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CONGRUENCE RELATIONS ON AND VARIETIES OF DIRECTED MULTILATTICES

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Foundations of the theory of multilattices were laid in the fifties by Benado (cf. [1]—[4]). The research was carried on by other authors (cf. e.g. [11]—[14], [16], [17], [23], [24]; [16] and [17] concern multilattice groups).

The notion of the variety of multialgebras was introduced by Schweigert in [22]. In this paper there are investigated some questions concerning varieties of directed multilattices. We prove that both the class of all modular directed multilattices and the class of all distributive directed multilattices are varieties (section 5). In contrast with the situation in the lattice of varieties of lattices we show that there are infinitely many varieties of distributive directed multilattices and infinitely many of them cover the variety of all distributive lattices (section 6). The results of sections 5 and 6 are obtained by applying those of sections 1—4 concerning congruence relations on directed multilattices. The notion of the congruence relation dealt with here is in accordance with the corresponding notions introduced by Pickett (see [18]) for multialgebras and by Schweigert for relational systems (see [22]). As to the notions of the regularity considered by Dubreil-Jacotin and Croisot in [6], every congruence relation on a directed multilattice M is a strongly regular and hence normally regular and regular equivalence on $(M, \leqslant)$, but a strongly regular equivalence on $(M, \leqslant)$ need not be a congruence relation on M.

0. Introduction

Let $(M, \leqslant)$ be a partially ordered set, $a, b \in M$. Denote by $u(a,b)$ and $l(a,b)$ the set of all upper and lower bounds of the set $\{a, b\}$ in M, respectively. Further let $a \vee b$ denote the set of all minimal elements of the set $u(a,b)$, $a \wedge b$ the set of all maximal elements of the set $l(a,b)$. If $a \vee b = \{h\}$, then we shall write $a \vee b = h$. The meaning of $a \wedge b = d$ will be analogous. If $A, B \subseteq M$, we define $A \vee B = \cup \{a \vee b \colon a \in A, b \in B\}$ and $A \wedge B$ is defined dually. For $h \in u(a,b)$ define $(a \vee b)_h = \{v \in a \vee b \colon v \leqslant h\}$ and for $d \in l(a,b)$ let $(a \wedge b)_d = \{w \in a \wedge b \colon w \geqslant d\}$.

0.1. Definition. *A partially ordered set $(M, \leqslant)$ is said to be a multilattice if the sets $(a \vee b)_h$, $(a \wedge b)_d$ are nonempty for all $a, b \in M$, $h \in u(a, b)$, $d \in l(a, b)$. If, moreover, $(M, \leqslant)$ is a directed set, i.e. the sets $u(a, b)$, $l(a, b)$ are nonempty for all $a, b \in M$, then $(M, \leqslant)$ is called a directed multilattice.*

A multilattice M is a lattice if and only if $\operatorname{card}(a \vee b) = \operatorname{card}(a \wedge b) = 1$ for all $a, b \in M$.

If M is a directed multilattice, then we can assign to every couple of elements $a, b \in M$ nonempty sets $a \wedge b$, $a \vee b$. Hence $(M, \{\wedge, \vee\})$ is a multialgebra with two binary multioperations.

1. Congruence relations

Let Θ be a binary relation on a set X, $A \subseteq X$, $B \subseteq X$. By $A\Theta B$ we mean that (1) for each $a \in A$ there exists $b \in B$ such that $a\Theta b$ and (2) for each $b \in B$ there exists $a \in A$ such that $a\Theta b$. If, e.g., $A = \{a\}$, we shall write $a\Theta B$ instead of $A\Theta B$.

1.1. Definition. *Let Θ be a binary relation on a directed multilattice M. Then Θ is called a congruence relation on M provided that:*

(i) *Θ is an equivalence relation on M,*

(ii) *for all $a, a', b, b' \in M$ the relations $a\Theta a'$, $b\Theta b'$ imply $a \vee b\Theta a' \vee b'$ and $a \wedge b\Theta a' \wedge b'$.*

Evidently the condition (ii) can be replaced by

(ii′) *for all $a, b, c \in M$ the relation $a\Theta b$ implies $a \vee c\Theta b \vee c$, $a \wedge c\Theta b \wedge c$.*

This definition is in accordance with that of an ideal (or ideal congruence relation) for a multialgebra (cf. [18], p. 329) and with the definition of a congruence relation on a relational system (cf. [22], Definition 1.1). It generalizes the definition of a congruence relation on a lattice.

