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A NONLINEAR DIFFUSION EQUATION WITH NONLINEAR BOUNDARY CONDITIONS: METHOD OF LINES

JÁN FILO

1. Introduction

In this paper we investigate the existence and some properties of solutions of the initial-boundary value problem

$$(1.1)\qquad \begin{aligned} &(\beta(u))_t - \Delta u = f(x, t, u) && x \in D,\ t > 0 \\ &\frac{\partial u}{\partial \nu} + g(x, t, u) = 0 && x \in \Gamma,\ t > 0 \\ &u(x, 0) = u_0(x) && x \in D, \end{aligned}$$

where $\beta(u) = |u|^m \operatorname{sign} u$ for some positive parameter m, $D \subset \mathbf{R}^N$ is a smoothly bounded domain with boundary Γ, $\partial u/\partial \nu$ denotes the outward directed normal derivative of u on Γ, f and g are functions satisfying some smoothness and growth conditions to be detailed later, nevertheles, g is nondecreasing in u and the initial function u_0 is allowed to change the sign.

The equation in (1.1) appears in various physical, chemical and biological models and without the reaction term f it is for $0 < m < 1$ well known as the porous medium or slow diffusion equation, for $m = 1$ as the heat conduction equation and for $m > 1$ as the plasma or fast diffusion equation. Many results are known in the case $m = 1$ and therefore we are primarily interested in the case $m \neq 1$. For these values of m, however, a "degeneracy" or "singularity" may occur. To see this let us rewrite the equation in (1.1) putting $v = \beta(u)$ into a more familiar form

$$v_t - \Delta(|v|^\alpha \operatorname{sign} v) = f, \qquad \alpha = 1/m$$

and one can see that the diffusion coefficient $D(v) = \alpha|v|^{\alpha - 1}$ tends to zero $(0 < m < 1)$ or to infinity $(m > 1)$ when v tends to zero. Therefore it is necessary to be careful and consider a suitable class of weak solutions (see, e.g., [13] for $0 < m < 1$ and [14] for $m > 1$).

The present paper is organized as follows. In Section 2 we prove the existence of the solution to Problem (1.1) for $0 < m \leqslant 1$ using the method of lines. This method was applied to the nondegenerate parabolic equations with the nonlinear boundary conditions by Chzhou Yui-Lin in [2], Kačur in [8], and in general it was intensively studied in [7]. As concerns the nonlinear diffusion problems, this method was used to treat the slow diffusion equation with homogeneous Dirichlet boundary conditions by Jerome in [6]. Here we improve his procedure using a more suitable inequality to estimate the time derivative. By means of that also the restriction on the sign of initial functions is removed and the L^∞ estimate is not necessary. Further, a certain type of the smoothing effect is established, namely, starting with the initial function $u_0 \in L^{m+1}(D)$ it is shown that the weak solution u of Problem (1.1) is actually in $L^{m+1}(D) \cap \cap H^1(D)$ at any later time (see also [12]).

In Section 3 some sufficient conditions for L^∞ solvability are presented and a type of the "maximum" principle is derived. The case of the fast diffusion is briefly considered in Section 4. We are able to introduce the uniqueness and comparison theorem for Problem (1.1) only for the L^∞ solutions, and for its proof the method discussed in [1] is adopted. In the end also a local existence theorem is stated.

In the sequel we shall adopt the following notations: Let $I = (0, T)$, $Q_T = = D \times I$, $S_T = \Gamma \times I$, $Q_{\varepsilon, T} = D \times (\varepsilon, T)$, $S_{\varepsilon, T}$ analogously and let $|D|$ denote the Lebesgue measure of the set D. The norms in the spaces $L^p(D)$, $W^{1,p}(D)$ ($W^{1,2} \equiv H^1$) are denoted by $|\cdot|_p$, $|\cdot|_{1,p}$ $1 \leqslant p \leqslant \infty$ and in $L^2(\Gamma)$ by $|\cdot|_{2,\Gamma}$ (the function spaces we use are rather familiar and we omit the definitions, see, e.g., [10]). C or $C(\eta)$ indicates various constants (even in the same discusion) depending on η and other known constants and set, suppressing integration variables,

$$\int_D u(t)\varphi(t) = \int_D u(x, t)\varphi(x, t)\,dx,$$

$$\int_\Gamma g(t, u(t))\varphi(t) = \int_\Gamma g(X, t, u(X, t))\varphi(X, t)\,dS$$

and

$$\iint_{Q_T} f(u)\varphi = \iint_{Q_T} f(x, t, u(x, t))\varphi(x, t)\,dx dt.$$

We shall frequently use Young's inequality, i.e. $ab \leqslant \varepsilon a^p + \varepsilon^* b^q$, where $a, b \geqslant 0$, $p > 1$, $p^{-1} + q^{-1} = 1$, $\varepsilon^* = ((p\varepsilon)^{q/p} q)^{-1}$, and denote $\gamma(u) = |u|^{(m+1)/2} \operatorname{sign} u$.

2. The case of slow diffusion

This part of the paper is concerned with the existence and some properties of solutions to Problem (1.1) for $0 < m \leqslant 1$ assuming the following:

(H_1) *$u_0 \in L^{m+1}(D)$ with no restriction on the* sign *of u_0.*

(H_2) *$f \in C(\bar{Q}_T \times \mathbf{R})$, where T is an arbitrarily fixed positive time, and there exists a constant $K(=K(T)) \in \mathbf{R}^+$ such that*

$$|f(x, t, u) - f(x, s, v)| \leqslant K(|t-s|(1+|\beta(u)|) + |\beta(u) - \beta(v)|)$$

for all (x, t), $(x, s) \in Q_T$, u, $v \in \mathbf{R}$.

(H_3) *$g \in C(\bar{S}_T \times \mathbf{R})$ and there exists a constant $L(=L(T)) \in \mathbf{R}^+$ such that*

$$|g(x, t, u) - g(x, s, v)| \leqslant L(|t-s|(1+|u|) + |u-v|)$$

for all (x, t), $(x, s) \in S_T$, u, $v \in \mathbf{R}$.

(H_4) *$g(x, t, u)$ is nondecreasing in u for all $(x, t) \in S_T$.*

We shall refer to these hypotheses collectively as (H). The assumption of continuity of f and g in the space variable may be relaxed but we omit such generalization for the sake of convenience.

The main result of this section reads then as follows.

Theorem 1. *Let* (H) *hold, and suppose that* $0 < m \leqslant 1$. *Then Problem* (1.1) *admits a solution u in the following sense*:

$$u \in L^\infty(I; L^{m+1}(D)) \cap L^2(I; H^1(D)), \quad t^{1/2} u \in L^\infty(I; H^1(D)), \\ t^{1/2} u_t \in L^2(I; L^{m+1}(D)), \quad t^{1/2}(\gamma(u))_t \in L^2(Q_T), \tag{2.1}$$

and

$$\int_D \beta(u(t))\varphi(t) - \iint_{Q_t} (\beta(u)\varphi_t - \nabla u \nabla \varphi + f(u)\varphi) + \\ + \iint_{S_t} g(u)\varphi = \int_D \beta(u_0)\varphi(0) \tag{2.2}$$

for all $t \in I$ and $\varphi \in H^1(Q_T)$.

If, in addition to (H), *$u_0 \in H^1(D)$, then instead of* (2.1) *we have*

$$u \in L^\infty(I; H^1(D)), \quad u_t \in L^2(I; L^{m+1}(D)), \quad (\gamma(u))_t \in L^2(Q_T), \tag{2.3}$$

and if we take $f(x, t, u) = f(x, u)$ and $g(x, t, u) = g(x, u)$ only, then

$$\frac{4m}{(m+1)^2} \int_s^t \int_D (\gamma(u))_t^2 + J(u(t)) \leqslant J(u(s)) \tag{2.4}$$

for all $0 \leqslant s \leqslant t \leqslant T$, where

$$J(u) = \tfrac{1}{2} \int_D |\nabla u|^2 + \int_\Gamma \int_O^u g(r)\, dr - \int_D \int_O^u f(r)\, dr. \tag{2.5}$$

Inequality (2.4) *is an energy inequality.*

Remark. It is not difficult to see that (2.1) ((2.3)) implies that $u \in C((0, T]; L^{m+1}(D))$, $\gamma(u) \in C((0, T]; L^2(D))$ ($u \in C([0, T]; L^{m+1}(D))$, $\gamma(u) \in C([0, T]; L^2(D))$, respectively).

We now prove a series of assertions, which contain most of the essential elements for the proof of Theorem 1. Because in Section 4 the case of the fast diffusion ($m > 1$) will be considered, we shall treat, where it is possible, also this value of parameter m.

