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Mathematica Slovaca, Vol. 38 (1988), No. 2, 139--145

Persistent URL: http://dml.cz/dmlcz/136467

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RADICALS AND THEIR LEFT IDEAL ANALOGUES IN A SEMIGROUP

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The first section of the present paper deals with an R^*NC-semigroup. It is known that an R^*NC-semigroup is a semilattice of archimedean semigroups (see [5]). We prove that the converse is also true (Theorem 1).

In the second section we prove that in a semigroup S for any left ideal L we have $L \subseteq r(L) \subseteq m(L) \subseteq r^*(L) \subseteq N(L) \subseteq c(L)$ (Theorem 2). This is a left-sided analogue of the known result about radicals of R. Šulka [9, Lemma 19] and J. Bosák [2].

We give some definitions (the others can be found in [2), [3], [7], or [9]). Let S be a semigroup.

A non-empty subset J of S is a two-sided (or left) ideal if $S^1JS^1 \subseteq J$ (or $S^1J \subseteq J$). The principal two-sided (or left) ideal of S generated by an element $a \in S$ is denoted by $J(a)$ (or $L(a)$).

An element $x \in S$ is nilpotent with respect to a subset A if $x^n \in A$ for some positive integer n. The set of all nilpotent elements of S with respect to A is denoted by $N(A)$.

A two-sided (or left) ideal A is a nilideal (or left nilideal) with respect to a two-sided (or left) ideal J if $A \subseteq N(J)$. The union of all two-sided (or left) nilideals with respect to a two-sided (or left) ideal J is denoted by $R^*(J)$ (or $r^*(J)$).

A two-sided (or left) ideal A is nilpotent with respect to a two-sided (or left) ideal J if $A^n \subseteq J$ for some positive integer n. The union of all two-sided (or left) nilpotent ideals of S with respect to a two-sided (or left) ideal J is denoted by $R(J)$ (or $r(J)$).

A two-sided (or left) ideal Q is prime (or left prime) if for any two-sided (or left) ideals A, B of S, $AB \subseteq Q$ implies that $A \subseteq Q$ or $B \subseteq Q$. We denote by $M(J)$ (or $m(J)$) the intersection of all two-sided (or left) prime ideals of S containing a two-sided (or left) ideal J.

A two-sided (or left) ideal P is completely prime (or left completely prime) if for any $a, b \in S$, $ab \in P$ implies that $a \in P$ or $b \in P$. We denote by $C(A)$ (or $c(A)$) the intersection of all two-sided (or left) completely prime ideals of S containing a given subset A.

If J is a two-sided (or left) ideal of S, then $R(J)$, $M(J)$, $R^*(J)$, $C(J)$ (or $r(J)$, $m(J)$, $r^*(J)$, $c(J)$) are two-sided (or left) ideals. The two-sided ideals $R(J)$, $M(J)$, $r^*(J)$ and $C(J)$ are called the radicals of Schwarz, McCoy, Clifford and Luh with respect to J.

A two-sided (or left) ideal J of S is semiprime (or left semiprime) if for any two-sided (or left) ideal A of S, $A^n \subseteq J$ for some positive integer n implies that $A \subseteq J$.

A two-sided (or left) ideal J of S is completely (or left completely) semiprime if for any $a \in S$, $a^n \in J$ for some positive integer n implies that $a \in J$.

Evidently, if J is a two-sided (or left) ideal, then $M(J)$ (or $m(J)$) is semiprime (or left semiprime) and $C(J)$ (or $c(J)$) is completely (or left completely) semiprime two-sided (or left) ideal.

A semigroup S has the Q_3-property (see M. S. Putcha [8]) if for any $a, b \in S$, $b \in J(a)$ implies that $b^n \in J(a^2)$ for some positive integer n.

A semigroup is called an R^*NC-semigroup if for any two-sided ideal $J \subseteq S$, $R^*(J) = N(J) = C(J)$ holds.

A commutative semigroup, each element of which is idempotent, is called a semilattice.

A congruence ϱ on S is a semilattice congruence if the factor semigroup S/ϱ is a semilattice.

