Mathematica Slovaca

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Mathematica Slovaca, Vol. 38 (1988), No. 2, 159--166

Persistent URL: http://dml.cz/dmlcz/136469

ON THE CONDITIONAL EXPECTATION IN A REGULAR ORDERED SPACE

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The conditional expectation of functions defined on an arbitrary probability space with values in a regular ordered space is defined in the paper presented. We use the Bochner integral defined in [2]. The integration theory in ordered spaces is elaborated in papers [4], [8], too. The conditional expectation of functions with values in a Banach lattice and its applications can be found in [1], [5], [6], [7].

1. Notations and notions

We suppose that $(\Omega, \mathscr{S}, P)$ is a probability space.

We say that X is a regular ordered space if X is a σ-complete vector lattice (that is X is a vector lattice such that every non-empty at most countable subset of X which is bouded from above has a supremum) and X has the diagonal property (that is for any $x_{n,k} \in X (n, k = 1, 2, \ldots)$, any $x_n \in X (n = 1, 2, \ldots)$ and any $x \in X$ such that $x_{n,k} \to x_n$ $(k \to \infty)$ for all n and $x_n \to x$, there exists for every n an appropriate $k = k(n)$ such that $x_{n,k(n)} \to x$). In the preceding we use the convergence with respect to the ordering ($x_n \to x$ iff there exist $a_n \in X$, $a_n \searrow 0$ and $|x_n - x| \leqq a_n$ for all n).

If $f_n, f\colon \Omega \to X$, then

(i) $f_n \to f$ uniformly on $A \in \mathscr{S}$ iff there exist $a_n \in X$, $a_n \searrow 0$ such that $|f_n(\omega) - f(\omega)| \leqq a_n$ for every $\omega \in A$ and every n; $f_n \to f$ uniformly iff $f_n \to f$ uniformly on Ω,

(ii) $f_n \to f$ almost uniformly iff for any $\varepsilon > 0$ there exists $A \in \mathscr{S}$ such that $P(\Omega - A) < \varepsilon$ and $f_n \to f$ uniformly on A,

(iii) $f_n \to f$ uniformly almost everywhere iff there exists $A \in \mathscr{S}$, $P(A) = 0$ and $f_n \to f$ uniformly on $\Omega - A$.

A function $f\colon \Omega \to X$ is a simple function iff $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $a_i \in X$, $A_i \in \mathscr{S}$ $(i = 1, 2, \ldots, n)$, $\bigcup_{i=1}^{n} A_i = \Omega$, $A_i \cap A_j = \emptyset$ $(i \neq j)$.

The integral of the simple function f is defined by $\int f \, \mathrm{d}P = \sum_{i=1}^{n} a_i P(A_i)$.

1.1. Definition. *A function f: $\Omega \to X$ is called Bochner integrable iff there exist f_n simple such that $f_n \to f$ almost uniformly and*

$$\bigvee_{i,j \geqq n} \int |f_i - f_j| \, \mathrm{d}P \to 0.$$

Then $\int f \, \mathrm{d}P = \lim_{n \to \infty} \int f_n \, \mathrm{d}P$.

If f is Bochner integrable and $A \in \mathscr{S}$, then $f\chi_A$ is Bochner integrable and we define

$$\int_A f \, \mathrm{d}P = \int f\chi_A \, \mathrm{d}P.$$

If f: $\Omega \to R$ (R is the set of the real numbers), f is measurable and integrable, $a \in X$, then af is Bochner integrable and

$$\int af \, \mathrm{d}P = a \int f \, \mathrm{d}P.$$

2. A random variable

2.1. Definition. *Denote $\mathscr{L}(\mathscr{S}) = \{f$: $\Omega \to X; f_n$ simple, $f_n \to f$ uniformly$\}$ and $L(\mathscr{S}) = \{f$: $\Omega \to X$; $\exists f' \in \mathscr{L}(\mathscr{S})$, $f = f'$ almost everywhere$\}$. A function f: $\Omega \to X$ is called an X-random variable iff $f \in L(\mathscr{S})$.*

2.2. Theorem. *If $f_n \in \mathscr{L}(\mathscr{S})$ $(n = 1, 2, \ldots)$ and $f_n \to f$ uniformly, then $f \in \mathscr{L}(\mathscr{S})$.*