Suppose now that an integer n is specified and set $h = T/n$. For $i = 1, 2, \ldots, n$ consider the sequence of semilinear elliptic problems obtained formally by applying to (1.1) an implicit time discretization formula

$$(2.6)\qquad \begin{aligned} &-\Delta u_i + h^{-1}(\beta(u_i) - \beta(u_{i-1})) = f(x, (i-1)h, u_{i-1}) && \text{in } D,\\ &\frac{\partial u_i}{\partial \nu} + g(x, ih, u_i) = 0 && \text{on } \Gamma, \end{aligned}$$

where u_0 is given by (H_1). At the end of this section we prove the following result concerning weak solutions of Problem (2.6).

Proposition 1. *Suppose that* $0 < m < \infty$ *and set* $V = H^1(D) \cap L^{m+1}(D)$. *Then for given* $F \in L^{(m+1)/m}(D)$, $c > 0$ *and* $G(x, v)$ *satisfying* (H_3)—(H_4), *the semilinear elliptic problem,*

$$(2.7)\qquad \int_D (\nabla v \nabla \varphi + (c\beta(v) - F)\varphi) + \int_\Gamma G(v)\varphi = 0 \quad \textit{for all} \quad \varphi \in V,$$

has a unique solution $v \in V$.

Using Proposition 1 one can immediately obtain the existence of the unique weak solution u_i to Problem (2.6) for any $i = 1, 2, \ldots, n$.

Consider the sequence of step functions $\bar{u}_n$ defined by

$$(2.8)\qquad \begin{aligned} &\bar{u}_n(x, 0) = u_0(x),\\ &\bar{u}_n(x, t) = u_i(x), \qquad \text{for } (i-1)h < t \leqslant ih, \quad i = 1, 2, \ldots, n. \end{aligned}$$

We start with

Lemma 1. *Suppose that* (H) *holds, and let* $0 < m < \infty$. *Then the sequence* $\{\bar{u}_n\}$ *is bounded in the space* $L^2(I; H^1(D))$ *and for any* η, $0 < \eta < \min(1, h^{-1})$, *the estimate*

$$(2.9)\qquad |\bar{u}_n(t)|_{m+1} \leqslant (|u_0|_{m+1} + C)\exp((t+h)(K+\eta)/(1-\eta h)m)$$

holds for all $t \in I$. *For the nonnegative constant*

$$C = C(N, D, L, K, m, \eta, |g(0, 0)|_{2, \Gamma}, |f(0, 0)|_{(m+1)/m})$$

see (2.17) *below. Moreover, there exist a subsequence of* $\{\bar u_n\}$ (*let us denote it again by* $\{\bar u_n\}$) *and a function* u *such that*

$$\bar u_n \rightharpoonup u \quad in \quad L^2(I; H^1(D)), \quad as\ n \to \infty.$$

Remark. If, in addition to (H), we suppose that $g(x, t, u)u \geq 0$ and $f(x, t, u) = f(x, u)$, then the constant C in (2.9) may be evaluated as

$$C = (2/(m+1))^{1/m(m+1)} (|f(0)|_{(m+1)/m}/\eta)^{1/m}.$$

Proof. Putting $F = f(x, (i-1)h, u_{i-1}) + h^{-1}\beta(u_{i-1})$, $c = h^{-1}$ and $G(x, u) = g(x, ih, u)$, (2.7) as a weak formulation of (2.6) yields for $\varphi = u_i$:

(2.10)

$$\int_D (|u_i|^{m+1} + h|\nabla u_i|^2) = \int_D (\beta(u_{i-1})u_i + hf((i-1)h, u_{i-1})u_i) - h\int_\Gamma g(ih, u_i)u_i.$$

It might be noted that our hypotheses imply

$$(2.11)\qquad \begin{aligned} &|f(x, t, u)| \leq |f(x, t, 0)| + K|u|^m && \text{(by } (H_2)),\\ &-g(x, t, u)u \leq |g(x, t, 0)|\,|u| && \text{(by } (H_4)),\end{aligned}$$

and

$$(2.12)\qquad \int_\Gamma |g(t, 0)|\,|u| \leq \eta \int_D |\nabla u|^2 + \frac{\eta}{2}\int_D |u|^{m+1} + C_1(\eta) \quad \text{for any } \eta,$$

$0 < \eta < \infty$, where

$$C_1(\eta) = \frac{C(D)}{\eta} |g(t, 0)|^2_{2,\Gamma} + \frac{C(m, N, D)}{\eta^{1/m}} |g(t, 0)|^{(m+1)/m}_{2,\Gamma}.$$

For $m \geq 1$, (2.12) follows from Hölder's inequality, the embedding $H^1(D)$ into $L^2(\Gamma)$ and Young's inequality. To obtain (2.12) for $0 < m < 1$ we need to estimate the L^2 norm by the L^{m+1} norm. For this purpose we introduce a special case of the Nirenberg—Gagliardo inequality (see [4, Theorem 10.1, p. 27]) in the form

$$(2.13)\qquad |v|_2 \leq C_F |v|^a_{1,2} |v|^{1-a}_{m+1}, \qquad a = (1-m)N/(2N - (N-2)(m+1)),$$

and (2.12) follows then by routine calculations if we use Young's inequality. Now, using (2.11—12), from (2.10) we get

$$(2.14)\qquad (1-\eta h)\int_D |u_i|^{m+1} + h(1-\eta)\int_D |\nabla u_i|^2 \leq (1+Kh)\int_D |u_{i-1}|^m |u_i| + C_2 h,$$

where $C_2 = C_1(\eta) + \eta^* \int_D |f((i-1)h, 0)|^{(m+1)/m}$. Choosing η sufficiently small we have further

$$\int_D |u_i|^{m+1} \leqslant (1+\xi h)\int_D |u_{i-1}|^m |u_i| + hC_2/(1-\eta h), \qquad \xi = (K+\eta)/(1-\eta h),$$

and applying Young's inequality

$$\int_D |u_i|^{m+1} \leqslant (1+\xi h)^{(m+1)/m}\int_D |u_{i-1}|^{m+1} + hC_3, \qquad C_3 = \frac{(m+1)\,C_2}{m(1-\eta h)}. \tag{2.15}$$

From (2.15) we obtain succesively

$$\int_D |u_i|^{m+1} \leqslant (1+\xi h)^{i(m+1)/m}\left(\int_D |u_0|^{m+1} + \frac{hC_3}{((1+\xi h)^{(m+1)/m}-1)}\right),$$

but as $\xi h/((1+\xi h)^{(m+1)/m}-1) \leqslant 1$, we get

$$|u_i|_{m+1} \leqslant (|u_0|_{m+1} + (C_3/\xi)^{1/(m+1)})\, e^{ih\xi/m}. \tag{2.16}$$

Now let $t \in I$ be arbitrary but fix and for each n let k be such that $t \in ((k-1)h, kh]$. Using the notation (2.8), (2.16) for $i = k$ yields

$$|\bar{u}_n(t)|_{m+1} \leqslant (|u_0|_{m+1} + ((m+1)\,C_2/m\eta)^{1/(m+1)}) \exp\left(\frac{(t+h)(K+\eta)}{m(1-\eta h)}\right), \tag{2.17}$$

hence (2.9).

To get the boundedness of $\{\bar{u}_n\}$ in $L^2(I;\, H^1(D))$, let us add up (2.14) for $i = 1, 2, \ldots, n$, and we obtain

$$(1-\eta)\sum_{i=1}^{n} h\int_D |\nabla u_i|^2 + (1-\eta h)\sum_{i=1}^{n}\int_D |u_i|^{m+1} \leqslant C_2 T +$$
$$+ (1-\eta h)\sum_{i=1}^{n}\int_D |u_{i-1}|^m |u_i| + h(K+\eta)\sum_{i=1}^{n}\int_D |u_{i-1}|^m |u_i|.$$

As the right hand side of the above inequality may be estimated by Young's and Hölder's inequalities, we have further

$$(1-\eta)\sum_{i=1}^{n} h\int_D |\nabla u_i|^2 + \frac{(1-\eta h)m}{m+1}\int_D |u_n|^{m+1} \leqslant \frac{(1-\eta h)m}{m+1}\int_D |u_0|^{m+1} +$$
$$+ C_2 T + (K+\eta)\sum_{i=1}^{n} h|u_{i-1}|_{m+1}^{m}\, |u_i|_{m+1},$$

and the proof of Lemma 1 is completed by (2.16).