If Θ is a congruence relation on a directed multilattice M and $a \in M$, then define $[a]\Theta = \{b \in M: b\Theta a\}$. It is easy to see that $[a]\Theta$ is a convex subset of M.

1.2. Lemma. *Let Θ be a congruence relation on a directed multilattice M and let $a, b \in M$. The following conditions are equivalent:*

(i) *$a\Theta b$;*

(ii) *$u\Theta v$ for all $u \in a \wedge b$, $v \in a \vee b$;*

(iii) *$a \wedge b\Theta a \vee b$;*

(iv) *$u\Theta v$ for some $u \in a \wedge b$, $v \in a \vee b$.*

Proof. Let $a\Theta b$. Then $a \wedge a\Theta a \wedge b$ and also $a \vee a\Theta a \vee b$. Since $a \wedge a = a \vee a = a$, we get (ii). The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are evident. Finally (iv) implies (i) by convexity of $[u]\Theta$.

Remark. If Θ is a congruence relation on a directed multilattice M, then the relation $a\Theta b$ implies $a \vee c\Theta b \vee c$ $(a \wedge c\Theta b \wedge c)$ for any $c \in M$, but it need not be $u\Theta v$ for each $u \in a \vee c, v \in b \vee c$ $(u \in a \wedge c, v \in b \wedge c)$. To show this, we can take M in Fig. 1 and Θ whose equivalence classes are $\{a, b\}$, $\{x, y\}$, $\{r, s\}$, $\{c\}$, $\{o\}$, $\{i\}$.

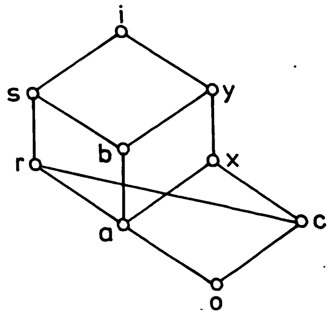

Fig. 1

Let M be a directed multilattice, Θ a reflexive binary relation on M. Consider the following conditions for Θ:

(i_1) *for all* $x, y \in M$, $x\Theta y$ *if and only if* $u\Theta v$ *for some* $u \in x \wedge y$, $v \in x \vee y$;

(ii_1) *for all* $x, y z \in M$, *if* $x \leqslant y \leqslant z$, $x\Theta y$, $y\Theta z$, *then* $x\Theta z$;

(iii_1) *for all* $x, y, t \in M$, *if* $x \leqslant y$, $x\Theta y$, *then* $x \vee t\Theta y \vee t$, $x \wedge t\Theta y \wedge t$.

Our aim is to prove that these conditions are necessary and sufficient for Θ to be a congruence relation.

1.3. Lemma. *Let* Θ *fulfil* (i_1)—(iii_1) *and let* a, b, c, d *be elements of* M *such that* $a \leqslant b \leqslant d$, $a \leqslant c \leqslant d$, $a\Theta d$. Then $b\Theta c$ *holds.*

Proof. Choose $u \in (b \wedge c)_a$, $v \in (b \vee c)_d$. The relation $a\Theta d$ gives $u\Theta d$ by (iii_1). Using (iii_1) once more we get $u\Theta v$. By (i_1) we have $b\Theta c$.

1.4. Lemma. *Let* Θ *fulfil* (i_1)—(iii_1) *and let* a, b *be elements of* M *satisfying* $a\Theta b$. *Then* $t \in a \wedge b$ *implies* $t\Theta a$, $t\Theta b$ *and* $r \in a \vee b$ *implies* $a\Theta r$, $b\Theta r$.

Proof. Let $t \in a \wedge b$. The relation $a\Theta b$ implies $t_1\Theta r_1$ for some $t_1 \in a \wedge b$, $r_1 \in a \vee b$, by (i_1). Then, using 1.3, we obtain $a\Theta r_1$. From this it follows that $t\Theta b$ by (iii_1). Analogously we can prove $t\Theta a$ and the second part of the statement.

