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IDEMPOTENTS OF COMPACT MONOTHETIC SEMICLOSURE SEMIGROUPS

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One of the most unusual things about compact monothetic semigroups is that a compact monothetic semigroup may have more than one idempotent [1]. It was shown, using deep results of functional analysis, that any finite lower semilattice is the set of idempotents for some compact monothetic semitopological semigroup. With respect to this result, Berglund [2] stated the following problem.

Problem: Find a topologico-algebraic construction of a compact semitopological semigroup with many idempotents.

We give a construction of a sequentially compact monothetic semiclosure semigroup with countably many idempotents. Unfortunately, the closure structure of this semigroup cannot be topologized.

We start with notes about terminology and definitions.

Let X be a nonempty set. With any point x in X there will be associated a collection of subsets of X denoted by $\mathcal{U}(x)$. The map $u(x \mapsto \mathcal{U}(x))$ is called a closure structure on X if the folowing conditions are satisfied for each x in X:

(i) $\mathcal{U}(x) \neq \emptyset$.

(ii) For each $U \in \mathcal{U}(x)$, $x \in U$.

(iii) For each U and V in $\mathcal{U}(x)$ there exists a W in $\mathcal{U}(x)$ with $W \subset U \cap V$.

(iv) If $x \neq y$, then there exists a $U \in \mathcal{U}(x)$ and a $V \in \mathcal{U}(y)$ with $U \cap V = \emptyset$.

The set X, together with a closure structure $\mathcal{U}$, is called a closure space. A sequence $\langle c_n\}$ of elements of X is said to converge to c iff for every $U_c \in \mathcal{U}(c)$ there exists an n_0 such that $c_n \in U_c$ for $n \geqq n_0$. The element c is called a limit of the sequence c_n and is denoted by $c = \lim_n c_n$ (or simply $c_n \to c$).

A closure space is called sequentially compact iff every sequence $\{a_n\}$ of elements from X contains a convergent subsequence.

A semiclosure semigroup S is a semigroup provided with a closure structure in which multiplication is continuous in one variable, i.e. if $a_n \to a$, then $a_n b \to ab$ and $ba_n \to ba$ (the elements a_n, a and b being in S).

The continuity of the multiplication in a commutative semigroup can be defined as follows: For every $U_{xy} \in \mathcal{U}(xy)$ there exists a $U_y \in \mathcal{U}(y)$ with $x \, . \, U_y \subset U_{xy}$.

The construction

Let $q_1, q_2, \dots$ be the increasing sequence of all prime numbers. Put

$$p_{i,n} = (q_1 \cdot q_2 \dots q_i)^n$$

for $i, n = 1, 2, 3, \dots$.

1. Lemma. *The double sequence $\{p_{i,n}\}$ has the following property: For every fixed integers i, j, r and s with $i, j > 0$ and $i \neq j$ there exists an integer n_0 such that the following two sets are disjoint*

$$\{p_{i,n} + r, \ -p_{i,n} + r; \ n \geqq n_0\}$$

$$\{p_{j,n} + s, \ -p_{j,n} + s; \ n \geqq n_0\}.$$

Proof. Since $i \neq j$ we may and do assume without loss of genrality that $i > j$.

Choose an n_0 with $p_{j,n_0} > |r - s|$, $p_{i,n_0} + r > 0$, $p_{j,n_0} + s > 0$, $-p_{i,n_0} + r < 0$ and $-p_{j,n_0} + s < 0$. Let $n, m \geqq n_0$, then

$$|p_{i,n} + r - (p_{j,m} + s)| \geqq p_{j,n_0} |p_{i,n-n_0}(q_{j+1} \dots q_i)^{n_0} - p_{j,m-n_0}| - |r - s|$$

$$\geqq p_{j,n_0} - |r - s| > 0,$$

where we used the fact that

$$|p_{i,n-n_0}(q_{j+1} \dots q_i)^{n_0} - p_{j,m-n_0}| \geqq 1.$$

Similarly we can get that

$$|-p_{i,n} + r - (-p_{j,m} + s)| > 0,$$

thus the assertion of the lemma is true.

