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ON VECTOR MEASURES AND DISTRIBUTIONS

MILOSLAV DUCHOŇ

Let T be the quotient group $R/2\pi Z$ (R and Z denoting the additive group of reals, integers, respectively). If k is an integer, $k \geqslant 0$, $C^k = C^k(T)$ will denote the set of all complex-valued functions with period 2π and with k continuous derivatives, and $C^\infty = C^\infty(T) = \bigcap_{k=1}^{\infty} C^k$; C is written instead of C^0. Let X be a sequentially complete locally convex Hausdorff topological vector space. Let F be a vector-valued distribution, i.e. F is a continuous linear mapping on C^∞ with values in X. The Fourier—Schwartz coefficients of F are, by definition, the elements of X of the form $\hat{F}(n) = \frac{1}{2\pi} F(e^{-int})$, $n \in Z$. If F is also continuous (weakly compact) on C into X, we say that F is the Radon mapping (Radon measure) with values in X. In this paper the relations are investigated between the trigonometric series

$$\text{(A)} \qquad \sum_{n \in Z} c_n e^{int},$$

c_n being elements of X, and the formally (without the member c_0) integrated series

$$\text{(B)} \qquad \sum_{n \neq 0} (in)^{-1} c_n e^{int},$$

It is shown, e.g., that (A) is the Fourier—Stieltjes series, i.e. $c_n = \hat{F}(n)$ for a Radon measure F if and only if (B) is the Fourier—Lebesgue series of some function z on $[-\pi, \pi]$ into X of weakly compact semivariation,

$$(in)^{-1} c_n = \frac{1}{2\pi} \int e^{-int} z(t)\, dt, \qquad n \in Z,\ n \neq 0,$$

or if and only if the coefficients c_n are expressible as the Riemann—Stieltjes integrals with respect to such a function z in the form

$$c_n = \frac{1}{2\pi} \int e^{-int}\, dz(t).$$

From this we also deduce that any Radon measure F on T with values in X is expressible in the form

$$F = c + \mathrm{D}z,$$

where c is a constant element of X and z is a function on $[-\pi, \pi]$ into X of a weakly compact semivariation, Dz denoting the distributional derivative of the vector-valued distribution determined by z. The converse is also true. These results are vector generalizations of the results obtained in [3, Ch. 12] for scalar-valued distributions.

1. Let the topology of our sequentially complete locally convex space X — with the dual X' and the bidual X'' — be defined by a family $P = (p)$ of continuous seminorms on X. Let $I = [a, b]$ be a real interval. If z is a function on I with values in X, we say that z is of bounded semivariation in I if the set $SV(z, I)$ consisting of all the elements of the form

$$\sum_{i=1}^{n} a_i[z(t_i) - z(t_{i-1})],$$

where $a = t_0 < t_1 < \ldots < t_n = b$ and $|a_i| \leqslant 1$, a_i being complex numbers, is a bounded set in X. Clearly z is of bounded semivariation in I if and only if for every p in P there is a positive finite number K_p such that

$$pSV(z, I) = \sup p\left(\sum_{i=1}^{n} a_i[z(t_i) - z(t_{i-1})]\right) \leqslant K_p,$$

where the supremum is taken over all $a \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant b$ and a_i are complex numbers with $|a_i| \leqslant 1$. We say that $pSV(z, I)$ is a p-semivariation of z in $I = [a, b]$. Recall also that the p-variation of z in I is defined by

$$pV(z, I) = \sup \sum_{i=1}^{n} p[z(t_i) - z(t_{i-1})],$$

where the supremum is taken over all $a \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant b$. Clearly $pSV(z, I) \leqslant pV(z, I)$ for all p in P. In general the inequality may be strict. It can be shown that if $X = K$, real or complex numbers, then (with absolute value for p)