A semigroup S is called archimedean if for any $a, b \in S$ there exists a positive integer n for which $a^n \in SbS$.

A semigroup S is a semilattice of archimedean semigroups if there exists a semilattice congruence σ on S such that each σ-class of the factor semigroup S/σ is an archimedean subsemigroup of S. Then σ is the least semilattice congruence on S, since an archimedean subsemigroup of S contains no proper completely prime ideals (see [7, Lemma II.4.2]).

1. On radicals

Theorem 1. *In a semigroup S the following conditions are equivalent:*

(1) $N(J(a)) = N(J(a^n))$ *for every $a \in S$ and every positive integer n.*

(2) *The set $N(J(a))$ is a two-sided ideal of S for every $a \in S$.*

(3) *The set $N(J)$ is a two-sided ideal of S for every two-sided ideal J of S.*

(4) *S is an R^*NC-semigroup.*

(5) *S is a semilattice of archimedean semigroups.*

(6) *S has the Q_3-property.*

Proof. We prove that (1) implies (2). Let $a \in S$, $b \in N(J(a))$. Then $b^k \in J(a)$ for some positive integer k, hence $J(b^k) \subseteq J(a)$ and $N(J(b^k)) \subseteq N(J(a))$. Let $x, y \in S^1$, then $xby \in J(b) \subseteq N(J(b))$, since by the assumption $N(J(b^k)) = N(J(b))$

we obtain that $xby \in N(J(b^k)) \subseteq N(J(a))$. Therefore $N(J(a))$ is a two-sided ideal of S.

We prove that (2) implies (3). Let J be any two-sided ideal of S. If $J = \{a_i, i \in I\}$, then evidently $J = \bigcup_{i \in I} J(a_i)$. Since each $N(J(a_i))$ is a two-sided ideal of S we obtain that $N(J) = N\left(\bigcup_{i \in I} J(a_i)\right) = \bigcup_{i \in I} N(J(a_i))$ is a two-sided ideal of S.

By Corollary 1 of [4] the condition (3) implies (4).

By Theorem 5 of [5] the condition (4) implies (5).

We prove that (5) implies (1). Let S be a semilattice of archimedean semigroups S_α, $\alpha \in \Lambda$. Then S is a disjoint union of archimedean subsemigroups S_α, $\alpha \in \Lambda$ and for every $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $S_\alpha S_\beta \cup S_\beta S_\alpha \subseteq S_\gamma$.

We show that $N(J(a^n)) \supseteq N(J(a))$ for every $a \in S$ and for every positive integer n. Let $a \in S$, $x \in N(J(a))$, then $x^k \in J(a)$ for some positive integer k, hence $x^k = sat$ for some $s, t \in S^1$. The elements a, a^n for every positive integer n belong to the same subsemigroup S_α for some $\alpha \in \Lambda$, hence there exists $\beta \in \Lambda$ such that $x^k = sat \in S_\beta$ and $sa^n t \in S_\beta$. Since x^k, $sa^n t \in S_\beta$ and S_β is an archimedean semigroup there exists a positive integer m such that

$$(x^k)^m = x^{km} \in S_\beta s a^n t S_\beta \subseteq J(a^n),$$

thus $x \in N(J(a^n))$ for every positive integer n.

In any semigroup $N(J(a^n)) \subseteq N(J(a))$ for every positive integer n, hence we have $N(J(a)) = N(J(a^n))$.

The equivalence of (5) and (6) was proved by M. S. Putcha [8, Theorem 2.1].

2. Left ideal analogues of radicals

From the definitions we immediately obtain

Lemma 1. *Let S be a semigroup with a left ideal L. Then $L \subseteq r(L) \subseteq r^*(L) \subseteq N(L)$ and $L \subseteq c(L) \cap m(L)$.*

Lemma 2. *Let S be a semigroup, L_1, L_2 left ideals of S with $L_1 \subseteq L_2$. Then*

a) $r(L_1) \subseteq r(L_2)$,

b) $m(L_1) \subseteq m(L_2)$,

c) $r^*(L_1) \subseteq r^*(L_2)$,

d) $N(L_1) \subseteq N(L_2)$,

e) $c(L_1) \subseteq c(L_2)$.