Lemma 2. *Under the same hypotheses as in Lemma 1 there exists a nonnegative constant C such that the estimate*

$$t\int_D |\nabla \bar{u}_n(t)|^2 \leqslant C \tag{2.18}$$

holds for all (sufficiently large) n and $0 < t \leqslant T$, *i.e.* $t^{1/2}|\bar{u}_n(t)|_{1,2} \leqslant C$ *for all* $0 < t \leqslant T$. *If, in addition to* (H), $u_0 \in H^1(D)$, *then*

$$\int_D |\nabla \bar{u}_n(t)|^2 \leqslant C$$

for all n and $0 \leqslant t \leqslant T$.

Proof. Analogously to the proof of Lemma 1, (2.6) for $\varphi = i(u_i - u_{i-1})$, $i \geqslant 2$ yields

$$i\int_D (\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}) + ih\int_D \nabla u_i \nabla(u_i - u_{i-1}) +$$
$$+ ih\int_\Gamma g(ih, u_i)(u_i - u_{i-1}) = ih\int_D f((i-1)h, u_{i-1})(u_i - u_{i-1}).$$

If we use the notation

$$\Psi_i(u) = \int_\Gamma \int_0^u g(ih, r)\,dr, \tag{2.19}$$

(H_4) implies the obvious inequality

$$\Psi_i(u_i) - \Psi_i(u_{i-1}) \leqslant \int_\Gamma g(ih, u_i)(u_i - u_{i-1}),$$

and we have further

$$i\int_D (\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}) + ih2^{-1}\left(\int_D |\nabla u_i|^2 - \int_D |\nabla u_{i-1}|^2\right) +$$
$$+ ih(\Psi_i(u_i) - \Psi_i(u_{i-1})) \leqslant ih\int_D \int_{u_{i-1}}^{u_i} f((i-1)h, r)\,dr + \tag{2.20}$$
$$+ ih\int_D \left(\int_{u_{i-1}}^{u_i} (f((i-1)h, u_{i-1}) - f((i-1)h, r))\,dr\right).$$

Now, the last term on the right-hand side of (2.20) may be estimated using (H_2) by

$$ih\int_D \left|K\int_{u_{i-1}}^{u_i} |\beta(u_{i-1}) - \beta(r)|\,dr\right| \leqslant ihK\int_D (\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}),$$

and if we add up (2.20) for $i = 2, 3, \ldots, k$ and perform recognizable calculations, we obtain

$$
\begin{aligned}
&(1 - Kh) \sum_{i=2}^{k} i \int_D (\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}) + \frac{kh}{2} \int_D |\nabla u_k|^2 - \sum_{i=1}^{k-1} h \int_D |\nabla u_i|^2 + \\
&\quad + kh \Psi_k(u_k) - h \Psi_1(u_1) - \sum_{i=1}^{k-1} h \Psi_i(u_i) + \sum_{i=2}^{k} ih(\Psi_{i-1}(u_{i-1}) - \Psi_i(u_{i-1})) \leqslant \\
&\quad \leqslant kh \int_D \int_0^{u_k} f((k-1)h, r)\, dr - \sum_{i=1}^{k-1} h \int_D \int_0^{u_i} f((i-1)h, r)\, dr + \\
&\quad + \sum_{i=2}^{k} ih \int_D \int_0^{u_{i-1}} (f((i-2)h, r) - f((i-1)h, r))\, dr.
\end{aligned}
\tag{2.21}
$$

Both the last term on the left-hand side and the last term on the right hand side of (2.21) may be estimated using (H_{2-3}) and Lemma 1 by Ckh, where the nonnegative constant C does not depend on k. If we choose h sufficiently small, (2.21), yields

$$
\begin{aligned}
&\frac{kh}{2} \int_D |\nabla u_k|^2 + kh \Psi_k(u_k) \leqslant Ckh + kh \int_D \int_0^{u_k} f((k-1)h, r)\, dr + \\
&+ \sum_{i=1}^{k-1} h \left(\int_D |\nabla u_i|^2 + \Psi_i(u_i) - \int_D \int_0^{u_i} f((i-1)h, r)\, dr \right) + h \Psi_1(u_1).
\end{aligned}
$$

As by (H_4) we have

$$
\int_\Gamma g(kh, 0)\, u_k \leqslant \Psi_k(u_k), \tag{2.22}
$$

(2.12), (H_{2-3}) and Lemma 1 gives

$$
kh \int_D |\nabla u_k|^2 \leqslant C_1 kh + C_2,
$$

where the nonnegative constants C_1, C_2 do not depend on k. Suppose now that $t \in I$ ranks among $((k-1)h, kh]$, then

$$
t \int_D |\nabla \bar{u}_n(t)|^2 \leqslant C,
$$

where C does not depend on t, hence (2.18).

If $u_0 \in H^1(D)$, let us multiply (2.20) by $1/i$ and add up such inequalities for $i = 1, 2, \ldots, k$. We obtain

$$(2.23)\quad \begin{aligned} &(1-kh)\sum_{i=1}^{k} h^{-1}\int_D (\beta(u_i)-\beta(u_{i-1}))(u_i-u_{i-1}) + \tfrac{1}{2}\int_D |\nabla u_k|^2 + \\ &+ \Psi_k(u_k) - \int_D\int_0^{u_k} f((k-1)h, r)\,dr \leqslant \tfrac{1}{2}\int_D |\nabla u_0|^2 + \Psi_0(u_0) - \\ &- \int_D\int_0^{u_0} f(0, r)\,dr + \sum_{i=1}^{k} |\Psi_i(u_{i-1}) - \Psi_{i-1}(u_{i-1})| + \\ &+ \sum_{i=1}^{n}\int_D \Big|\int_0^{u_i} |f(ih, r) - f((i-1)h, r)|\,dr\Big|. \end{aligned}$$

Now, by virtue of (2.22), (2.12), (H_{2-3}) and Lemma 1, we arrive at

$$\int_D |\nabla u_k|^2 \leqslant C,$$

where the constant C does not depend on k, hence the conclusion.

As regards an a priori estimate for the time derivative, we begin with the proposition giving the inequality, which plays a key role in our considerations.

Proposition 2. *Let* $0 < m < \infty$ *and* $\gamma(x) = |x|^{(m+1)/2}\operatorname{sign} x$ *for* $x \in \mathbf{R}$. *Then the inequality*

$$(2.24)\qquad \frac{4m}{(m+1)^2}(\gamma(x)-\gamma(y))^2 \leqslant (\beta(x)-\beta(y))(x-y)$$

holds for any $x, y \in \mathbf{R}$.

Proof. If $\operatorname{sign} x \neq \operatorname{sign} y$, (2.24) may be verified by direct computations and we omit the details.

If $\operatorname{sign} x = \operatorname{sign} y$, it is sufficient to verify the inequality

$$\frac{4m}{(m+1)^2}(\gamma(z)-1)^2 \leqslant (\beta(z)-1)(z-1) \qquad \text{for } z > 1,$$

assuming $m > 1$ (as for $0 < m < 1$ we put $m' = m^{-1} > 1$, $z' = \beta(z)$), which may be rewritten into

$$(\sqrt{\beta(z)} - \sqrt{z})^2 \leqslant \frac{(m-1)^2}{(m+1)^2}(\gamma(z)-1)^2 \qquad \text{for } z > 1.$$

Putting $\lambda = \sqrt{z}$, it is not difficult to demonstrate that the function

$$f(\lambda) = \frac{m-1}{m+1}(\lambda^{m+1} - 1) - \lambda(\lambda^{m-1} - 1)$$

is nonnegative for $\lambda \geqslant 1$, hence the conclusion.

Let us now consider the sequence of piecewise linear functions $\{U_n\}$, defined by

$$(2.25) \qquad U_n(x, t) = \gamma(u_{i-1}(x)) + (\gamma(u_i(x)) - \gamma(u_{i-1}(x)))\,\frac{t-(i-1)h}{h}$$

for $(i-1)h \leqslant t \leqslant ih$, $i = 1, 2, \ldots, n$,
and $\{u_n\}$, constructed analogously to $\{U_n\}$ by means of u_i. Then we have the following assertions.

Lemma 3. *Let the hypotheses of Lemma 1 be satisfied. Then the sequence* $\{t^{1/2}(U_n)_t\}$ *is bounded in the space* $L^2(Q_T)$.

If, in addition to (H), $u_0 \in H^1(D)$, *then* $(U_n)_t$ *is uniformly (with regard to n) bounded in* $L^2(Q_T)$.