$$pSV(z, I) = \sup\left|\sum_{i=1}^{n} a_i[z(t_i) - z(t_{i-1})]\right| =$$

$$= \sup \sum_{i=1}^{n} |z(t_i) - z(t_{i-1})| = pV(z, I),$$

that is the semivariation of z for scalar-valued functions z is the same thing as the variation of z. This makes it possible to deduce that

$$pSV(z, I) = \sup_{x' \leqslant p} V(x'z, I),$$

where we will write $x' \leqslant p$ whenever $|x'x| \leqslant p(x)$ for all x in X, $x'z(t) = x'(z(t))$. We say that the function z on I into X is of a weakly compact (compact) semivariation if the set $SV(z, I)$ is contained in a weakly compact (compact) subset W of X; clearly then z is bounded semivariation.

In the context of the locally convex space X it is possible to define the Riemann – Stieltjes integral

$$\int_a^b f(t)\,\mathrm{d}z(t)$$

of a scalar-valued function f with respect to the function z on I into X, of bounded semivariation, as an element of X to which Riemann sums

$$\sum_{i=1}^{n} f(s_i)\,[z(t_i) - z(t_{i-1})],$$

where $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = b$ and $t_{i-1} \leqslant s_i \leqslant t_i$, $i = 1, \ldots, n$, converge with respect to the topology of X. This integral has properties analogical to those in the Banach space setting, cf. [4]. For example, this integral exists for any continuous function f on I; for any two continuous functions f and g and complex numbers c and d we have

$$\int_a^b [cf(t) + dg(t)]\,\mathrm{d}z(t) = c\int_a^b f(t)\,\mathrm{d}z(t) + d\int_a^b g(t)\,\mathrm{d}z(t)$$

and for any x' in X' we have

$$x'\left(\int_a^b f(t)\,\mathrm{d}z(t)\right) = \int_a^b f(t)\,\mathrm{d}x'z(t).$$

If (f_n) is a sequence of continuous functions converging to f uniformly on I, then

$$\int_a^b f(t)\,\mathrm{d}z(t) = \lim_{n\to\infty}\int_a^b f_n(t)\,\mathrm{d}z(t).$$

Note also that

$$p\left(\int_a^b f(t)\,\mathrm{d}z(t)\right) \leqslant \|f\|\,pSV(z, I), \quad \text{for } p \text{ in } P,$$

where $\|f\| = \sup_{t\in I} |f(t)|$.

If the function z on I into X is of a bounded semivariation, then we can see that

$$L(f) = \int_a^b f(t)\,\mathrm{d}z(t)$$

defines a continuous linear mapping on $C(I)$ — the space of all scalar-valued continuous functions on I with the supremum norm — into X.

Denote by $NBV(I)$ the space of all normalized complex-valued functions on I of bounded variation. Recall that $NBV(I)$ may be taken as the dual space of $C(I)$. If L is a continuous linear mapping on $C(I)$ into X, then for every x' in X' we obtain a continuous linear form $x'L$ on $C(I)$ and so there is a unique function $u_{x'}$ in $NBV(I)$ such that

$$x'L(f) = \int f(t)\,\mathrm{d}u_{x'}(t)$$

for all f in $C(I)$. By this a function z on I into X'' of bounded semivariation is defined, $z(t)x' = u_{x'}(t)$. Moreover the following result can be proved, the proof being analogical to that of the result in [2, 9.4.14 A]. We omit the details.

Proposition. *Let L be a continuous linear mapping on $C(I)$ into X. Then there exists a unique function z on I into X'' of bounded semivariation such that*

a) *for each x' in X' the function $s \to zx'(s) = z(s)x'$ belongs to $NBV(I)$ and*

b) *the mapping $x' \to zx'$ of X' into $NBV(I)$ is continuous in the $\sigma(X', X)$-topology and the $\sigma(NBV(I), C(I))$-topology and*

$$L(f) = \int f(t)\,\mathrm{d}z(t) \tag{!}$$

in the sense that

$$x'L(f) = \int f(t)\,\mathrm{d}z(t)\,x', \quad \text{for } x' \text{ in } X'.$$

Conversely, every such function z on I into X'' defines a continuous linear mapping on $C(I)$ into X.