Let $G = \{\dots a^{-3}, a^{-2}, a^{-1}, a^0, a, a^2, a^3, \dots\}$ be a group and $E = \{0, e_1, e_2, \dots\}$ be a commutative semigroup of idempotents with $e_i e_j = 0 . e_i = 0$ if $i \neq j$. Let

$$S = \{a, a^2, a^3, \dots\} \cup E \cup \{a^n e_i; \ i = 1, 2, \dots, n = 0, 1, -1, 2, -2, \dots\}.$$

Define a commutative binary operation on S as follows: $a^n . e_i = a^n e_i$, $a^n . a^m e_i = a^{n+m} e_i$, $a^n e_i . a^m e_j = (a^n . a^m)(e_i . e_j)$ and $0 . x = 0$ for every x in S. Then

$$S = \{a, a^2, a^3, \dots\} \cup E \cup \bigcup_{i=1}^{\infty} G e_i,$$

E is exactly the set of all idempotents of S and Ge_i is a maximal subgroup of S for every $i = 1, 2, \dots$.

Define now the closure structure on S. Let r be a positive integer. Put

$$B_r(a^n) = \{a^n\},$$

$$B_r(e_i) = \{e_i, a^{p_{i,s}}, a^{p_{i,s}}e_i;\ s \geqq r\},$$

$$B_r(a^n e_i) = a^n B_r(e_i),$$

$$\mathcal{U}(x) = \{B_r(x);\ r = 1, 2, \ldots\} \qquad \text{for } x \neq 0,$$

$$\mathcal{U}(0) = S - \{B_1(x_1) \cup \ldots \cup B_1(x_n);\ x_i \neq 0,\ i = 1, 2, \ldots, n,\ n = 1, 2, \ldots\}.$$

2. Lemma. *$(S, \mathcal{U})$ is a closure space.*

Proof. (i) Clearly, $\mathcal{U}(x) \neq \emptyset$ for every x in S.

(ii) For each U in $\mathcal{U}(x)$, $x \in U$ by definition.

(iii) Let U and V be in $\mathcal{U}(x)$, we have to show that there exists a W in $\mathcal{U}(x)$ with $W \subset U \cap V$.

(iii)$_1$ If $x \neq 0$, then $U = B_r(x)$ and $V = B_s(x)$ for suitable r and s. It is enough to put $W = B_q(x)$ with $q = \max\{r, s\}$.

(iii)$_2$ If $x = 0$, then

$$U = S - B_1(x_1) \cup \ldots \cup B_1(x_n)$$

$$V = S - B_1(y_1) \cup \ldots \cup B_1(y_m)$$

for suitable n, m and suitable $x_i, y_j \in S - \{0\}$.

Put $W = S - B_1(x_1) \cup \ldots \cup B_1(x_n) \cup B_1(y_1) \cup \ldots \cup B_1(y_m)$, then $W \subset U \cap V$.

(iv) Let $x, y \in S$ with $x \neq y$; we have to show that there exists $U \in \mathcal{U}(x)$ and $V \in \mathcal{U}(y)$ with $U \cap V = \emptyset$.

(iv)$_1$ If $x, y \in S - \{0\}$, then by Lemma 1 there exists an r with $B_r(x) \cap B_r(y) = \emptyset$. It is sufficient to put $U = B_r(x)$ and $V = B_r(y)$.

(iv)$_2$ If $x = 0$, then put $U = B_1(y)$ and $V = S - B_1(y)$.

3. Lemma. *S is a sequentially compact closure space.*

Proof. Let $\{c_n\}$ be a sequence in S.

(i) If $\{c_n\} \cap B_1(x)$ is an infinite set for some $x \neq 0$, then clearly $\{c_n\}$ contains a subsequence which converges to x.

(ii) Let $\{c_n\} \cap B_1(x)$ be finite for every $x \in S - \{0\}$. We shall show that $c_n \to 0$. Let $U \in \mathcal{U}(0)$ with

$$U = S - B_1(x_1) \cup \ldots \cup B_1(x_k).$$

Since $\{c_n\} \cap B_1(x_i)$ is finite for every i, there exists an n_i with $\{c_n\}_{n \geqq n_i} \cap B_1(x_i) = \emptyset$. Put $n_0 = \max\{n_1, n_2, \ldots, n_k\}$. Then clearly $c_n \in U$ for $n \geqq n_0$.

4. Lemma. *The set $\{a, a^2, a^3, \ldots\}$ is dense in S.*

Proof. We have to show that to every $x \in S$ there exists a sequence $\{n_k\}$ with $a^{n_k} \to x$.

(i) If $x = e_i$, then put $n_k = p_{i,k}$.
(ii) If $x = a^s e_i$, then put $n_k = p_{i,k} + s$.
(iii) If $x = 0$, then put $n_k = (q_1 \cdot q_2 \cdots q_k)^k$.
Before proving the continuity of multiplication in one variable we prove some technical lemmas.