1.5. Lemma. *If* Θ *is a reflexive relation satisfying* (i_1)—(iii_1), *then* Θ *is an equivalence relation.*

Proof. The symmetry of Θ is evident from (i_1). We will prove the transitivity of Θ. Let $x\Theta y$, $y\Theta z$. Choose $u \in x \vee y$, $v \in y \vee z$, $w \in u \vee v$, $r \in x \wedge y$, $s \in y \wedge z$, $t \in r \wedge s$. Then 1.4 gives $r\Theta y$ and using (iii_1) we obtain $t\Theta s$. Analogously $y\Theta u$ implies $v\Theta w$. Using again 1.4 and (ii_1) we get $t\Theta w$. Then by 1.3 we have $x\Theta z$.

1.6. Lemma. *Let Θ be a reflexive relation satisfying* (i_1)—(iii_1). *Then for all $a, b, c \in M$ the relation $a\Theta b$ implies $a \vee c\Theta b \vee c$, $a \wedge c\Theta b \wedge c$.*

Proof. We will prove, e.g., that $a\Theta b$ implies $a \vee c\Theta b \vee c$. Let $a\Theta b$ hold. Choose $t \in a \vee c$. By (i_1) there exist $i \in a \wedge b, j \in a \vee b$ satisfying $i\Theta j$. In view of (iii_1) we have $i \vee c\Theta j \vee c$. Since t is an upper bound of $\{i, c\}$, there exists $m \in (i \vee c)_t$ and we can choose $n \in j \vee c$ such that $m\Theta n$. Now let $r \in (b \vee c)_n$. We will show that $t\Theta r$. Since $i\Theta a$ by 1.4, using (iii_1) we get $m\Theta t$. Analogously $b\Theta j$ implies $r\Theta n$. Finally $m\Theta t$, $m\Theta n$ and $r\Theta n$ give $t\Theta r$ by 1.5. The proof of the fact that for every $r \in b \vee c$ there exists $t \in a \vee c$ satisfying $t\Theta r$ would be analogous.

The following theorem generalizes the result of G. Grätzer and E. T. Schmidt (cf. [9]) and F. Maeda (cf. [15]) for lattices.

1.7. Theorem. *Let M be a directed multilattice, Θ a reflexive binary relation on M. Then Θ is a congruence relation if and only if it fulfils the conditions* (i_1), (ii_1) *and* (iii_1).

Proof. Let Θ be a congruence relation on M. The validity of (ii_1), (iii_1) is evident, while (i_1) follows from 1.2. If Θ fulfils (i_1)—(iii_1), then it is a congruence relation by 1.5 and 1.6.

2. The lattice of all congruence relations

Denote by $\mathrm{Con}\, M$ the system of all congruence relations on a directed multilattice M. Applying a known result for relational systems (cf. [22], Theorem 1.4) we obtain that $\mathrm{Con}\, M$ is a complete lattice. The join is the same as in the lattice of all equivalence relations. The intersection of two congruence relations on a relational system need not be a congruence relation in general. But we will show that the intersection of an arbitrary nonempty system of congruence relations on a directed multilattice is a congruence relation, exactly as in the case of lattices. The proof is based on the following two lemmas.

2.1. Lemma. *Let $\Theta \in \mathrm{Con}\, M$, $a, b, c \in M$, $a \leqslant b$, $a\Theta b$. Then*
(i) *every couple of the elements $v \in b \vee c$, $u \in (a \vee c)_v$ fulfils $u\Theta v$;*
(ii) *every couple of the elements $r \in a \wedge c$, $s \in (b \wedge c)_r$ fulfils $r\Theta s$.*
The proof is evident.

2.2. Lemma. *Let $\Theta \in \mathrm{Con}\, M$, $a, b, c \in M$, $a \leqslant b$, $a\Theta b$.*
(i) *If $u \in a \vee c$, $w \in b \vee u$, $v \in (b \vee c)_w$, then $u\Theta v$;*
(ii) *If $s \in b \wedge c$, $t \in a \wedge s$, $r \in (a \wedge c)_t$, then $r\Theta s$.*

Proof. We will prove (i), the proof of (ii) would be dual. Choose $u' \in (a \vee c)_v$, $p \in (u \wedge u')_a$, $q \in (u \wedge u')_c$. The relation $a\Theta b$ implies $u\Theta w$, which gives $u \wedge u'\Theta u'$. Hence $p\Theta q$ and we infer $p \vee q\Theta q$. Since $u, u' \in p \vee q$, we obtain $u\Theta u'$. Finally 2.1 gives $u'\Theta v$ and this together with $u\Theta u'$ yields $u\Theta v$.