So (!) gives a representation of the continuous linear mapping L on $C(I)$ into X be means of a function z with the values in $Y = X''$. In particular we may have z with the values in X as above.

If f is a continuous function and z is a function on I into X of the bounded semivariation on I, then the integral

$$\int_a^b z(t)\,\mathrm{d}f(t)$$

can be defined and the formula for integration by parts holds. Namely

$$\int_a^b f(t)\,\mathrm{d}z(t) = f(b)\,z(b) - f(a)\,z(a) - \int_a^b z(t)\,\mathrm{d}f(t),$$

the proof being similar as that in the case X is a Banach space [4]. A similar formula for integration by parts can be given for a continuous function f on I and a function z on I into X'' of the bounded semivariation. We shall confine ourselves to the space $SV(I, X'')$, the vector space of all functions z on I into X'' such that for each x' in X' the scalar function $z(.)x'$ is in $NBV(I)$ and the mapping from x' into $z(.)x'$ is continuous in the $\sigma(X', X)$ and the $\sigma(C', C)$-topologies, cf. Proposition. Note that then by the relation

$$M(f)x' = \int_a^b f(t)\,dz(t)\,x', \qquad f \in C(I),$$

a continuous linear mapping on $C(I)$ into X is defined. For the map $x' \to M(f)x'$ is a linear form on X' which is continuous in the topology $\sigma(X'. X)$ and hence $M(f)$ belongs to X and not only to X'', since, by assumption, the mapping x' into $z(.)x'$ is a continuous mapping in $\sigma(X', X)$ and the $\sigma(C', C)$-topologies.

2. From now on let $I = [a, b] = [-\pi, \pi]$. We shall consider integrals only over I and hence we will omit the limits of integration.

If z is a function on I into X of bounded semivariation, $z \in SV(I, X)$, then the integral

$$A_z(f) = \int f(t)\,dz(t)$$

defines a continuous linear mapping on $C(I)$ into X and so also a continuous linear mapping on C^∞ into X, i.e. a vector-valued distribution on T with the values in X. We need here only the elementary properties of vector-valued distributions as contained in [5, Ch. IV.]. The same as for z in $SV(I, X)$ can be said for a function z in $SV(I, X'')$ and the integral

$$B_z(f) = \int f(t)\,dz(t).$$

We say that A_z and B_z are Radon mappings on T with the values in X.

If z on I into X is of weakly compact semivariation, $z \in C_w SV(I, X)$, the relation

$$C_z(f) = \int f(t)\,dz(t)$$

defines a weakly compact mapping on $C(I)$ into X, cf. [1] or [6] for Banach spaces, the proof for locally convex spaces being similar, and so a weakly compact Radon mapping on T into X, called the Radon vector measure on T in the context of this paper. Similarly if z is of bounded variation, $z \in BV(I, X)$, the relation

$$D_z(f) = \int f(t)\,dz(t)$$

defines a continuous linear mapping on $C(I)$ into X with bounded variation and so — as a vector-valued distribution on T — a Radon vector measure with bounded variation. Note that for every p in P we have

$$p\left(\int f(t)\,\mathrm{d}z(t)\right) \leqslant \int |f(t)|\,\mathrm{d}pV(z)(t), \qquad f \text{ in } C(I),$$

where $pV(z)(t) = pV(z, [0, t])$.

If z is in $SV(I, X)$, then by means of the formula for integrations by parts we show that the relation

$$U_z(f) = \int f(t)\,z(t)\,\mathrm{d}t$$

defines a continuous linear mapping on $C(I)$ into X and hence on C^∞ into X and so a vector-valued distribution on T with the values in X. Its distributional derivative is, by using the formula for integration by parts,

$$\mathrm{D}U_z(u) = -[u(t)\,z(t)]_{-\pi}^{\pi} + \int u(t)\,\mathrm{d}z(t), \qquad u \in C^\infty.$$

So $\mathrm{D}U_z$ is a vector-valued distribution with the values in X which is also a continuous linear mapping on $C(I)$, i.e. is a Radon mapping on T with the values in X. Hence from the preceding the following can be proved.