Proof. If we take into account the proof of Lemma 2, (2.21) yields

$$(1 - Kh)\sum_{i=1}^{n} i\int_D (\beta(u_i) - \beta(u_{i-1}))(u_i - u_{i-1}) \leqslant C,$$

where the constant C does not depend on n. Now, with the assistance of Lemma 1 and (2.24), the above inequality gives

$$\iint_{Q_T} t(U_n)_t^2 \leqslant \sum_{i=1}^{n} i\int_D (\gamma(u_i) - \gamma(u_{i-1}))^2 \leqslant C,$$

hence the first conclusion.

If $u_0 \in H^1(D)$, the assertion follows from (2.23) analogously as above.

Lemma 4. *Suppose that* (H) *holds, and let* $0 < m \leqslant 1$. *Then the sequence* $\{t^{1/2}(u_n)_t\}$ *is bounded in the space* $L^2(I; L^{m+1}(D))$, *and*

$$(2.26) \qquad \begin{aligned} &\text{(i)}\ \ u_n \to u && \text{in } C([\varepsilon, T]; L^{m+1}(D)),\\ &\text{(ii)}\ \ \bar{u}_n \to u && \text{in } L^2(\varepsilon, T; L^{m+1}(D)), \text{ as } n \to \infty, \end{aligned}$$

for any ε, $0 < \varepsilon < T$ *(through a subsequence depending on* ε*). Moreover,* u *satisfies*

$$(2.27) \qquad |u(t)|_{m+1} \leqslant (|u_0|_{m+1} + C(\eta))\exp((K + \eta)\,t/m)$$

for all $t \in I$ *and* $0 < \eta \leqslant 1$, *where the constant* $C(\eta)$ *is from* (2.17).

If, in addition to (H), $u_0 \in H^1(D)$, *then* $\{(u_n)_t\}$ *is bounded in* $L^2(I; L^{m+1}(D))$ *and* $\varepsilon = 0$ *is allowed in* (2.26).

Corollary 1. *Let the hypotheses of Lemma* 4 *be satisfied, then there exists a subsequence of* $\{n\}$ *such that, by relabelling, we have:*

(2.28)

(i) *For* $t \in I$ *let* $k = k_n$ *be such positive integer that* $(k-1)h < t \leqslant kh$. *Then*
$u_n(kh) \to u(t)$ *in* $L^{m+1}(D)$,

(ii) $\beta(u_n(kh)) \to \beta(u(t))$ *in* $L^{(m+1)/m}(D)$,

(iii) $\bar{u}_n(\cdot + h) \to u$ *in* $L^{m+1}(Q_{\varepsilon, T})$,

(iv) $\beta(\bar{u}_n(\cdot - h)) \to \beta(u)$ *in* $L^{(m+1)/m}(Q_{\varepsilon, T})$, *and*

(v) $(U_n)_t \rightharpoonup (\gamma(u))_t$ *in* $L^2(Q_{\varepsilon, T})$, *as* $n \to \infty$,

for any ε, $0 < \varepsilon < T$ *(in* (iii) *for simplicity we put* $\bar{u}_n(t+h) = u_n$ *for* $t \in (T-h, T]$.

If $u_0 \in H^1(D)$, *then* $t = 0$ *is allowed in* (i—ii) *and* $\varepsilon = 0$ *is allowed in* (iii—v).

Moreover, the function u satisfies (2.1). *If, however,* $u_0 \in H^1(D)$, *then* (2.5) *is satisfied.*

Proof of Lemma 4. We start with the calculation:

$$\int_I |t^{1/2}(u_n)_t|^2_{m+1} = \sum_{i=1}^{n} \int_{(i-1)h}^{ih} t|(u_i - u_{i-1})/h|^2_{m+1} dt \leqslant$$

(2.29)
$$\leqslant \left(\frac{2}{m+1}\right)^2 \sum_{i=1}^{n} (|u_i|_{m+1}^{(m+1)/2} + |u_{i-1}|_{m+1}^{(m+1)/2})^{2(1-m)/(m+1)}.$$

$$\cdot \int_{(i-1)h}^{ih} t \left|\frac{\gamma(u_i) - \gamma(u_{i-1})}{h}\right|_2^2 dt \leqslant C(m, \|\bar{u}_n\|_{L^\infty(I, L^{m+1}(D))}) \int_I t|(U_n)_t|_2^2 dt.$$

Now, taking into account Lemmas 1 and 3, we have the first assertion of Lemma 4. As for any ε, $0 < \varepsilon < T$ it holds that

$$\int_\varepsilon^T |(u_n)_t|^2_{m+1} \leqslant C \quad \text{and} \quad |\bar{u}_n(t)|_{1,2} \leqslant C \qquad \text{for all } t \in [\varepsilon, T],$$

using a standard argument (see e.g. [7, Lemma 1.3.13, p. 25]) we arrive at (2.26)(i). (2.26)(ii) follows from (2.26)(i) and the estimate

(2.30)
$$\int_\varepsilon^T |\bar{u}_n - u_n|^2_{m+1} \leqslant Ch^2 \qquad \text{(see (2.29))}.$$

The estimate (2.27) is a simple consequence of (2.9) and (2.26)(i), and the conclusion for $u_0 \in H^1(D)$ follows from the corresponding part of Lemma 3.

Proof of Corollary 1. As

$$(2.31)\qquad |u_n(t)-u_n(s)|_{m+1}\leqslant \left|\int_s^t |(u_n)_t|_{m+1}\right| \leqslant C(\varepsilon)\,|t-s|^{1/2},$$

for $t,\ s\in[\varepsilon,\ T],\ \varepsilon>0$, (i) follows easily from (2.26)(i).

To prove (ii), let us note that $f_n = |\beta(u_n(kh))-\beta(u(t))|^{\frac{m+1}{m}}$ tends to zero a.e. in D (through a subsequence). Because $\int_D |f_n|^{2(m+1)}$ are uniformly bounded by Lemma 2, we see that

$$\int_B f_n \leqslant |f_n|_{2(m+1)}|B|^{(1-m)2}.$$

Hence, the hypotheses of Vitali's theorem are satisfied (see, e.g., [10, Theorem 2.1.4, p. 60]) and we can conclude that

$$\int_D f_n \to 0 \qquad \text{as } n\to\infty, \text{ hence (2.28)(ii).}$$

The assertion (iii) follows from (2.30—31) and (2.26)(i).

The assertion (iv) can be proved analogously to (ii).

To prove (v) let us note that Lemma 3 implies $(U_n)_t \rightharpoonup \xi$ in $L^2(Q_{\varepsilon,T})$ as $n\to\infty$, and that by the same way as in (ii) we obtain that $U_n\to\gamma(u)$ in $L^2(Q_{\varepsilon,T})$, hence $\xi=\gamma(u)$.

Now let us show that $t^{1/2}(\gamma(u))_t\in L^2(Q_T)$. First by Lemma 3 we have that $t^{1/2}(U_n)_t \rightharpoonup \vartheta$ in $L^2(Q_T)$ and by (2.28)(v) that $t^{1/2}(U_n)_t \rightharpoonup t^{1/2}(\gamma(u))_t$ in $L^2(Q_{\varepsilon,T})$, hence $\vartheta = t^{1/2}(\gamma(u))_t$ a.e. in Q_T. The rest of the proof is not difficult and is omitted.

Consider now that the test function φ is from $C^{2,1}(\bar{Q}_{T+1})$ and set

$$\varphi_i(x) = \frac{1}{h}\int_{ih}^{(i+1)h} \varphi(x,s)\,ds \qquad \text{for } i=0,1,\dots,n.$$

By means of φ_i we can construct the sequences of functions $\{\varphi_n\}$ and $\{\bar{\varphi}_n\}$ analogously to $\{u_n\}$ and $\{\bar{u}_n\}$ as above. Then we have the following lemma, the proof of which is left to the reader.

Lemma 5. *Let the function φ, φ_n and $\bar{\varphi}_n$ be as above and for fixed $t\in I$ let $k=k_n$ be such that $(k-1)h<t\leqslant kh$. Then*

$$\begin{aligned}
&\varphi_n(kh)\to\varphi(t),\quad \varphi_n(h)\to\varphi(0) && \text{in } L^{\infty}(D),\\
(2.32)\qquad &(\varphi_n)_t\to\varphi_t,\ \bar{\varphi}_n(\cdot+h)\to\varphi && \text{in } L^{\infty}(Q_T),\\
&\bar{\varphi}_n\to\varphi && \text{in } L^2(I;H^1(D)),\ \text{as } n\to\infty.
\end{aligned}$$

After these preparations it is now easy to prove Theorem 1.