Using 1.7, 2.1 and 2.2 we obtain the following theorem.

2.3. Theorem. If $\emptyset \neq \{\Theta_i : i \in I\} \subseteq \operatorname{Con} M$, *then* $\cap \{\Theta_i : i \in I\} \in \operatorname{Con} M$.

We will prove two lemmas to obtain a simple description of the join in $\operatorname{Con} M$.

2.4. Lemma. *Let* $\{\Theta_1, \ldots, \Theta_n\} \subseteq \operatorname{Con} M$, $\{z_0, z_1, \ldots, z_n\} \subseteq M$ *and let* $z_0\Theta_1 z_1 \Theta_2 z_2 \ldots z_{n-1}\Theta_n z_n$. *Then there exist* $u_0, u_1, \ldots, u_n \in M$ *such that* $u_0 = z_0 \leqslant u_1 \leqslant \ldots \leqslant u_n$, $u_0\Theta_1 u_1 \Theta_2 u_2 \ldots u_{n-1}\Theta_n u_n$ *and* $u_j \geqslant z_j$ *for each* $j \in \{0, \ldots, n\}$.

Proof. It suffices to set $u_0 = z_0$ and to choose $u_1 \in u_0 \vee z_1, \ldots, u_n \in u_{n-1} \vee z_n$.

2.5. Lemma. *Let* $\{\Theta_1, \ldots, \Theta_n\} \subseteq \operatorname{Con} M$, $u_0 \leqslant u_1 \leqslant \ldots \leqslant u_n$ *be a chain of elements of M such that* $u_0\Theta_1 u_1 \Theta_2 u_2 \ldots u_{n-1}\Theta_n u_n$. *Further let* $u_0 \leqslant v \leqslant u_n$. *Then there exists a chain* $v_0 = u_0 \leqslant v_1 \leqslant \ldots \leqslant v_n = v$ *such that* $v_0\Theta_1 v_1 \Theta_2 v_2 \ldots v_{n-1}\Theta_n v_n$ *and* $u_j \geqslant v_j$ *for each* $j \in \{0, \ldots, n\}$.

Proof. Define $v_0 = u_0$, $v_1 \in (u_1 \wedge v)_{v_0}$, $v_2 \in (u_2 \wedge v)_{v_1}, \ldots, v_n \in (u_n \wedge v)_{v_{n-1}} = \{v\}$. Then the statement holds true by 2.1.

Using 2.4, 2.5 and their duals we obtain the following theorem.

2.6. Theorem. *Let Θ be the join of a nonempty subsystem* $\{\Theta_i : i \in I\}$ *of* $\operatorname{Con} M$. *Then for* $a, b \in M$ $a\Theta b$ *holds true if and only if there exists a chain* $z_0 \leqslant z_1 \leqslant \ldots \leqslant z_n$ *in M and congruence relations* $\Theta_{i_1}, \ldots, \Theta_{i_n} \in \{\Theta_i : i \in I\}$ *such that* $z_0 \in a \wedge b$, $z_n \in a \vee b$, $z_0\Theta_{i_1} z_1 \Theta_{i_2} z_2 \ldots z_{n-1}\Theta_{i_n} z_n$.

Now using 2.3, 2.6 and 1.2 we obtain (in the same way as in the case of lattices) that for any directed multilattice M the lattice $\operatorname{Con} M$ is algebraic and satisfies the infinite distributive law

$$\Theta \wedge (\vee \{\Phi_i : i \in I\}) = \vee \{\Theta \wedge \Phi_i : i \in I\}.$$

We can ask whether the congruence lattice of any directed multilattice is isomorphic to the congruence lattice of a lattice. This question is open. However, since several sufficient conditions for an algebraic distributive lattice to be a congruence lattice of a lattice are known (see e.g. [7], [8], [10], [19], [20], [21]), in some special cases the answer is affirmative. E.g. to any directed multilattice M with a finite congruence lattice there exists a lattice L with $\operatorname{Con} L$ isomorphic to $\operatorname{Con} M$.

3. Congruence relations generated by sets of quotients

Results of this section have a rather technical character; they will be applied by investigating varieties of directed multilattices. We will describe the least congruence relation on a directed multilattice M that collapses a given set of quotients. (For the case of lattices cf. [5], 10.2, p. 77).