Theorem 1. *Let z be a function on I into X and U_z the linear mapping defined by the relation*

$$U_z(u) = \int u(t)\,z(t)\,\mathrm{d}t, \qquad u \in C^\infty.$$

a) *If z is of bounded semivariation ($z \in SV(I, X)$), then U_z and its distributional derivative (as X-valued distribution) are Radon mappings on T into X.*

b) *If z is of weakly compact semivariation ($z \in C_w SV(I, X)$), then U_z and its distributional derivative (as X-valued distribution) are Radon measures on T into X.*

c) *If z is of bounded variation ($z \in BV(I, X)$), then U_z and its distributional derivative (as X-valued distribution) are Radon mappings with bounded variation on T into X-Radon measures with bounded variation.*

Remark. The functions u in C^∞ are periodic by definition, the functions z in $SV(I, X)$ and so on are, however, in general not periodic.

Similarly we have the following.

Theorem 2. *Let z be in $SV(T, X'')$. Then the relation*

$$U_z(f) = \int f(t)\,z(t)\,\mathrm{d}t$$

defines a continuous linear mapping on $C(I)$ into X'' and so a vector-valued distribution on T with the values in X'', a Radon mapping on T with the values in X''. Its distributional derivative $\mathrm{D}U_z$ *is, as a vector-valued distribution, a Radon mapping on T with the values in X''.*

We may restate the preceding theorems in the language of distributions.

Theorem 3. *Let z be a function on I into X or into X''.*

a) *If z is in $SV(I, X)$, then its distributional derivative* $\mathrm{D}z$ $(=\mathrm{D}U_z)$ *is a Radon mapping on T into X.*

b) *If z is in $C_w SV(I, X)$, then its distributional derivative* $\mathrm{D}z$ $(=\mathrm{D}U_z)$ *is a weakly compact Radon mapping or a Radon measure on T with values in X.*

c) *If z is in $BV(I, X)$, then its distributional derivative* $\mathrm{D}z$ $(=\mathrm{D}U_z)$ *is a Radon mapping with bounded variation (a Radon measure with bounded variation or a majorized Radon mapping).*

d) *If z is in $SV(I, X'')$, then its distributional derivative* $\mathrm{D}z$ $(=\mathrm{D}U_z)$ *is a Radon mapping on T with the values in X''.*

3. In the rest of the paper we shall write e_n instead of the function $t \to \exp(int)$. Let F be a vector-valued distribution on T, i.e. a continuous linear mapping on C^∞ with the values in X. The Fourier–Schwartz coefficients of F are, by definition, the elements of X of the form

$$\hat{F}(n) = \frac{1}{2\pi} F(\bar{e}_n), \qquad n \in Z.$$

If F is also a continuous linear mapping on C into X we will say that $\hat{F}(n)$ are the Fourier–Stieltjes coefficients of F. Then

$$\hat{F}(n) = \frac{1}{2\pi} \int \bar{e}_n(t)\,\mathrm{d}z(t)$$

for the Radon mapping on T with the values in X represented by z in $SV(I, X)$ or in $SV(I, X'')$.

An application of the formula for integration by parts shows that

$$\hat{F}(n) = \frac{1}{2\pi}[z(\pi) - z(-\pi)](-1)^n + in\frac{1}{2\pi}\int z(t)\,\bar{e}_n(t)\,\mathrm{d}t,$$

that is

$$\hat{F}(n) = \frac{1}{2\pi}[z(\pi) - z(-\pi)](-1)^n + in\hat{z}(n),$$

where $\hat{z}(n)$ are the Fourier–Lebesgue coefficients of z, $\hat{z}(n)$ being elements of X for z in $SV(I, X)$ and those of X'' for z in $SV(I, X'')$.

