, 2.4.

Theorem 3.3. *Let* (X, τ_X) *be a topological space,* (X, σ_X) *a measurable space and* $\mathcal{U} \subset \sigma_X$, *with* $\emptyset \in \mathcal{U}$, *If we consider the following*:

1) *X is countably R-separated by* $\mathcal{U}$,

2) *conditions of theorem* 2.2,

3) *conditions of teorem* 2.4,

then we have: 1) ⇒ 3) ⇒ 2).

If, also, $\mathcal{U}$ *satisfies the property* 2.1, *then* 2) ⇒ 1) *and hence* 1) ⇔ 2) ⇔ 3).

As an example of how theorem 3.3 can be used, we conclude with the following two theorems which state some relations among cR-separation, cT_0R-separation and the second axiom of countability.

Theorem 3.4. *The following conditions are equivalent for a topological space* (X, τ_X):

1) *X is cR-separated*;

2) *X is* cT_0R*-separated and every open set is a* F_σ.

Proof. 1) ⇒ 2): it follows from the definitions.

2) ⇒ 1): for each measurable space (T, σ_T) and each closed-valued and measurable multifunction $F: T \to X$, F is weakly measurable (because every open set is a F_σ, see theorem 1 of [8]); then, X being cT_0R-separated, it follows, by 1) ⇒ 3) of theorem 3.3 (in which $\mathcal{U} = \tau_X \cup \mathcal{F}(X)$) that $Gr(F) \in \sigma_T \times \sigma(\tau_X)$. Moreover, X is R_0 (still because every open set is a F_σ) and hence τ_X satisfies the property 2.1. Thus 1) follows by 3) ⇒ 1) of theorem 3.3 (in which $\mathcal{U} = \tau_X$). □

Theorem 3.5. *A second countable topological space in which every open set is a* F_σ *is cR-separated.*

Proof. It follows by 2) ⇒ 1) of theorem 3.4, taking into account that a second countable topological space is, obviously, cT_0R-separated. □

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