If $a \geqslant b$ in M, then the subset $\{x \in M: a \geqslant x \geqslant b\}$ is called a *quotient* and is denoted by a/b. A quotient c/d is a *subquotient* of a/b if $a \geqslant c \geqslant d \geqslant b$. Given two quotients a/b and c/d, we say that a/b is an *upper transpose* of c/d and $c\ d$ is a *lower transpose* of a/b if $d \in b \wedge c$, $a \in b \vee c$. A quotient a/b will be called a *transpose* of c/d if a/b is either an upper or a lower transpose of c/d. Two quotients a/b and c/d are *projective* if there is a finite sequence of quotients $a/b = x_0/y_0, x_1/y_1, \dots, x_n/y_n = c/d$ such that each x_{i-1}/y_{i-1} is a transpose of x_i/y_i. More generally, a/b is said to be *weakly projective* into c/d in n steps if there is a finite sequence of quotients $a/b = x_0/y_0, x_1/y_1, \dots, x_n/y_n = c/d$ such that each x_{i-1}/y_{i-1} is a transpose of a subquotient of x_i/y_i.

3.1. Lemma. *Let $e_0 \geqslant e_1 \geqslant \dots \geqslant e_n$ be a chain in M and let a quotient r/s be a transpose of e_0/e_n. Then there exists a chain $r_0 = r \geqslant r_1 \geqslant \dots \geqslant r_n = s$ such that for each $i \in \{1, \dots, n\}$ the quotient r_{i-1}/r_i is weakly projective into e_{i-1}/e_i.*

Proof. Let us suppose, e.g., that r/s is a lower transpose of e_0/e_n. It suffices to set $r_0 = r$, $r_1 \in (r_0 \wedge e_1)_s$, $r_2 \in (r_1 \wedge e_2)_s, \dots, r_{n-1} \in (r_{n-2} \wedge e_{n-1})_s$, $r_n \in (r_{n-1} \wedge e_n)_s = \{s\}$.

3.2. Lemma. *Let $e_0 \geqslant e_1 \geqslant \dots \geqslant e_n$ be a chain in M and let $e_0 \geqslant t \geqslant e_n$. Then there exist chains $f_0 = t \geqslant f_1 \geqslant \dots \geqslant f_n = e_n$, $g_0 = e_0 \geqslant g_1 \geqslant \dots \geqslant g_n = t$ such that for each $i \in \{1, \dots, n\}$ the quotients f_{i-1}/f_i* and *g_{i-1}/g_i are weakly projective into e_{i-1}/e_i.*

Proof. We can set $f_0 = t$, $f_1 \in (f_0 \wedge e_1)_{e_n}$, $f_2 \in (f_1 \wedge e_2)_{e_n}, \dots, f_n \in (f_{n-1} \wedge e_n)_{e_n} = \{e_n\}$, $g_n = t$, $g_{n-1} \in (g_n \vee e_{n-1})_{e_0}$, $g_{n-2} \in (g_{n-1} \vee e_{n-2})_{e_0}, \dots, g_0 \in (g_1 \vee e_0)_{e_0} = \{e_0\}$. Then the quotients f_{i-1}/f_i, g_{i-1}/g_i are such as we need.

Using 3.2 and the transitivity of the property "to be weakly projective into" we get:

3.3. Lemma. *Let $e_0 \geqslant e_1 \geqslant \dots \geqslant e_n$ be a chain in M and let u/v be a subquotient of e_0/e_n. Then there exists a chain $t_0 = u \geqslant t_1 \geqslant \dots \geqslant t_n = v$ such that for each $i \in \{1, \dots, n\}$ the quotient t_{i-1}/t_i is weakly projective into e_{i-1}/e_i.*

3.4. Theorem. *Let Q be a nonempty set of quotients of a directed multilattice M. Define the relation Θ on M by the rule: $a\Theta b$ if there exists a finite chain $e_0 \geqslant e_1 \geqslant \dots \geqslant e_n$ in M such that $e_0 \in a \vee b$, $e_n \in a \wedge b$ and for each $i \in \{1, \dots, n\}$ the quotient e_{i-1}/e_i is weakly projective into some quotient in Q. Then Θ is the least congruence relation that collapses the quotients in Q.*

Proof. First we will show by 1.7 that Θ is a congruence relation. To show that Θ is reflexive, take $c \in M$ and a quotient $a/b \in Q$. If $p \in c \vee a$, $q \in (c \vee b)_p$, then c/c is weakly projective into p/q and the last quotient is weakly projective into a/b. Hence c/c is weakly projective into a/b and we have $c\Theta c$. Further it suffices to verify the conditions (i_1)—(iii_1) of 1.7.