Proof of Theorem 1. Putting φ_i into a weak formulation of (2.6) and adding up such identities for $i = 1, 2, \ldots, k$ we obtain, after recognizable arrangements:

$$\int_D \beta(u_n(kh))\,\varphi_n(kh) - \int_h^{kh}\int_D \beta(\bar{u}_n(\cdot - h))\,(\varphi_n)_t + \int_0^{kh}\int_D \nabla\bar{u}_n\nabla\bar{\varphi}_n -$$

$$(2.33)\qquad - \int_0^{(k-1)h}\int_D \bar{f}_n(\bar{u}_n)\,\bar{\varphi}_n(\cdot + h) + \int_0^{kh}\int_\Gamma \bar{g}_n(\bar{u}_n)\,\bar{\varphi}_n =$$

$$= \int_D \beta(u_0)\,\varphi_n(h) + h\int_D f(0, u_0)\,\varphi_n(h),$$

where

$$\left.\begin{aligned}\bar{g}_n(x, t, u) &= g(x, ih, u)\\ \bar{f}_n(x, t, u) &= f(x, ih, u)\end{aligned}\right\}\qquad \text{for } (i-1)h < t \leqslant ih.$$

Now, letting $n \to \infty$ in (2.33), with the help of Lemmas 1 and 4, Corollary 1, (H_{2-3}) and Lemma 5, we obtain

(2.34)

$$\int_D \beta(u(t))\,\varphi(t) - \iint_{Q_t} (\beta(u)\,\varphi_t - \nabla u\nabla\varphi + f(u)\,\varphi) + \iint_{S_t} \chi\varphi = \int_D \beta(u_0)\,\varphi(0),$$

where $\chi \in L^2(S_T)$ is such that $\bar{g}_n(\bar{u}_n) \rightharpoonup \chi$ in $L^2(S_T)$ as $n \to \infty$ (through a subsequence). We still have to show that $\chi = g(x, t, u)$. To that purpose let us first note that (2.34) yields

(2.35)

$$\int_D \beta(u(t))\,\varphi(t) - \int_s^t\int_D (\beta(u)\,\varphi_t - \nabla u\nabla\varphi + f(u)\,\varphi) + \int_s^t\int_\Gamma \chi\varphi = \int_D \beta(u(s))\,\varphi(s)$$

for $0 < s < t \leqslant T$, and that this identity continues to hold also for $\varphi = u$. In fact, we have already shown above that $u \in W^{1, m+1}(Q_{\varepsilon, T})$ for any ε, $0 < \varepsilon < T$, so u can be approximated by smooth functions and the assertion follows easily.

Now let $0 < s < t \leqslant T$ be arbitrary and let $l = l_n$, $k = k_n$ be such that $(l-1)h < s \leqslant lh$ and $(k-1)h < t \leqslant kh$. Then, by the same way as in (2.33), we obtain

$$\int_{lh}^{kh}\int_\Gamma \bar{g}_n(\bar{u}_n)\,\bar{u}_n = \int_{lh}^{kh}\int_D \beta(\bar{u}_n(\cdot - h))\,(u_n)_t - \int_{lh}^{kh}\int_D |\nabla\bar{u}_n|^2 +$$

$$+ \int_{(l-1)h}^{(k-1)h}\int_D \bar{f}_n(\bar{u}_n)\,\bar{u}_n(\cdot + h) + \int_D \beta(u_n(lh))\,u_n(lh) - \int_D \beta(u_n(kh))\,u_n(kh).$$

By Lemmas 1 and 4, (H_2) and Corollary 1 we have further

$$\limsup_{n\to\infty}\int_s^t\int_\Gamma \bar{g}_n(\bar{u}_n)\,\bar{u}_n \leqslant \int_s^t\int_D (\beta(u)\,u_t - |\nabla u|^2 + f(u)\,u) + \tag{2.36}$$
$$+ \int_D \beta(u(s))\,u(s) - \int_D \beta(u(t))\,u(t).$$

Next, considering (H_{3-4}) and taking Lemma 1 into account, we obtain

$$0 \leqslant \limsup_{n\to\infty}\int_s^t\int_\Gamma (\bar{g}_n(\bar{u}_n) - \bar{g}_n(w))(\bar{u}_n - w) = \tag{2.37}$$
$$= \limsup_{n\to\infty}\int_s^t\int_\Gamma \bar{g}_n(\bar{u}_n)\,\bar{u}_n - \int_s^t\int_\Gamma \chi w - \int_s^t\int_\Gamma g(w)(u - w),$$

which holds for all $w \in L^2(S_T)$. So, if we estimate the first term on the right-hand side of (2.37) by (2.36) and apply (2.35), it can be seen that

$$0 \leqslant \int_s^t\int_\Gamma (\chi - g(w))(u - w) \qquad \text{for all } w \in L^2(S_T).$$

Now, by standard arguments we may conclude that $\chi = g(x, t, u)$ a.e. on S_T.

To prove the energy inequality (2.4), let us note (2.23), which with the help of (2.24) yields

$$\frac{(1 - Kh)\,4m}{(m+1)^2}\int_{lh}^{kh}\int_D (U_n)_t^2 + J(\bar{u}_n(t)) \leqslant J(u_0)$$

and the assertion follows easily from (2.28)(v), Lemmas 2 and 4, (H_{2-3}) and the weak lower semicontinuity of a norm. The proof of Theorem 1 is complete.

Proof of Proposition 1. Put

$$I(w) = \int_D (\tfrac{1}{2}|\nabla w|^2 + c(m+1)^{-1}|w|^{m+1} - Fw) + \int_\Gamma\int_0^w G(r)\,dr \qquad \text{for } w \in V.$$

We merely show that I is a continuous, strictly convex and coercive functional over V. The existence of the unique solution v of Problem (2.7) then follows immediately if the classical results concerning the minimization of I, namely the existence and the characterization of the solution, are taken into account (see e.g. [5, Theorem 26.8]). Due to the assumptions (H_3) and (2.13), the continuity of I in V with the norm $|\cdot|_{1,2}$ if $0 < m \leqslant 1$ or $|\nabla\cdot|_2 + |\cdot|_{m+1}$ if $m > 1$ is evident. The convexity of I is guaranteed by (H_4) and strong convexity by the term $|\cdot|_{m+1}^{m+1}$. To show the coercivity let us note that (H_{3-4}) and (2.12) implies

$$\int_\Gamma\int_0^w G(r)\,dr \geqslant \int_\Gamma G(0)\,w \geqslant -\eta\Big(\int_D |\nabla w|^2 + \int_D |w|^{m+1}\Big) - C(\eta).$$

Choosing η sufficiently small, we arrive at

$$\mathrm{I}(w) \geqslant c\left(\int_D |\nabla w|^2 + \int_D |w|^{m+1}\right) - C,$$

hence the conclusion.

3. L^∞ solvability

In this section we shall discuss Problem (1.1) providing that the initial function u_0 is bounded. The question arises whether the solution u would be bounded for any later time under our hypothesis (H). To answer this question it seems to be necessary (for our way of proof) to add an additional condition on the function g. Therefore, let us assume instead of (H):

$(H_1)^*$ $u_0 \in L^\infty(D)$.

(H_{2-4}) *as in Section* 2.

(H_5) *there exists a function* $r \in L^\infty(S_T)$ *such that* $g(x, t, r(x, t)) = 0$ *on* S_T.

We shall refer to these hypotheses as (H)*.

Now we can state the main result of this section.

Lemma 6. *Suppose that* (H)* *holds, and let* $0 < m < \infty$. *Then the estimate*

$$\begin{aligned} |\bar{u}_n(t)|_\infty \leqslant (|u_0|_\infty + \|r\|_{L^\infty(S_{t+h})} + (\varepsilon^{-1}\|f(0)\|_{L^\infty(Q_{t+h})})^{1/m})\cdot \\ \cdot\exp\left(((K+\varepsilon)(t+h)/m(1-\varepsilon h)\right) \end{aligned} \tag{3.1}$$

holds for any t, $0 \leqslant t < T$ *and* ε, $0 < \varepsilon < h^{-1}$.

As the simple consequence of Lemma 6 we have

Corollary 2. *Suppose that* (H)* *holds, and let* $0 < m \leqslant 1$. *Then the solution* u *of Problem* (1.1) *satisfies*

$$|u(t)|_\infty \leqslant (|u_0|_\infty + \|r\|_{L^\infty(S_t)} + (\varepsilon^{-1}\|f(0)\|_{L^\infty(Q_t)})^{1/m}) \exp\left(\frac{(K+\varepsilon)t}{m}\right) \tag{3.2}$$

for all $0 \leqslant t \leqslant T$ *and* $0 < \varepsilon < \infty$.