Proof. a) Let $x \in r(L_1)$. Then for some positive integer n, $L(x)^n \subseteq \subseteq L_1 \subseteq L_2$, therefore $L(x) \subseteq r(L_2)$ and so $x \in r(L_2)$.

The assertions b), d) and e) are evident.

c) Let $x \in r^*(L_1)$, then $L(x) \subseteq r^*(L_1) \subseteq N(L_1) \subseteq N(L_2)$. Hence $L(x)$ is a left nilideal with respect to L_2, thus $x \in r^*(L_2)$.

The next lemmas 3 and 4 are analogous to Lemma 1 and Theorem 4 of [6], where the statements are proved for two-sided ideals.

Lemma 3. *Let S be a simigroup with a left ideal L. If $H = \{x, x^2, x^3, \dots\}$ is a cyclic subsemigroup of S with $H \cap L = \emptyset$, then there exists a left prime ideal $Q \supseteq L$ such that $Q \cap J = \emptyset$ and $Q = r^*(Q)$.*

Proof. The set of all left ideals which contain L and do not meet H is non-empty since it contains L. We denote this set by $\boldsymbol{T}$. The set $\boldsymbol{T}$ is closed under unions of increasing chain, thus we can apply Zorn's lemma and we obtain a maximal element $Q \in \boldsymbol{T}$.

We prove that Q is a left prime ideal of S. Suppose that for some left ideals A, B of S we have $AB \subseteq Q$, however $A \nsubseteq Q$ and $B \nsubseteq Q$. Then the left ideal $Q \cup A$ contains some x^r and the left ideal $Q \cup B$ contains some x^s of H. Since $x^r \notin Q$, $x^s \notin Q$ we have $x^r \in A$, $x^s \in B$ and so $x^{r+s} \in AB \subseteq Q$, which contradicts $H \cap Q = \emptyset$. Therefore Q is a left prime ideal.

We prove that $r^*(Q) = Q$. By Lemma 1 we have $Q \subseteq r^*(Q)$.

Suppose that $Q \neq r^*(Q)$. Then $H \cap r^*(Q) \neq \emptyset$, hence for some positive integer m we have $x^m \in H \cap r^*(Q)$. Since $x^m \in r^*(Q)$ there exists a positive integer n with $(x^m)^n = x^{mn} \in Q$. This contradicts $H \cap Q = \emptyset$.

Lemma 4. *Let S be a semigroup with a left ideal L. If $\{Q_i, i \in I\}$ is the set of all left prime ideals of S containing L such that $r^*(Q_i) = Q_i$, then $r^*(L) = \bigcap_{i \in I} Q_i$.*

Proof. By Lemma 2, $L \subseteq Q_i$ implies $r^*(L) \subseteq r^*(Q_i) = Q_i$ for each $i \in I$. Therefore $r^*(L) \subseteq \bigcap_{i \in I} Q_i$.

Conversely, we prove that $\bigcap_{i \in I} Q_i \subseteq r^*(L)$. If $r^*(L) = S$, then the statement holds. Suppose therefore that $r^*(L) \neq S$. We prove that $S - r^*(L) \subseteq S - \bigcap_{i \in I} Q_i$. Let $x \in S - r^*(L)$. Then $x \notin r^*(L)$, therefore the principal left ideal $L(x) \nsubseteq r^*(L)$ and so there exists an element $y \in L(x)$ such that $y^n \notin L$ for all positive integers n. Denote $H = \{y, y^2, y^3, \dots\}$. We have $H \cap L = \emptyset$. By Lemma 3 there exists a left prime ideal $Q_j = r^*(Q_j) \supseteq L$ such that $Q_j \cap H = \emptyset$ for some $j \in I$. Then $x \notin Q_j$ since $x \in Q_j$ implies $L(x) \subseteq Q_j$, hence $y \in Q_j$, a contradiction with $H \cap Q_j = \emptyset$. Thus $x \notin \bigcap_{i \in I} Q_i$ and so $x \in S - \bigcap_{i \in I} Q_i$.