Proof. Since $f_n \in \mathscr{L}(\mathscr{S})$, there exist $f_{n,k}$ simple $(n, k = 1, 2, \ldots)$ and $a_{n,k} \in X$ $(n, k = 1, 2, \ldots)$, $a_{n,k} \searrow 0$ $(k \to \infty)$ such that for every $\omega \in \Omega$ and any k

$$|f_{n,k}(\omega) - f_n(\omega)| \leqq a_{n,k} \quad \text{for} \quad n = 1, 2, \ldots.$$

By the diagonal property for every n there exist $k = k(n)$ such that $a_{n,k(n)} \to 0$, which implies that there exist $b_n \in X$, $b_n \searrow 0$ such that $a_{n,k(n)} \leqq b_n$. Now from the fact that $f_n \to f$ uniformly there exist $c_n \in X$, $c_n \searrow 0$ such that for every $\omega \in \Omega$ and any n

$$|f_{n,k(n)}(\omega) - f(\omega)| \leqq |f_{n,k(n)}(\omega) - f_n(\omega)| + |f_n(\omega) - f(\omega)| \leqq b_n + c_n.$$

We see that $f_{n,k(n)}$ are simple and $f_{n,k(n)} \to f$ uniformly. Hence $f \in \mathscr{L}(\mathscr{S})$.

2.3. Remark. From Theorem 2.2 and the definition of $L(\mathscr{S})$ it follows that if $f_n \in L(\mathscr{S})$ and $f_n \to f$ uniformly almost everywhere, then $f \in L(\mathscr{S})$ and it is easy to show that $f, g \in \mathscr{L}(\mathscr{S})$ $(L(\mathscr{S}))$ implies $f + g$, $f \vee g$, $f \wedge g$, cf, $|f| \in \mathscr{L}(\mathscr{S})$ $(L(\mathscr{S}))$.

3. A conditional expectation

Let $\mathcal{S}_0 \subset \mathcal{S}$ be a σ-algebra. The version of the conditional expectation $E(f/\mathcal{S}_0)$ of a random variable f in the real case is an $\mathcal{S}_0$-measurable function for which

$$\int_A f \, \mathrm{d}P = \int_A E(f/\mathcal{S}_0) \, \mathrm{d}P$$

holds for any $A \in \mathcal{S}_0$. If g, h are two versions of the conditional expectation of f, then $g = h$ $P/\mathcal{S}_0$ almost everywhere. Further, the operator $E(./\mathcal{S}_0)$ has the following properties:

(1) if f, g are random variables and c is real, then
$E(f+g/\mathcal{S}_0) = E(f/\mathcal{S}_0) + E(g/\mathcal{S}_0)$ $P/\mathcal{S}_0$ a.e. and
$E(cf/\mathcal{S}_0) = cE(f/\mathcal{S}_0)$ $P/\mathcal{S}_0$ a.e.,

(2) if f, g are random variables and $f \geqq g$, then
$E(f/\mathcal{S}_0) \geqq E(g/\mathcal{S}_0)$ $P/\mathcal{S}_0$ a.e.,

(3) if f_n are random variables and $f_n \to 0$, then
$E(f_n/\mathcal{S}_0) \to 0$ $P/\mathcal{S}_0$ a.e.

We shall define the conditional expectation for the functions from $L(\mathcal{S})$ now.

3.1. Definition. *(i) If $f\colon \Omega \to X$ is simple, $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, then we define the version of a conditional expectation of f with respect to $\mathcal{S}_0$ by the formula $J_0(f) = E(f/\mathcal{S}_0) = \sum_{i=1}^{n} a_i P(A_i/\mathcal{S}_0)$, where $P(A_i/\mathcal{S}_0)$ is the conditional probability of the set A_i.*

(ii) If $f \in \mathcal{L}(\mathcal{S})$ and f_n are simple such that $f_n \to f$ uniformly, then we define

$$J(f) = E(f/\mathcal{S}_0) = \lim_{n\to\infty} E(f_n/\mathcal{S}_0).$$

(iii) If $f \in L(\mathcal{S})$ and $f' \in \mathcal{L}(\mathcal{S})$, $f = f'$ a.e., then we put

$$J'(f) = E(f/\mathcal{S}_0) = J(f').$$

3.2. Remark. It is evident that if f is simple, $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $f = \sum_{j=1}^{m} b_j \chi_{B_j}$, then