It is well known that $(-1)^n(in)^{-1}$, $n \neq 0$, are the Fourier–Lebesgue coefficients of a scalar function of bounded variation, in other words,

$$\sum_{n \neq 0} (-1)^n (in)^{-1} e_n$$

is the Fourier – Lebesgue series of a scalar function of bounded variation, say h. Hence $\hat{F}(n)(in)^{-1}$, $n \neq 0$, are the Fourier – Lebesgue coefficients of a function z_1 from $SV(I, X)$ or $SV(I, X'')$. From this, using Theorem 3, we may conclude the following. Consider the trigonometric series

$$(\mathrm{A}) = \sum_{n \in Z} c_n e_n$$

and the (without the constant term c_0) formally integrated series, i.e. the trigonometric series

$$(\mathrm{B}) = \sum_{n \neq 0} (in)^{-1} c_n e_n$$

c_n being elements of X.

Theorem 4. a) *A trigonometric series* (A) *is the Fourier – Stieljes series of a Radon mapping F on T with values in X if and only if the trigonometric series* (B) *is the Fourier – Lebesgue series of a function z from $SV(I, X'')$, in particular, from $SV(I, X)$.*

b) *A trigonometric series* (A) *is the Fourier – Stieltjes series of a Radon measure with the values in X if and only if the trigonometric series* (B) *is the Fourier – Lebesgue series of a function z from $C_{w^*}SV(I, X)$.*

c) *A trigonometric series* (A) *is the Fourier – Stieltjes series of a Radon measure with the values in X with bounded variation if and only if the trigonometric series* (B) *is the Fourier – Lebesgue series of a function z from $BV(I, X)$.*

Since the distributional derivative of (B) is the series (A) without its constant term, we may infer that a) any Radon mapping F on T with the values in X; b) any Radon measure F on T with the values in X; c) any Radon measure F with bounded variation on T with the values in X is expressible in the form

$$F = C + \mathrm{D}z,$$

where a) $c \in X$ is a constant and z is a function in $SV(I, X'')$, in particular a function in $SV(I, X)$; b) $c \in X$ and z is a function in $C_{w^*}SV(I, X)$; c) $c \in X$ and z is a function in $BV(I, X)$. According to Theorem 3 the converse is also true.

We may also say that a trigonometric series (A) is the Fourier – Stieltjes series of a) Radon mapping F on T with the values in X; b) a Radon measure on T with the values in X; c) a Radon measure with bounded variation on T with the values in X if and only if the coefficients c_n are expressible as Riemann – Stieltjes integrals with respect to a function z from a) $SV(I, X'')$, in particular $SV(I, X)$; b) $C_{w^*}SV(I, X)$; c) $BV(I, X)$ in the following manner

$$c_n = \frac{1}{2\pi} \int \bar{e}_n(t)\, \mathrm{d}z(t).$$

Note that if X is a semireflexive space or, equivalently, weakly quasicomplete, then the function z takes its values in X for any Radon mapping F on T with the values in X.

In conclusion we may say that all a) Radon mappings; b) Radon measures; b) majorized Radon measures F on T into X are of the form

$$F = c + \mathrm{D}z,$$

where c is a constant element and z is a function in a) $SV(I, X'')$, in particular in $SV(I, X)$; b) in $C_{w} SV(I, x)$; c) in $BV(I, X)$.

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О ВЕКТОРНЫХ МЕРАХ И ОБОБЩЕННЫХ ФУНКЦИЯХ

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Резюме

В работе рассматриваются некоторые отношения между функциями z на отрезке $[a, b]$ со значениями в локально выпуклом пространстве X с ограниченной полувариацией и векторными обобщенными функциями. Получено представление векторных отображений Радона при помощи производной некоторой функции z со значениями в X с ограниченной полувариацией.