The validity of (i_1) and (ii_1) is evident. Let us prove (iii_1). Let $x, y, t \in M$, $x \leqslant y$, $x\Theta y$. We will prove that $x \vee t\Theta y \vee t$. The proof of $x \wedge t\Theta y \wedge t$ would be dual. By assumption there exists a chain $y = e_0 \geqslant e_1 \geqslant \ldots \geqslant e_m = x$ in M such that for each $i \in \{1, \ldots, m\}$ the quotient e_{i-1}/e_i is weakly projective into some quotient in Q.

Take $v \in y \vee t$ and $f_0 = v, f_1 \in (e_1 \vee t)_{f_0}, f_2 \in (e_2 \vee t)_{f_1}, \ldots, u = f_m \in (e_m \vee t)_{f_{m-1}}$. Then $v = f_0 \geqslant f_1 \geqslant \ldots \geqslant f_m = u$ and for each $i \in \{1, \ldots, m\}$ f_{i-1}/f_i is weakly projective into e_{i-1}/e_i. Hence $u\Theta v$, where $u \in x \vee t$.

Conversely, let $u \in x \vee t$. We will find $v \in y \vee t$ such that $u\Theta v$. Define the elements $u_m, u_{m-1}, \ldots, u_0$ as follows: $u_m = u \in e_m \vee t$, $u_{m-1} \in e_{m-1} \vee u_m$, $u_{m-2} \in e_{m-2} \vee u_{m-1}, \ldots, u_1 \in e_1 \vee u_2$, $u_0 \in e_0 \vee u_1 = y \vee u_1$. Take $v \in (y \vee t)_{u_0}$. Construct the elements $v_0 = v$, $v_1 \in (v_0 \wedge u_1)_t$, $v_2 \in (v_1 \wedge u_2)_t, \ldots, v_m \in (v_{m-1} \wedge u_m)_t$ (see Fig. 2). Evidently v_{i-1}/v_i is weakly projective into u_{i-1}/u_i and u_{i-1}/u_i is weakly projective into e_{i-1}/e_i for each $i \in \{1, \ldots, m\}$. Hence we have $v\Theta v_m$. Take $r \in (x \vee v_m)_v$, $s \in x \wedge v_m$. Then $r\Theta v_m$ by 3.3 and using 3.1 we obtain $x\Theta s$. Using again 3.1 we get $u\Theta v_m$, since $u \in x \vee v_m$. Now let $p \in (u \vee v)_{u_0}$, $q \in (u \wedge v)_{v_m}$. Then $p\Theta q$ by 3.3 and consequently $u\Theta v$. We have proved that Θ is a congruence relation.

Evidently Θ collapses the quotients in Q. Finally observe that if a congruence relation collapses a quotient a/b and x/y is weakly projective into a/b, then this congruence collapses also x/y. This fact together with 1.2 yields that $\Theta \leqslant \Phi$ whenever Φ is a congruence relation on M which collapses all the quotients in Q. The proof is complete.

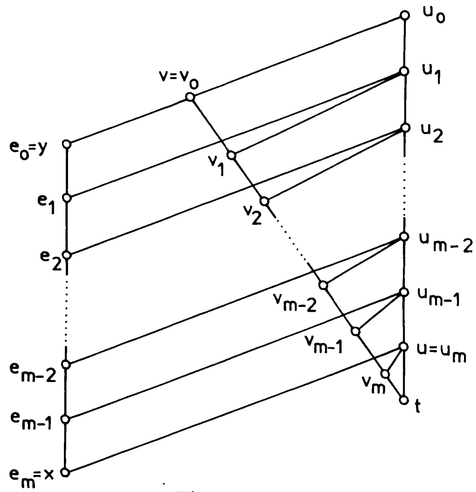

Fig. 2

Remark. Let $\{(a_i, b_i): i \in I\}$ be any nonempty subset of $M \times M$. Choose $v_i \in a_i \wedge b_i$, $u_i \in a_i \vee b_i$ for each $i \in I$. In view of 1.2 the least congruence relation that collapses all quotients u_i/v_i turns out to be the least congruence relation satisfying $a_i \Theta b_i$ for each $i \in I$.

