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LOCALLY LINEAR GRAPHS

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In the article presented we shall study some local properties of graphs. P. Erdös and M. Simonovits [1] have shown the maximal number of edges of a locally acyclic graph, Z. Ryjáček and B. Zelinka [2] have constructed a locally disconected graph with a large number of edges. A survey article on local properties was written by J. Sedláček [3].

We shall describe one class of locally disconnected and locally acyclic graphs and show the bounds of number of edges in these graphs. All graphs are considered finite undirected without loops and multiple edges.

Let G be a graph and x be its vertex, then by $N_G(x)$ we denote the subgraph of G induced by the set of all vertices adjacent to x. If $N_G(x)$ is a linear graph (i.e. $N_G(x)$ is regular of degree 1) for each vertex x of G, then G is called locally linear. It is evident that each vertex of G is of an even degree.

Theorem 1. *A graph G is locally linear if and only if each edge of G belongs to exactly one triangle.*

Proof. ($\Rightarrow$) Let G be a locally linear and let e be an edge with end vertices x_1, x_2. As each vertex of $N_G(x_1)$ is of degree 1, then there is exactly one vertex x_3 adjacent to x_1 and also to x_2. Hence the edge e belongs to exactly one triangle with vertices x_1, x_2, x_3.

($\Leftarrow$) Let x_1 be any vertex of G, let x_2 be a vertes of $N_G(x_1)$. The edge e with the end vertices x_1, x_2 belongs to exactly one triangle; let its third vertex be x_3. Then the vertex x_2 (and analogously x_3) is in $N_G(x_1)$ of degree 1. As x_2 was chosen arbitrarily, the graph $N_G(x_1)$ is a linear graph.

Theorem 2. *Let G be a locally linear graph. Let T be a triangle with the edges e_1, e_2, e_3. Let G_0 be the graph resulting from G by omitting the edges $e_1, e_2. e_3$. Then each component of G_0 is either a locally linear graph or an isolated vertex.*

Proof. The assertion of this Theorem follows easily from Theorem 1.

Now we describe the construction of a regular locally linear graph.

Construction. Let k be an even integer. We denote by H_k a regular locally linear graph of degree k.

The graph H_2 is a triangle. Now let us have a graph H_k. Then we construct H_{k+2} in the following way:

$H_{k+2} = H_k \square C_3$, where $\square$ denotes the cartesian product of the graphs. In this graph every edge belongs to exactly one triangle. Hence H_{k+2} is a locally linear graph.

By this we have proved the following

Theorem 3. *Let k be an even integer. Then there exists a regular locally linear graph of degree k.*

Now we give the bounds for the number of edges contained in a locally linear graph.

Theorem 4. *Let G_n be a connected locally linear graph with n vertices and m edges. Then*

$$3\left[\frac{n}{2}\right] \leqq m. \tag{1}$$

Proof. Let in G_n exist the vertex x_n of degree 2 (in the opposite case the number of edges is at least $2n$). Let x_n be adjacent to x_1 and x_2. Now we omit the vertex x_n and the edges $e_1 = (x_1, x_n)$, $e_2 = (x_2, x_n)$, $e_3 = (x_1, x_2)$ from G_n. Thus, a graph G_{n-1} consisting of one or two components results. Now the inequality (1) is easy to prove by induction with respect to n and can be left to the reader.

To determine the upper bound of the number of edges we prove the following lemmas.

Lemma 1. *Let G be a locally linear graph with n vertices and let d_i be a degree of the vertex x_i ($i = 1, 2, \ldots, n$). Let*

$$\sum_{i=1}^{n} d_i = nk. \tag{2}$$

Then

$$\sum_{i=1}^{n} d_i^2 \geqq nk^2, \tag{3}$$

and the equality holds if and only if G is k-regular.

Proof. Let $d_i = k + z_i$ ($i = 1, 2, \ldots, n$). Then $\sum_{i=1}^{n} d_i = nk + \sum_{i=1}^{n} z_i$ and hence

$$\sum_{i=1}^{n} z_i = 0. \tag{4}$$

Then $\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (k + z_i)^2 = nk^2 + 2k \sum_{i=1}^{n} z_i + \sum_{i=1}^{n} z_i^2$. From (4) we can see that $\sum_{i=1}^{n} d_i^2 = nk^2 + \sum_{i=1}^{n} z_i^2 \geqq nk^2$. It is evident that $\sum_{i=1}^{n} z_i^2 = 0$ if and only if $z_i = 0$ for each $i \in \{1, 2, \ldots, n\}$.