Proof of Lemma 6. First we have to show that $u_i \in L^\infty(D)$ for any $i = 1, 2, \ldots, n$. To prove this, it is sufficient to demonstrate that $v \in L^\infty(D)$ whenever $F \in L^\infty(D)$ and g satisfies (H_{3-5}) (see Proposition 1). We shall follow the idea from Kačur's paper [9].

Suppose the contrary, i.e. there exist a sequence $\{c_j\}$, $c_j \leqslant c_{j+1}$, $c_j \to \infty$ as $j \to \infty$ and a sequence of sets $K_j = \{x \in D: |v(x)| > c_j\}$ such that $|K_j| > 0$. Put

$$v_j(x) = \begin{cases} v(x) & x \in D \backslash K_j \\ c_j \operatorname{sign} v & x \in K_j \end{cases}.$$

We obtain the contradiction by showing that $I(v_j) < I(v)$ for sufficiently large j, which contradicts the minimum property of v. In fact, $v \in H^1(D)$ implies that $v_j \in H^1(D)$ too and we can compute

$$I(v) - I(v_j) \geqslant c(m+1)^{-1} \int_{K_j} (|v|^{m+1} - c_j^{m+1}) - \int_{K_j} F(v - c_j) + \\ + \int_\Gamma \int_{c_j \operatorname{sign} v}^{v} G(r)\, dr \geqslant (Cc_j^m - |F|_\infty) \int_{K_j} (|v| - c_j) > 0$$

for sufficiently large j, as $\int_{c_j \operatorname{sign} v}^{v} G(x, t, \xi)\, d\xi \geqslant 0$ if $c_j \geqslant \|r\|_{L^\infty(S_T)}$, hence the conclusion.

Now we can proceed to deriving the estimate (3.1). Putting $\varphi = |u_i|^k \operatorname{sign} u_i$ into a weak formulation of (2.6), where the positive integer k is arbitrarily large, we obtain

$$\int_D |u_i|^{k+m} + kh \int_D |u_i|^{k-1} |\nabla u_i|^2 + h \int_\Gamma g(ih, u_i) |u_i|^k \operatorname{sign} u_i = \\ = h \int_D (f((i-1)h, u_{i-1}) + h^{-1} \beta(u_{i-1})) |u_i|^k \operatorname{sign} u_i .$$

Now, the last term on the left-hand side of the equality above can be estimated using (H_{4-5}) as follows:

$$- \int_\Gamma (g(ih, u_i) \operatorname{sign} u_i) |u_i|^k \leqslant - \int_\Gamma g(ih, 0) |r(ih)|^k,$$

and using (H_2) we obtain further

$$\int_D |u_i|^{k+m} \leqslant (1 + Kh) \int_D |u_i|^k \sqrt{|u_{i-1}|^m} + |f((i-1)h, 0)|_\infty h \int_D |u_i|^k + \\ + |g(ih, 0)|_{\infty, \Gamma} |\Gamma|_{n-1} |r(ih)|^k. \tag{3.3}$$

Next, due to Young's inequality, (3.3) yields

$$(1 - \varepsilon h) \int_D |u_i|^{k+m} \leqslant (1 - \varepsilon h)(1 + \eta h) \int_D |u_i|^k |u_{i-1}|^m + C_1 h, \tag{3.4}$$

where $\eta = (K + \varepsilon)/(1 - \varepsilon h)$ and

$$C_1 = |g(ih, 0)|_{\infty, \Gamma} |\Gamma|_{n-1} \|r\|^k_{L^\infty(S_{t+h})} + \frac{m}{m+k} \left(\frac{k}{\varepsilon(m+k)} \right)^{k/m} |D| \, \|f(0)\|^{(k+m)/m}_{L^\infty(Q_t)} .$$

Again, using Young's inequality in (3.4), we arrive at

$$\int_D |u_i|^{k+m} \leqslant (1+\eta h)^{(k+m)/m} \int_D |u_{i-1}|^{k+m} + C_2 h, \qquad C_2 = \frac{(m+k)\,C_1}{m(1-\varepsilon h)},$$

from which we obtain successively

$$\int_D |u_i|^{k+m} \leqslant (1+\eta h)^{i(m+k)/m}\left(\int_D |u_0|^{k+m} + \frac{hC_2}{((1+\eta h)^{(m+k)/m}-1)}\right),$$

and further (see (2.16))

$$\int_D |u_i|^{k+m} \leqslant (1+\eta h)^{i(m+k)/m}\left(\int_D |u_0|^{k+m} + C_2/\varepsilon\right). \tag{3.5}$$

Now, taking the $(k+m)$-th root of (3.5) and letting $k \to \infty$, we obtain

$$|u_i|_\infty \leqslant (1+\eta h)^{i/m}(|u_0|_\infty + \|r\|_{L^\infty(S_{t+h})} + (\varepsilon^{-1}\|f(0)\|_{L^\infty(Q_t)})^{1/m}),$$

and then (3.1), as $t \leqslant ih < t+h$, proving Lemma 6.

4. The case of fast diffusion

We start this section by stating its result.

Theorem 2. *Let $m > 1$ and suppose that* (H)* *holds. Then Problem (1.1) admits a unique solution u such that*

$$\begin{gathered} u \in L^2(I;\, H^1(D)) \cap L^\infty(Q_T), \quad t^{1/2}u \in L^\infty(I;\, H^1(D)), \\ t^{1/2}(\gamma(u))_t \in L^2(Q_T), \end{gathered} \tag{4.1}$$

the equation is satisfied in the sense that

$$\int_D ((\beta(u(t)))_t w + \nabla u(t)\nabla w - f(t, u(t))\, w) + \int_\Gamma g(t, u(t))\, w = 0 \tag{4.2}$$

for any $w \in H^1(D)$ and a.e. on I, and u satisfies the "maximum" principle (3.2).

If, in addition to (H)*, $u_0 \in H^1(D)$, *then*

$$u \in L^\infty(I;\, H^1(D)) \cap L^\infty(Q_T), \quad (\gamma(u))_t \in L^2(Q_T), \tag{4.3}$$

and if we consider $f(x, t, u) = f(x, u)$, $g(x, t, u) = g(x, u)$ only, then the energy inequality (2.4) *remains to hold.*

The development of the proof which we present here is closely related to the one in Section 2. As this proof is even easier, we shall proceed briefly (see also [3], where the problem with homogeneous Dirichlet boundary conditions is discussed). First, let us recall that Proposition 1 and Lemmas 1—3 yield the following.

Corollary. 3. *Under the hypotheses of Theorem 2 there exists the function u such that* $\bar{u}_n \rightharpoonup u$ *in* $L^2(I; H^1(D))$ *as* $n \to \infty$, $t^{1/2}|\bar{u}_n(t)|_{1,2} \leqslant C$ *for all* $0 < t \leqslant T$ *and the sequence* $\{t^{1/2}(U_n)_t\}$ *is bounded in* $L^2(Q_T)$.

Next, similar arguments as in Lemma 4 prove (with the assistance of Lemma 6 and Corollary 3)

Lemma 7. *Let* $m > 1$, *and suppose that* (H)* *holds. Then for any* ε, $0 < \varepsilon < T$ *there exists a subsequence of* $\{n\}$ *such that, by relabelling, we have*

$$\begin{aligned} U_n &\to U && \text{in } C([\varepsilon, T]; L^2(D)), \\ \bar{U}_n &\to U && \text{in } L^2(Q_{\varepsilon, T}), \\ (U_n)_t &\rightharpoonup U_t && \text{in } L^2(Q_{\varepsilon, T}), \text{ as } n \to \infty, \end{aligned} \tag{4.4}$$

and by the monotonicity argument $U = \gamma(u)$. *Moreover, also*

$$\begin{aligned} W_n &\to \beta(u) && \text{in } C([\varepsilon, T]; L^2(D)), \\ \bar{W}_n &\to \beta(u) && \text{in } L^2(Q_{\varepsilon, T}) \text{ and} \\ (W_n)_t &\rightharpoonup (\beta(u))_t && \text{in } L^2(Q_{\varepsilon, T}), \text{ as } n \to \infty, \end{aligned} \tag{4.5}$$

where W_n, $\bar{W}_n$ *are constructed analogously to* u_n, $\bar{u}_n$ *(see* (2.8), (2.25)) *by means of* $\beta(u_i)$.