$$\sum_{i=1}^{n} a_i P(A_i/\mathcal{S}_0) = \sum_{j=1}^{m} b_j P(B_j/\mathcal{S}_0) \; P/\mathcal{S}_0 \text{ a.e.}$$

and that J_0 fulfils the properties (1) and (2). If f_n are simple and $f_n \to 0$ uniformly, then there exists $A \in \mathscr{S}_0$, $P(A) = 0$ and there exist $a_n \in X$, $a_n \searrow 0$ such that for any n and every $\omega \in \Omega - A$ we have (by (2))

$$|J_0(f_n)(\omega)| \leqq J_0(|f_n|)(\omega) \leqq J_0(a_n) = a_n.$$

Then $J_0(f_n) \to 0$ uniformly $P/\mathscr{S}_0$ almost everywhere.

3.3. Lemma. *If f is simple, then $J_0(f) \in L(\mathscr{S}_0)$ and for all $A \in \mathscr{S}_0$*

$$\int_A f \, \mathrm{d}P = \int_A J_0(f) \, \mathrm{d}P.$$

Proof. Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$. Then $J_0(f) = \sum_{i=1}^{n} a_i P(A_i/\mathscr{S}_0)$, where $P(A_i/\mathscr{S}_0)$ $(i = 1, 2, \ldots, n)$ is an $\mathscr{S}_0$-measurable, almost everywhere bounded real-valued function. Then there exists a sequence $(g_{i,k})_k$ $(i = 1, 2, \ldots, n)$ of simple $\mathscr{S}_0$-measurable real-valued functions such that $g_{i,k} \to P(A_i/\mathscr{S}_0)$ $(k \to \infty)$ uniformly a.e. $(i = 1, 2, \ldots, n)$. Hence

$$h_k = \sum_{i=1}^{n} a_i g_{i,k} \quad (k = 1, 2, \ldots)$$

is simple and $h_k \to J_0(f)$ uniformly a.e., which implies $J_0(f) \in L(\mathscr{S}_0)$.
If $A \in \mathscr{S}_0$, then

$$\int_A J_0(f) \, \mathrm{d}P = \int_A \sum_{i=1}^{n} a_i P(A_i/\mathscr{S}_0) \, \mathrm{d}P = \sum_{i=1}^{n} a_i \int_A P(A_i/\mathscr{S}_0) \, \mathrm{d}P =$$
$$= \sum_{i=1}^{n} a_i P(A_i \cap A) = \int_A f \, \mathrm{d}P.$$

3.4. Lemma. *Let f_n be simple functions and let $f_n \to f$ uniformly. Then the following conditions hold*:

(*i*) $$\bigvee_{i,j \geqq n} \int |f_i - f_j| \, \mathrm{d}P \to 0 \quad (n \to \infty)$$

(*that is f is Bochner integrable*),

(*ii*) $$\bigvee_{i,j \geqq n} J_0(|f_i - f_j|) \to 0 \quad (n \to \infty) \text{ uniformly a.e.}$$

Proof. There exist $a_n \in X$, $a_n \searrow 0$ such that for every $\omega \in \Omega$ and any n we have $|f_n(\omega) - f(\omega)| \leqq a_n$. Then for $i, j \geqq n$ and for any $\omega \in \Omega$ we get

$$|f_i(\omega) - f_j(\omega)| \leqq |f_i(\omega) - f(\omega)| + |f(\omega) - f_j(\omega)| \leqq a_i + a_j \leqq 2a_n \quad \text{and} \quad 2a_n \searrow 0.$$

The proofs of (*i*) and (*ii*) follow from the positivity of the integral and the property (2) of J_0.

3.5. Remark. It is easy to show that if $f \in L(\mathscr{S})$, then f is Bochner integrable.

3.6. Lemma. *If $f \in \mathscr{L}(\mathscr{S})$ and f_n, g_n are simple such that $f_n \to f$ uniformly and $g_n \to f$ uniformly, then*

$$\lim_{n\to\infty} J_0(f_n) = \lim_{n\to\infty} J_0(g_n)\, P/\mathscr{S}_0 \text{ a.e..}$$

If f is simple, then $J(f) = J_0(f)\, P/\mathscr{S}_0$ a.e..