Lemma 2. *Let G be a connected locally linear graph. Let*

$$2m = nk. \tag{5}$$

Then just one of the following assertions holds:

(i) *G is k-regular*,

(ii) *G contains as a subgraph the triangle with vertices x_1, x_2, x_3, for which there holds*

$$d_1 + d_2 + d_3 > 3k. \tag{6}$$

Proof. Let (i) hold. Then the invalidity of (ii) is evident. The existence of a k-regular graph follows from Theorem 3.

Now let (i) be invalid. Denote by t the number of the triangles of G. From Theorem 1 it follows that $3t = m$. As each vertex is of an even degree, then $m = \frac{nk}{2}$, and this yields $t = \frac{nk}{6}$. Denote by T_j ($j = 1, 2, \ldots, t$) the triangles of G. Let $\sum_{x_i \in T_j} d_i = 3k + r_j$ for each T_j. We show that there exists a triangle T_l, for which $r_l > 0$. We see that

$$\sum_{j=1}^{t} \sum_{x_i \in T_j} d_i = \frac{nk^2}{2} + \sum_{j=1}^{t} r_j. \tag{7}$$

In this sum the degree of x_i is included $\frac{d_i}{2}$ times, thus for every edge incident to x_i we have to subtract the expression $\frac{d_i}{2} - 1$ from (7). As the vertex x_i is incident to d_i edges, we have to subtract the expression $\frac{d_i(d_i - 2)}{2}$ for each vertex x_i from the sum (7). Hence

$$\sum_{i=1}^{n} d_i = \frac{nk^2}{2} + \sum_{j=1}^{t} r_j - \sum_{i=1}^{n} \frac{d_i(d_i - 2)}{2} = \frac{nk^2}{2} - \sum_{i=1}^{n} \frac{d_i^2}{2} + \sum_{i=1}^{n} d_i + \sum_{j=1}^{t} r_j.$$

This yields $\sum_{j=1}^{t} r_j = \sum_{i=1}^{n} \frac{d_i^2}{2} - \frac{nk^2}{2}$. However, as we have assumed that G is not regular, by our Lemma 1 there holds $\sum_{j=1}^{t} r_j > 0$ $\left(\text{as } \sum_{i=1}^{n} \frac{d_i^2}{2} > \frac{nk^2}{2}\right)$ and there exists a triangle T_l for which $\sum_{x_i \in T_l} d_i > 3k$.

We simply prove that the sum of the degrees of the vertices belonging to one triangle is at most equal to $n + 3$. Let us choose a triangle and denote it by $T(x_1, x_2, x_3)$. If $d_1 + d_2 + d_3 > n + 3$, then there exists a vertex x_j ($j \in \{4, 5, \ldots, \ldots, n\}$) adjacent to two vertices of $T(x_1, x_2, x_3)$ (e.g. x_1, x_2). Thus the edge (x_1, x_2) belongs both to triangles $T(x_1, x_2, x_3)$ and $T(x_1, x_2, x_j)$. Hence, G is not locally linear. From Lemma 2 it follows that in the graph where $m > \frac{n(n+3)}{6}$ there must exist a triangle $T(x_i, x_j, x_k)$ for which $d_i + d_j + d_k > n + 3$.

Now we see that the following holds.

Theorem 5. *Let G be a locally linear graph with n vertices and m edges. Then*

$$m \leqq \frac{n(n+3)}{6}. \tag{8}$$

Remark. For $n > 9$ no graph is known for which in (8) an equality holds.

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ЛОКАЛЬНО ЛИНЕЙНЫЕ ГРАФЫ

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Резюме

Если G есть граф и x есть его вершина, то $N_G(x)$ обозначает подграф графа G, порожденный множеством всех вершин, смежных с x в G. Граф G называется локально линейным, если $N_G(x)$ является регулярным графом степени 1 для всех x из G. В статье изучаются основные свойства таких графов. Показаны тоже минимальное и максимальное числа ребер этих графов.