4. Factor multilattices

In this section we assume that Θ is a fixed congruence relation on a directed multilattice M. For the denotation of the Θ-class to which an element $a \in M$ belongs, we shall use the symbol $[a]$ instead of the symbol $[a]\,\Theta$ introduced in the section 1.

4.1. Lemma. *Let $a, b \in M$. The following conditions are equivalent:*

(i) *there exist $a' \in [a]$, $b' \in [b]$ satisfying $a' \leqslant b'$;*

(ii) *for any $a' \in [a]$ there exists $b' \in [b]$ such that $a' \leqslant b'$;*

(iii) *for any $b' \in [b]$ there exists $a' \in [a]$ such that $a' \leqslant b'$.*

Proof. Let (i) hold. We prove (ii). Take $a'' \in [a]$. The condition $a' \Theta a''$ implies $b' \Theta a'' \vee b'$. Let b'' be an element of $a'' \vee b'$. Then $b'' \in [b]$ and evidently $a'' \leqslant b''$. The implication (ii) $\Rightarrow$ (i) is evident, hence (i) and (ii) are equivalent. Analogously it can be proved that (i) and (iii) are equivalent.

Define the relation $\leqslant$ in the set $M/\Theta = \{[a]: a \in M\}$ as follows: $[a] \leqslant [b]$ if and only if any of the conditions (i)—(iii) of 4.1 is fulfilled.

4.2. Theorem. *The relation $\leqslant$ defined above is a partial order and M/Θ with this relation is a directed multilattice.*

Proof. It is easy to see that $\leqslant$ is a partial order and that $(M/\Theta, \leqslant)$ is a directed set. Now let $[c] \geqslant [a], [b]$. Then there exist $a_1 \in [a]$, $b_1 \in [b]$ such that $c \geqslant a_1, b_1$. Let $d \in (a_1 \vee b_1)_c$. We prove that $[d]$ is a minimal upper bound of $[a]$ and $[b]$ under $[c]$. Evidently $[a], [b] \leqslant [d] \leqslant [c]$. Let $[a], [b] \leqslant [u] \leqslant [d]$. Then there exist $u_1, u_2, u' \in [u]$ satisfying $a_1 \leqslant u_1$, $b_1 \leqslant u_1$, $u' \leqslant d$. Let $u_1' \in (u_1 \wedge d)_{a_1}$, $u_2' \in (u_2 \wedge d)_{b_1}$. Then $u_1' \Theta u'$ and $u_2' \Theta u'$, hence $u_1' \in [u]$, $u_2' \in [u]$. Since $a_1 \leqslant u_1' \leqslant d$, $b_1 \leqslant u_2' \leqslant d$ and $d \in a_1 \vee b_1$, we have also $d \in u_1' \vee u_2'$. Consequently $[d] = [u]$.

4.3. Theorem. *In the directed multilattice M/Θ the following conditions are equivalent:*

(i) $[c] \in [a] \vee [b]$;

(ii) *there exists $c' \in [c]$ such that $c' \in a \vee b$;*

(iii) *there exist $c' \in [c]$, $a' \in [a]$, $b' \in [b]$ satisfying $c' \in a' \vee b'$.*

Proof. Let (i) hold. The conditions $[c] \geqslant [a]$, $[c] \geqslant [b]$ yield the existence of $u, v \in [c]$ with $a \leqslant u$, $b \leqslant v$. Let $d \in u \vee v$, $c' \in (a \vee b)_d$. Evidently $[a]$, $[b] \leqslant [c'] \leqslant [d] = [c]$ and we have $[c'] = [c]$. We have proved (ii). Obviously (ii) implies (iii). The implication (iii) $\Rightarrow$ (i) can be proved in the same way as in the proof of the preceding theorem.

The directed multilattice M/Θ of 4.2 is called the *factor multilattice* of M by Θ. Theorem 4.3 and its dual ensure that the factor multilattice M/Θ is a factor multialgebra of the multialgebra $(M, \{\wedge, \vee\})$ (cf. [22] and [18]).

The following definition can be considered a particular case of the definition of a relational homomorphism between two relational systems (cf. [22]) or an ideal homomorphism between two multialgebras (cf. [18]).