Proof of Theorem 2. To prove (4.2), let us note that using our notations, (2.7) may be rewritten in the form

$$\int_D ((W_n(t))_t w + \nabla \bar{u}_n(t) \nabla w - \bar{f}_n(t-h, \bar{u}_n(t-h))\, w) + \int_\Gamma \bar{g}_n(t, \bar{u}_n(t))\, w = 0$$

where $t \in I$ and n is sufficiently large. Multiplying this identity by $q \in L^\infty(I)$ and integrating over (ε, T) we obtain, after letting $n \to \infty$,

$$\iint_{Q_{\varepsilon, T}} ((\beta(u))_t w + \nabla u \nabla w - f(u)\, w)\, q + \iint_{S_{\varepsilon, T}} \chi w q = 0,$$

where $\chi \in L^2(S_T)$ is such that $\bar{g}_n(\bar{u}_n) \rightharpoonup \chi$ in $L^2(S_T)$. The fact that $\chi = g(x, t, u)$ a.e. on S_T can be demonstrated by the same way as the corresponding part in Section 2. So we arrive at (4.2). The rest of the proof follows easily and we omit further details.

5. Comparison, continuous dependence and local existence

We begin by proving the comparison principle and the continuous dependence of solutions of (1.1) on initial data. For this purpose we will make the following assumptions:

(A_1) *$f \in C(\bar{Q}_T \times \mathbf{R})$ and for any $M \in \mathbf{R}^+$ there exists a constant $K = K(T, M) \in \mathbf{R}^+$ such that*

$$|f(x, t, u) - f(x, t, v)| \leqslant K|\beta(u) - \beta(v)|$$

for all $(x, t) \in Q_T$, $|u|, |v| \leqslant M$.

(A_2) *$g \in C(\bar{S}_T \times \mathbf{R})$, nondecreasing in u, and for any $M \in \mathbf{R}^+$ there exists a constant $L = L(T, M) \in \mathbf{R}^+$ such that*

$$|g(x, t, u) - g(x, t, v)| \leqslant L|u - v|$$

for all $(x, t) \in S_T$, $|u|, |v| \leqslant M$.

Definition 1. *By a weak solution of Problem* (1.1) *on I we mean a function $u \in L^2(I; H^1(D)) \cap L^\infty(Q_T)$ such that*

(5.1)
$$\int_D \beta(u(t))\varphi(t) - \iint_{Q_t} (\beta(u)\varphi_t - \nabla u \nabla \varphi + f(u)\varphi) + \iint_{S_t} g(u)\varphi = \int_D \beta(u_0)\varphi(0)$$

for all $\varphi \in C^{2,1}(\bar{Q}_T)$ and each $t \in I$.

A function u is a weak subsolution (supersolution) if $\leqslant$ ($\geqslant$) replaces the equality in (5.1) *whenever $\varphi \geqslant 0$.*

Clearly, the solution u of (1.1) in the sense of Theorem 1 or 2, provided that (H)* holds, is also a weak solution of (1.1).

Theorem 3. *Suppose that* (A_1) *and* (A_2) *are satisfied, and $u_0, v_0 \in L^\infty(D)$.*

(i) *Let u, v be weak solutions of Problem* (1.1) *with initial data u_0, v_0, respectively. Put $M = \max(\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)})$. Then*

$$|\beta(u(t)) - \beta(v(t))|_1 \leqslant |\beta(u_0) - \beta(v_0)|_1 \exp(Kt), \qquad t \in I. \tag{5.2}$$

(ii) *Let u be a subsolution and v a supersolution of Problem* (1.1) *with initial data u_0 and v_0, respectively. Then $u_0 \leqslant v_0$ implies that $u \leqslant v$ a.e. on Q_T.*

Proof. We consider two cases. First suppose that β^{-1} is locally Lipschitz continuous ($0 < m \leqslant 1$).

We start with (ii). For u and v, (5.1) gives

$$\int_D (\beta(u(t)) - \beta(v(t)))\varphi(t) - \iint_{Q_t} (\beta(u) - \beta(v))(\varphi_t + a\Delta\varphi) + \tag{5.3}$$

$$+\iint_{S_t}(u-v)\left(b\varphi+\frac{\partial\varphi}{\partial\nu}\right)\leqslant\int_D(\beta(u_0)-\beta(v_0))\,\varphi(0)+\iint_{Q_t}(f(u)-f(v))\,\varphi$$

where $0\leqslant t\leqslant T$ is arbitrary but now fixed, $a(=a(x,t))=(u-v)/(\beta(u)-\beta(v))$ and $b(=b(x,t))=(g(x,t,u)-g(x,t,v))/(u-v)$. It is not difficult to see that both functions a and b are non-negative and bounded. Now we choose a sequence a_n of smooth functions such that

$$(5.4)\qquad \frac{1}{n}\leqslant a_n\leqslant\|a\|_{L^\infty(Q_T)}+\frac{1}{n},\quad \frac{(a_n-a)}{\sqrt{a_n}}\to 0\qquad\text{in } L^2(Q_T)\text{ as } n\to\infty$$

(see [1]), and for arbitrary ε, $0<\varepsilon<1$ we choose a function b_ε, say $b_\varepsilon\in C^2(\bar{S}_T)$ such that

$$(5.5)\qquad 0<b\leqslant\|b\|_{L^\infty(S_t)}+1,\quad \|b_\varepsilon-b\|_{L^2(S_t)}<\varepsilon.$$

Next, let φ_n be a solution of the backward problem

$$(5.6)\qquad \begin{aligned} &(\varphi_n)_s+a_n\Delta\varphi_n=\lambda\varphi_n && x\in D,\ s\in[0,t),\ \lambda>0,\\ &\varphi_n(x,t)=\chi(x) && x\in D,\\ &\frac{\partial\varphi_n}{\partial\nu}+b_\varepsilon\varphi_n=0 && x\in\Gamma,\ s\in[0,t), \end{aligned}$$

where $\chi(x)\in C_0^\infty(D)$, $0\leqslant\chi\leqslant 1$. Putting $s=t-\tau$, one can obtain the following result:

(5.7)
(i) *Problem* (5.6) *has an unique solution* $\varphi_n\in C^{2,1}(\bar{Q}_t)$ *for any* $n=1,2,\ldots$ (*see* [11, *Theorem* 7.4, p. 560]).
(ii) $0\leqslant\varphi_n(x,s)\leqslant\exp(-\lambda(t-s))$ *on* $\bar{D}$, $0\leqslant s\leqslant t$ (*see* [11, *Theorem* 7.3, p. 556]).
(iii) $\iint_{Q_t}a_n(\Delta\varphi_n)^2\leqslant C(\varepsilon,\chi)$ *for all* $n=1,2,\ldots$.

To prove (5.7) (iii) we multiply the equation in (5.6) by $\Delta\varphi_n$ and integrate it over Q_t, and one can then find that

$$(5.8)\qquad \begin{aligned} &\iint_{Q_t}(a_n(\Delta\varphi_n)^2+\lambda|\nabla\varphi_n|^2)+\iint_{S_t}\lambda b_\varepsilon\varphi_n^2+\tfrac{1}{2}\int_D|\nabla\varphi_n(0)|^2=\tfrac{1}{2}\int_D|\nabla\chi|^2+\\ &\quad+\tfrac{1}{2}\int_\Gamma(b_\varepsilon(t)\,\varphi_n^2(t)-b_\varepsilon(0)\,\varphi_n^2(0))-\tfrac{1}{2}\iint_{S_t}\varphi_n^2(b_\varepsilon)_s. \end{aligned}$$

As the left hand side of (5.8) is a sum of positive terms and with the help of

(5.7)(ii), the right-hand side may be estimated indepedently of n, (5.7)(iii) follows.

Now, if we put $\varphi = \varphi_n$ into (5.3), we obtain

$$\int_D (\beta(u(t)) - \beta(v(t)))\chi \leqslant \int_D (\beta(u_0) - \beta(v_0))\varphi_n(0) + \iint_{S_t} (u - v)(b_\varepsilon - b)\varphi_n + $$
$$+ \iint_{Q_t} (\beta(u) - \beta(v))(a - a_n)\Delta\varphi_n + \iint_{Q_t} (f(u) - f(v) + \lambda(\beta(u) - \beta(v)))\varphi_n .$$

Next, with the assistance of (5.4), (5.7)(ii), (iii), and letting $n \to \infty$, we arrive at

$$\int_D (\beta(u(t)) - \beta(v(t)))\chi \leqslant \int_D (\beta(u_0) - \beta(v_0)^+ \exp(-\lambda t) + \tag{5.9}$$
$$+ \iint_{Q_t} (f(u) - f(v) + \lambda(\beta(u) - \beta(v)))^+ \exp(\lambda(s - t)) + C\|b - b_\varepsilon\|_{L^2(S_t)},$$

where $\xi^+ = \max(\xi, 0)$ and the constant C does not depend on n. As the last term on the right-hand side of (5.9) may be arbitrarily small by the choice of b_ε (c.f. (5.5)), it can be omitted.