Proof. If f_n, g_n are simple and $f_n \to f$ uniformly, $g_n \to f$ uniformly, then $f_n - g_n \to 0$ uniformly, that is there exist $a_n \in X$, $a_n \searrow 0$ such that for every $\omega \in \Omega$ and any n we have $|f_n(\omega) - g_n(\omega)| \leqq a_n$. Then from the property (2) of J_0 there exist $A \in \mathscr{S}_0$, $P(A) = 0$ and

$$|J_0(f_n)(\omega) - J_0(g_n)(\omega)| = |J_0(f_n - g_n)(\omega)| \leqq J_0(|f_n - g_n|)(\omega) \leqq a_n$$

for any n and every $\omega \in \Omega - A$, which implies that

$$\lim_{n\to\infty} J_0(f_n) = \lim_{n\to\infty} J_0(g_n)\, P/\mathscr{S}_0 \text{ a.e..}$$

From the preceding $J(f) = J_0(f)\, P/\mathscr{S}_0$ a.e. for f simple holds.

We shall show the existence of $J(f)\, P/\mathscr{S}_0$ a.e..

3.7. Lemma. *If f_n are simple and $f_n \to f$ uniformly, then $\lim_{n\to\infty} J_0(f_n)$ exists almost everywhere.*

Proof. It is sufficient to show that

$$\bigwedge_{n=1}^{\infty} \bigvee_{i=n}^{\infty} J_0(f_i)(\omega) \leqq \bigvee_{n=1}^{\infty} \bigwedge_{j=n}^{\infty} J_0(f_j)(\omega)$$

for almost every $\omega \in \Omega$.

By Lemma 3.4 and Remark 3.2 we have

$$|J_0(f_i)(\omega) - J_0(f_j)(\omega)| \leqq J_0(|f_i - f_j|)(\omega) \leqq a_n$$

for $i,j \geqq n$ and almost every $\omega \in \Omega$, where $a_n \in X$, $a_n \searrow 0$. Now $J_0(f_i)(\omega) \leqq J_0(f_j)(\omega) + a_n$ for $i,j \geqq n$ and almost every $\omega \in \Omega$. Then

$$\bigvee_{i \geqq n} J_0(f_i)(\omega) \leqq \bigwedge_{j \geqq n} J_0(f_j)(\omega) + a_n$$

for any n and almost every $\omega \in \Omega$ and hence

$$\bigwedge_{n=1}^{\infty} \bigvee_{i \geqq n} J_0(f_i)(\omega) \leqq \bigvee_{n=1}^{\infty} \bigwedge_{j \geqq n} J_0(f_j)(\omega)$$

for almost every $\omega \in \Omega$.

3.8. Lemma. *The operator J fulfils the properties* (1), (2) *and if $f_n \in \mathscr{L}(\mathscr{S})$, $f_n \to 0$ uniformly, then $J(f_n) \to 0$ uniformly a.e..*

Proof. If f_n, g_n are simple, $f_n \to f$ uniformly, $g_n \to g$ uniformly and c is real, then $f_n + g_n \to f + g$ uniformly and $cf_n \to cf$ uniformly. Then by the property (1) of J_0 there exists $A \in \mathscr{S}_0$, $P(A) = 0$ such that for every $\omega \in \Omega - A$ we have

$$J(f+g)(\omega) = \lim_{n\to\infty} J_0(f_n + g_n)(\omega) = \lim_{n\to\infty} J_0(f_n)(\omega) +$$

$$+ \lim_{n\to\infty} J_0(g_n)(\omega) = J(f)(\omega) + J(g)(\omega).$$

Similarly we can show it for cf and hence J fulfils (1).

If f_n, g_n are simple, $f_n \to f$ uniformly, $g_n \to g$ uniformly and $f \leqq g$, then $h_n = f_n \wedge g_n \to f \wedge g = f$ uniformly, $h_n \leqq g_n$. Now (by the property (2) of J_0)

$$J(f) = \lim_{n\to\infty} J_0(h_n) \leqq \lim_{n\to\infty} J_0(g_n) = J(g) \text{ a.e.}$$

and J fulfils (2).

If $f_n \in \mathscr{L}(\mathscr{S})$, $f_n \to 0$ uniformly, then there exist $a_n \in X$, $a_n \searrow 0$ such that $|f_n(\omega)| \leqq a_n$ for every $\omega \in \Omega$ and any n. Then there exists $A \in \mathscr{S}_0$, $P(A) = 0$ such that for every $\omega \in \Omega - A$

$$|J(f_n)(\omega)| \leqq J(|f_n|)(\omega) \leqq a_n$$

for any n. Hence $J(f_n) \to 0$ uniformly a.e..