Theorem 2. *Let S be a semigroup with a left ideal L. Then we have*:

$$L \subseteq r(L) \subseteq m(L) \subseteq r^*(L) \subseteq N(L) \subseteq c(L).$$

Proof. By Lemma 1 we have $L \subseteq r(L)$.

We prove that $r(L) \subseteq m(L)$. Let $\{Q_k, k \in K\}$ be the set of all left prime ideals of S containing L. Then $m(L) = \bigcap_{k \in K} Q_k$. Let $a \in r(L)$. Then $L(a)^n \subseteq L$ for some positive integer n. However, $L \subseteq m(L)$ and so $L(a)^n \subseteq m(L)$. Since $m(L)$ is left semiprime we obtain $L(a) \subseteq m(L)$ and so $a \in m(L)$.

We prove that $m(L) \subseteq r^*(L)$. Let $\{Q_i, i \in I\}$ be the set of all left prime ideals containing L with the property $r^*(Q_i) = Q_i$ for any $i \in I$. Then evidently $\{Q_i, i \in I\} \subseteq \{Q_k, k \in K\}$ and so by Lemma 4 we obtain

$$m(L) = \bigcap_{k \in K} Q_k \subseteq \bigcap_{i \in I} Q_i = r^*(L).$$

Evidently, $r^*(L) \subseteq N(L)$.

We prove that $N(L) \subseteq c(L)$. Let $a \in N(L)$, then $a^n \in L$ for some positive integer n. Since $L \subseteq c(L)$ we have $a^n \in c(L)$. However, $c(L)$ is a left completely semiprime ideal, hence $a \in c(L)$.

Lemma 5. *Let S be a semigroup with a two-sided ideal J. Then*

a) $r(J) = R(J)$,

b) $m(J) \subseteq M(J)$,

c) $r^*(J) = R^*(J)$,

d) $c(J) \subseteq C(J)$.

Proof. a) Evidently $R(J) \subseteq r(J)$. Conversely, we show that $r(J) \subseteq R(J)$. Let $a \in r(J)$. Then a belongs to some nilpotent left ideal A with respect to J. If $A^n \subseteq J$ for some positive integer n, then $(AS^1)^n = A(S^1A)^{n-1}S^1 \subseteq A^nS^1 \subseteq J$. Therefore $a \in A \subseteq AS^1 \subseteq R(J)$, hence $r(J) \subseteq R(J)$ and thus $r(J) = R(J)$.

b) If Q is a two-sided prime ideal of S, then Q is left prime. If, namely, for left ideals A, B of S, $AB \subseteq Q$, then for the two-sided ideals AS^1, BS^1 we have $AS^1BS^1 \subseteq ABS^1 \subseteq Q$ and so $AS^1 \subseteq Q$ or $BS^1 \subseteq Q$, thus $A \subseteq Q$ or $B \subseteq Q$. This immediately implies that $m(J) \subseteq M(J)$.

c) Evidently $R^*(J) \subseteq r^*(J)$. We show that $r^*(J) \subseteq R^*(J)$. Let $a \in r^*(J)$, then $L(a) \subseteq r^*(J)$ and so $L(a)$ is the principal left nilideal with respect to J. We have $J(a) = L(a)S^1$. Choose $x \in J(a)$. Then $x = ys$, where $y \in L(a)$ and $s \in S^1$. Since $sy \in L(a)$ we have $(sy)^n \in J$ for some positive integer n. Then $x^{n+1} = (ys)^{n+1} = y(sy)^n s \in J$ and so x is nilpotent with respect to J. Hence $a \in R^*(J)$ and $r^*(J) \subseteq R^*(J)$, thus $r^*(J) = R^*(J)$.

d) Evidently, each two-sided completely prime ideal of S is left completely prime thus $c(J) \subseteq C(J)$.

The following examples show that the sets of Theorem 2 can be different.

Example 1. Let $S_1 = \{0, e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}, e_{31}, e_{32}, e_{33}\}$ with the multiplication $e_{ik} \cdot e_{kn} = e_{in}$, $e_{ik} \cdot e_{jn} = e_{ik} \cdot 0 = 0 \cdot e_{ik} = 0$ for $i, j, k, n \in \{1, 2, 3\}$, $j \neq k$. Then for the left ideal $L = \{0, e_{11}, e_{21}, e_{31}\}$ we have $L = r^*(L) \subset N(L) = S_1 - \{e_{22}, e_{33}\} \subset c(L) = S_1$.