5. Lemma. *If $xy \neq 0$, then $x \,.\, B_r(y) \subset B_r(xy)$.*

Proof. (i) Let $x = e_i$ and $y = e_i$, then

$$x \,.\, B_r(y) = e_i \,.\, B_r(e_i) = \{e_i,\ a^{p_{i,s}} e_i,\ a^{-p_{i,s}} e_i;\ s \geqq r\} \subset B_r(e_i) = B_r(xy)\,.$$

(ii) If $x = a^n e_i$ and $y = a^m e_i$, then

$$x \,.\, B_r(y) = a^n e_i \,.\, B_r(a^m e_i) = a^{n+m} \,.\, e_i \,.\, B_r(e_i) \subset a^{n+m} \,.\, B_r(e_i) =$$
$$= B_r(a^n e_i \,.\, a^m e_i) = B_r(xy)\,.$$

The other cases are trivial.

6. Lemma. *If $x, y \in S - \{0\}$ and $xy = 0$, then for every nonzero $z \in S$ there exists an integer $r = r(z)$ such that*

$$x \,.\, B_r(y) \cap B_1(z) = \emptyset\,.$$

Proof. Let $x = e_i$, $y = e_j$ $(i \neq j)$, then

$$x \,.\, B_r(y) = e_i \,.\, B_r(e_j) = \{0,\ a^{p_{j,s}} e_i;\ s \geqq r\}\,.$$

If $z = a^n e_i$, then the assertion follows by Lemma 1.
If $z \neq a^n e_i$, then the assertion holds trivially. The case $x = a^n e_i$, $y = a^m e_j$ is similar.

7. Lemma. *For every $x, z \neq 0$, there exists a $t \neq 0$ such that*

$$x \,.\, (S - B_1(t)) \cap B_1(z) = \emptyset\,.$$

Proof. If $z = a^n$, then put $t = a^{n-m}$ if $x = a^n$, else t is arbitrary.
If $x = a^m$ and $z = a^n e_i$, then it is enough to put $t = a^{n-m} e_i$.
If $x = a^m e_i$ and $z = a^n e_i$, then put $t = a^{n-m} e_i$.
If $x = a^m e_i$, $z = a^n e_j$ and $i \neq j$ then the assertion holds true for every $t \neq 0$.

8. Theorem. *$(S, \mathcal{U})$ is a sequentially compact monothetic semiclosure semigroup.*

Proof. The sequential compactness of S was proved in Lemma 3. S is monothetic by Lemma 4. We have to show that the multiplication in S is continuous in each variable separately, i.e. to every $U_{xy} \in \mathcal{U}(xy)$ there exists a $U_y \in \mathcal{U}(y)$ with $x \,.\, U_y \subset U_{xy}$.
(i) If $xy \neq 0$, then it is enough to show that for every $B_r(xy)$, $x \,.\, B_r(y) \subset B_r(xy)$, but this is true because of Lemma 5.
(ii) Let $xy = 0$ and $x \neq 0 \neq y$. Let

$$U_{xy} = S - B_1(z_1) \cup \ldots \cup B_1(z_n)$$

be from $\mathscr{U}(0)$. Then by Lemma 6 for every z_i ($i = 1, 2, \ldots, n$) there exists an integer r_i with

$$x \,.\, B_{r_i}(y) \cap B_1(z_i) = \emptyset;$$

put $r = \max\{r_i;\ i = 1, 2, \ldots, n\}$ and put $U_y = B_r(y)$, then clearly $x \,.\, U_y \subset U_{xy}$.
(iii) Let $x \neq 0$ and $y = 0$, let

$$U_{xy} = S - B_1(z_1) \cup \ldots \cup B_1(z_n).$$

By Lemma 7 for every z_i ($i = 1, 2, \ldots, n$) there exists a t_i such that

$$x \,.\, (S - B_1(t_i)) \cap B_1(z_i) = \emptyset$$

and so

$$x \,.\, (S - B_1(t_i)) \subset S - B_1(z_i)$$

for every $i = 1, 2, \ldots, n$. Put

$$U_y = S - B_1(t_1) \cup \ldots \cup B_1(t_n),$$

then $x \,.\, U_y \subset U_{xy}$.

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ИДЕНПОТЕНТЫ МОНОТЕТИЧНОЙ КОМПАКТНОЙ ПОЛУГРУППЫ ПОЛУСХОДИМОСТИ

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Резюме

В этой статье дана конструкция монотетичной компактной полугруппы полусходимости, которая содержит бесконечно много иденпотентов.