3.9. Lemma. *If $f \in \mathscr{L}(\mathscr{S})$, then $J(f) \in L(\mathscr{S}_0)$ and for all $A \in \mathscr{S}_0$ $\int_A f \, dP = \int_A J(f) \, dP$.*

Proof. $J(f) = \lim_{n\to\infty} J_0(f_n)$, where f_n are simple, $f_n \to f$ uniformly, $J_0(f_n) \in L(\mathscr{S}_0)$ (Lemma 3.3). Hence there exist $a_n \in X$, $a_n \searrow 0$ and $|f(\omega) - f_n(\omega)| \leqq a_n$ for every $\omega \in \Omega$ and any n. Then there exists $A \in \mathscr{S}_0$, $P(A) = 0$ such that for every $\omega \in \Omega - A$

$$|J(f)(\omega) - J_0(f_n)(\omega)| = |J(f - f_n)(\omega)| \leqq J(|f - f_n|)(\omega) \leqq a_n$$

for any n and hence we have $J_0(f_n) \to J(f)$ uniformly a.e.. By Remarks 2.3 and 3.5 we get $J_0(f_n)$, $J(f)$ are Bochner integrable. Let $A \in \mathscr{S}_0$. Then by Lemma 3.3 and by the continuity of the integral we get

$$\int_A f \, dP = \lim_{n\to\infty} \int_A f_n \, dP = \lim_{n\to\infty} \int_A J_0(f_n) \, dP = \int_A J(f) \, dP.$$

3.10. Lemma. *Let $f \in L(\mathscr{S})$, $h, g \in \mathscr{L}(\mathscr{S})$* and *$f = g$ a.e., $f = h$ a.e., then*

$J'(f) = J(g) = J(h) \quad P/\mathscr{S}_0$ a.e..

Proof. There exists $A \in \mathscr{S}$, $P(A) = 0$ such that $g\chi_{A'} = h\chi_{A'}$ and there exist f_n simple such that $f_n \to g\chi_{A'} = h\chi_{A'}$ uniformly. Then

$$J(h) = J(h\chi_{A'} + h\chi_A) = J(h\chi_{A'}) + J(h\chi_A) = \lim_{n\to\infty} J_0(f_n) + J(h\chi_A) \quad P/\mathcal{S}_0 \quad \text{a.e.},$$

$$J(g) = J(g\chi_{A'} + g\chi_A) \lim_{n\to\infty} J_0(f_n) + J(g\chi_A) \quad P/\mathcal{S}_0 \quad \text{a.e..}$$

We shall show that if $A \in \mathcal{S}$ and $P(A) = 0$, $h \in \mathcal{L}(\mathcal{S})$, then $J(h\chi_A) = 0$ $P/\mathcal{S}_0$ a.e.. Let h be simple, $h = \sum_{i=1}^{n} a_i \chi_{A_i}$. Then

$$J(h\chi_A) = J_0(h\chi_A) = \sum_{i=1}^{n} a_i P(A_i \cap A/\mathcal{S}_0) = 0 \quad P/\mathcal{S}_0 \quad \text{a.e..}$$

If $h \in \mathcal{L}(\mathcal{S})$, then $J(h\chi_A) = \lim_{n\to\infty} J_0(h_n\chi_A)$, where h_n are simple, $h_n \to h$ uniformly and hence $J(h\chi_A) = 0$ $P/\mathcal{S}_0$ a.e.. From the preceding we get $J(h) = J(g)$ $P/\mathcal{S}_0$ a.e..

3.11. Theorem. *The operator J' fulfils the property* (1), (2) *and* (3) *for the uniform convergence almost everywhere. If $f \in L(\mathcal{S})$, then $J'(f) \in L(\mathcal{S}_0)$ and $\int_A f \, dP = \int_A J'(f) \, dP$ for every $A \in \mathcal{S}_0$.*

Proof. The proof is wvident.

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Received December 16, 1985

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УСЛОВНОЕ МАТЕМАТИЧЕСКОЕ ОЖИДАНИЕ В РЕГУЛЯРНОМ ПРОСТРАНСТВЕ

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Резюме

В статье определено условное математическое ожидание случайной величины со значениями в регулярном пространстве, которая является равномерным пределом последовательности простых случайных величин.