Example 2. Let S_2 be the semigroup generated by the set $\{0, a_1, a_2, a_3, \ldots\}$ subject to the generating relations $0 \cdot x = x \cdot 0 = x^2 = 0$ for any $x \in S_2$. Then we have $0 = M(0) \subset R^*(0) = S_2$ (see [1, p. 232]). By the preceding Lemma 5 we obtain $m(0) = 0 \subset r^*(0) = S_2$.

Example 3. Let $S_3 = \{0, e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}\}$ be the subsemigroup of the semigroup S_1 of Example 1. Then for the left ideal $L = \{0, e_{12}, e_{22}\}$ of S_3 we have $L \subset r(L) = S - \{e_{11}, e_{33}\}$.

The author does not know an example of a semigroup S with a left ideal L such that $r(L) \subset m(L)$.

Lemma 6. *Let S be a semigroup with a left ideal L. Then $r(L) = r(L^2)$ holds.*

Proof. From $L^2 \subseteq L$ we have $r(L^2) \subseteq r(L)$. Conversely, we prove that $r(L) \subseteq r(L^2)$. Let $x \in r(L)$. Then $L(x)^n \subseteq L$ for some positive integer n. From this we obtain that $L(x)^{2n} \subseteq L^2$, therefore $L(x) \subseteq r(L^2)$ and $x \in r(L^2)$.

We recall that an ideal L is idempotent if $L^2 = L$.

Theorem 3. *Let S be a semigroup. Then the following conditions are equivalent:*

(1) *Each principal left ideal of S is idempotent.*

(2) *Each left ideal of S is idempotent.*

(3) *For every left ideal L of S, $L = r(L)$ holds.*

Proof. We prove that (1) implies (2). Let L be a left ideal of S. If $L = \{a_i, i \in I\}$, then $L = \bigcup_{i \in I} L(a_i)$. Then we have $L = \bigcup_{i \in I} L(a_i) = \bigcup_{i \in I} L(a_i)^2 \subseteq$
$$\subseteq \left[\bigcup_{i \in I} L(a_i)\right]^2 = L^2 \subseteq L, \text{ thus } L^2 = L.$$

We prove that (2) implies (3). Let L be a left ideal of S. By Lemma 1, $L \subseteq r(L)$, we show that $r(L) \subseteq L$. Let $a \in r(L)$. Then a belongs to some left ideal A having the property $A^n \subseteq L$ for some positive integer n. Then $a \in A = A^n \subseteq L$, since by the assumption A is idempotent. Hence $r(L) \subseteq L$ and thus $r(L) = L$.

We prove that (3) implies (1). Let $L(a)$ be a principal left ideal of S. Then the assumption and Lemma 6 imply that $L(a) = r(L(a)) = r(L(a)^2) = L(a)^2$. Therefore every principal left ideal of S is idempotent.

Remark. We note that a semigroup S is semisimple if and only if each two-sided ideal of S is idempotent (see e.g. [3; §2.6; Exercise 7(a)]). Hence a semigroup having the property (1), (2) or (3) of Theorem 3 belongs to the class of all semisimple semigroups.

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Received July 17, 86

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РАДИКАЛЫ И ИХ ЛЕВОИДЕАЛЬНЫЕ АНАЛОГИ В ПОЛУГРУППЕ

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Резюме

Сначала доказано, что R^*NC — полугруппа является полуструктурой архимедовых полугрупп и наоборот.

Кроме того в статье доказано, что в полугруппе для произвольного левого идеала L имеем: $L \subseteq r(L) \subseteq m(L) \subseteq r^*(L) \subseteq N(L) \subseteq c(L)$, где $r(L)$, $m(L)$, $r^*(L)$, $c(L)$ — левые идеалы, определеные аналогично радикалам Шварца, Маккойа, Клиффорда, Луга и $N(L)$ — множество всех нильпотентных элементов полугруппы относительно L.