The proof of our assertions now may proceed by the same way as in the corresponding part of [1] and we leave it out here. The case of $m > 1$ (see [3]) is similar and is left to the reader.

From now on, let us take $f(x, t, u) = f(u)$ and $g(x, t, u) = g(u)$ only so that the problem is

$$(5.10) \qquad \begin{aligned} &(\beta(u))_t = \Delta u + f(u) && x \in D,\ t > 0 \\ &\frac{\partial u}{\partial \nu} + g(u) = 0 && x \in \Gamma,\ t > 0 \\ &u(x, 0) = u_0(x) \in L^\infty(D). \end{aligned}$$

Define $F(u) = f(\beta^{-1}(u))$.

We will use the following hypothesis:

(B) F, $g \in C^1(\mathbf{R})$, $g(\cdot)$ *is nondecreasing on* $\mathbf{R}$ *and there exists* $r \in \mathbf{R}$ *such that* $g(r) = 0$.

The following theorem is a relatively simple consequence of the results discussed above, but may be useful in the study of (5.10).

Theorem 4. *Let* (B) *be satisfied, and suppose that* $0 < m < \infty$ *and* $u_0 \in L^\infty(D)$. *Then there exists a time* $t_{\max}$, $0 < t_{\max} \leqslant \infty$ *such that Problem* (5.10) *has a unique weak solution* (*in the sense of Definition* 1) *on any* $[0, T]$, $0 < T < t_{\max}$. *Moreover,*

$$t^{1/2}u \in L^\infty(0, T; H^1(D)), \quad t^{1/2}(\gamma(u))_t \in L^2(Q_T)$$

and the energy inequality (2.4) *remains to hold for all* $0 < s \leqslant t < t_{\max}$. *In the case* $t_{\max} < \infty$ *we have*

$$(5.11) \qquad \lim_{t \to t_{\max}^-} |u(t)|_\infty = +\infty.$$

Besides, if f satisfies

$$(5.12) \qquad (f(u) - f(0)) \operatorname{sign} u \leqslant K|\beta(u)| \qquad \text{for all } u \in \mathbf{R},$$

where K is a nonnegative constant, then $t_{\max} = \infty$, *i.e. there exists a global solution of* (5.10).

Remarks. (i) The preceding theorem can be easily extended to the case when the functions f and g depend also on x.

(ii) We shall denote the solution u of Problem (5.10) with the initial function u_0 at time t by $u(t, u_0)$.

The proof of Theorem 4 proceeds in a standard way: Put $M = |u_0|_\infty + r + (f(0))^{1/m}$ and define

$$f_M(u) = \begin{cases} f(u) & \text{for } |u| \leqslant M + 1, \\ f(M+1) & \text{otherwise.} \end{cases}$$

Then Problem (5.10) with f replaced by f_M has a unique global solution u_M (Theorem 1—2, Corollary 2), and it satisfies

$$|u_M(t, u_0)|_\infty \leqslant M \exp((K+1)\,t/m) \qquad \text{for all } 0 \leqslant t < \infty,$$

where $K = \max_{|r| \leqslant M+1} F'(\beta(r))$. Now we take $t_1 = \dfrac{m}{K+1} \ln\left(1 + \dfrac{1}{M}\right)$ and one can see that $u_M(t, u_0)$ is a solution of the original Problem (5.10) on $[0, t_1]$. By using the continuation procedure we obtain $t_{\max}$, $0 < t_{\max} \leqslant \infty$ so that (5.10) has a unique maximally defined solution $u(t, u_0)$ on $[0, t_{\max})$.

To prove the energy inequality (2.4) for all $0 < s \leqslant t < t_{\max}$, let now $0 < s < t < t_{\max}$ be arbitrary but fixed. We have already evidenced that $u(s, u_0) \in H^1(D) \cap L^\infty(D)$ and denote $\tilde{u}_0 = u(s, u_0)$. By Theorems 1 and 2 we immediately obtain

$$\int_0^{t-s} \int_D (\gamma(u(\cdot, \tilde{u}_0)))_t^2 + J(u(t-s, \tilde{u}_0)) \leqslant J(\tilde{u}_0),$$

but as $u(\tau, u(s, u_0)) = u(\tau + s, u_0)$ for all $0 \leqslant \tau \leqslant t - s$, the energy inequality (2.4) follows easily.

Now let $t_{\max} < \infty$. First, we show that

$$(5.13) \qquad \limsup_{t \to t_{\max}^-} |u(t, u_0)|_\infty = \infty.$$

If it does not hold, then $|u(t, u_0)|_\infty \leq C$ for all $0 \leq t < t_{\max}$ and (2.4) yields $\int_{s_0}^{t_{\max}} \int_D (\gamma(u))_t^2 \leq C$ for fixed $0 < s_0 < t_{\max}$. So we have $|\gamma(u(t)) - \gamma(u(s))|_2 \leq$ $\leq C|t - s|^{1/2}$ for all $t, s \in [s_0, t_{\max})$, which implies that $\lim_{t \to t_{\max}^-} \gamma(u(t, u_0))$ exists in $L^2(D)$. Let us denote it by V. Next, (2.4) gives that $u(t, u_0) \rightharpoonup v$ in $H^1(D)$ as $t \to t_{\max}^-$ and from the monotonicity of γ, that $V = \gamma(v)$. Then we have $v \in$ $\in H^1(D) \cap L^\infty(D)$ contradicting the maximality of $t_{\max}$. Now suppose that (5.11) does not hold. Then there exists a sequence $\{t_n\}$, $t_n \to t_{\max}^-$ as $n \to \infty$ with $|u(t_n, u_0)|_\infty \leq C$. Let K be a Lipschitz constant of f on $[0, M + 1]$, where $M =$ $= C + r + (f(0))^{1/m}$. Then for all n we have by (3.2)

$$|u(t_n + t, u_0)|_\infty = |u(t, u(t_n, u_0))|_\infty \leq M \exp((K + 1)\, t/m)$$

for $0 \leq t \leq t^*$, where $M \exp((K + 1)\, t^*/m) = M + 1$. So $|u(t_n + t, u_0)|_\infty \leq$ $\leq M + 1$ for all n, $0 \leq t \leq t^*$. But for sufficiently large n we have $t_{\max} < t_n + t^*$, therefore $|u(\tau, u_0)|_\infty \leq M + 1$ for $t_n \leq \tau < t_{\max}$, which contradicts (5.13).

Now let f satisfy (5.12). Define

$$f^+(u) = \begin{cases} \max(f(0), 0) + Ku^m & \text{for } u \geq 0 \\ \max(f(0), 0) & \text{for } u < 0 \end{cases}, \qquad g^+(u) = \begin{cases} 0 & \text{for } u \geq r \\ g(u) & \text{for } u < r \end{cases} \quad \text{and}$$

$$f^-(u) = \begin{cases} \min(f(0), 0) - K(-u)^m & \text{for } u \leq 0 \\ \min(f(0), 0) & \text{for } u > 0 \end{cases}, \qquad g^-(0) = \begin{cases} g(u) & \text{for } u \geq r \\ 0 & \text{for } u < r \end{cases}.$$

Then Problem (5.10) with f, g and u_0 replaced by f^+, g^+ and $u_0^+ (= \max(u_0, 0))$, respectively, has a unique global solution u^+ (Theorem 1—2, Corollary 2), and it is not difficult to check that it satisfies

$$0 \leq u^+(t, u_0^+) \leq (|u_0|_\infty + r + (f(0))^{1/m}) \exp((K + 1)\, t/m)$$

for all $0 \leq t < \infty$, and analogously,

$$-(|u_0|_\infty + r + (f(0))^{1/m}) \exp((K + 1)\, t/m) \leq u^-(t, u_0^-) \leq 0$$

for all $0 \leq t < \infty$. Now, it can be seen from the construction of f^+, g^+ and f^-, g^- above that u^+ is a supersolution and u^- is a subsolution of Problem (5.10). Hence

$$u^-(t, u_0^-) \leq u(t, u_0) \leq u^+(t, u_0^+)$$

and the solution u exists globally.

The proof of Theorem 4 is complete.

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НЕЛИНЕЙНОЕ ДИФУЗНОЕ УРАВНЕНИЕ С НЕЛИНЕЙНЫМИ КРАЕВЫМИ УСЛОВИЯМИ: МЕТОД ПРЯМЫХ

Ján Filo

Резюме

В работе рассматривается возмущенное уравнение типа нестационархой фильтрации газа и возмущенное уравнение типа быстрой дифузии с многими пространственными переменными в ограниченой области с нелинейными граничными условиями (1.1).

В статье доказано существование, единственность и некоторые особенности обобщенных решений.