4.4. Definition. *Let* $(M_1, \{\wedge, \vee\})$, $(M_2, \{\wedge, \vee\})$ *be multialgebras with binary multioperations. A mapping* $\varphi\colon M_1 \to M_2$ *is called a homomorphism if* $\varphi(a \wedge b) = (\varphi(a) \wedge \varphi(b)) \cap \varphi(M_1)$ *and* $\varphi(a \vee b) = (\varphi(a) \vee \varphi(b)) \cap \varphi(M_1)$ *for each* $a, b \in M_1$. *A one-to-one homomorphism of* M_1 *onto* M_2 *is an isomorphism.*

We shall use the following two homomorphism theorems.

4.5. Theorem. *Let* M_1 *be a directed multilattice,* M_2 *a multialgebra with two binary multioperations and let* φ *be a homomorphism of* M_1 *onto* M_2. *Then* Ker φ *is a congruence relation on* M_1 *and the factor multilattice* $M_1/\mathrm{Ker}\,\varphi$ *is isomorphic to* M_2.

4.6. Theorem. *Let* M *be a directed multilattice,* Θ *a congruence relation on* M. *For* $\Phi \in \mathrm{Con}\, M$, $\Phi \geqslant \Theta$ *define a relation* Φ/Θ *on* M/Θ *as follows*:

$$[a]\,\Theta \;\; \Phi/\Theta \;\; [b]\,\Theta \Leftrightarrow a\Phi b.$$

Then Φ/Θ *is a congruence relation on* M/Θ *and the factor multilattice* $M/\Theta/\Phi/\Theta$ *is isomorphic to* M/Φ.

Theorem 4.5 is a particular case of Theorem 1 in [18].

5. Varieties

A class of multialgebras of the same type which is closed under the construction of submultialgebras, homomorphic images and direct products is called a *variety* (cf. [22], Definition 3.1). First we want to show that the class $\mathcal{M}$ of all directed multilattices is a variety. Applying the definition of a Birkhoff subalgebra (see [18]) to the case of directed multilattices we get the following definition of a subalgebra of a directed multilattice.

5.1. Definition. *A nonempty subset* K *of a directed multilattice* M *is a subalgebra of* M *if* $a \vee b \subseteq K$, $a \wedge b \subseteq K$ *whenever* $a, b \in K$.

A subalgebra of a directed multilattice is again a directed multilattice.

If $(M_i\colon i \in I)$ is a family of directed multilattices, then the Cartesian product of $(M_i\colon i \in I)$ ordered componentwise is evidently a directed multilattice. This multilattice will be denoted by $\Pi(M_i: i \in I)$. Evidently the construction of the direct product of multialgebras M_i gives the same result.

In view of the foregoing remarks and Theorem 4.5 we have:

5.2. Theorem. *The class $\mathcal{M}$ of all directed multilattices is a variety.*

Now we will show that the class of all modular directed multilattices is a variety as well as the class of all distributive directed multilattices.

5.3. Definition (cf. [1] 4.41, 6.1). *A multilattice M is said to be modular (distributive) if the conditions $v \in a \vee b$, $v \in a \vee b'$, $u \in a \wedge b$, $u \in a \wedge b'$, $b \geqslant b'$ ($v \in a \vee b$, $v \in a \vee b'$, $u \in a \wedge b$, $u \in a \wedge b'$) imply $b = b'$.*

5.4. Theorem. *The class of all modular directed multilattices is a variety.*

Proof. It is easy to see that a subalgebra of a modular directed multilattice and also the direct product of modular directed multilattices are modular multilattices. It remains to show that M/Θ is modular whenever M is a modular directed multilattice and $\Theta \in \operatorname{Con} M$. Let $[v] \geqslant [a] \geqslant [u]$, $[v] \geqslant [b] \geqslant [b'] \geqslant [u]$, $[v] \in [a] \vee [b]$, $[v] \in [a] \vee [b']$, $[u] \in [a] \wedge [b]$, $[u] \in [a] \wedge [b']$. ($[a], [b], \ldots$ mean Θ-classes containing $a, b, \ldots$). We can suppose that $b' \leqslant b$, $v \in a \vee b$, $u \in a \wedge b'$. Let $u_1 \in (a \wedge b)_u$. Then $[u] \leqslant [u_1] \leqslant [a], [b]$ and since $[u] \in [a] \wedge [b]$, we have $[u] = [u_1]$. Take any $b'_1 \in (u_1 \vee b')_b$. The relation $u\Theta u_1$ implies $b'\Theta b'_1$. Now let $v_1 \in (a \vee b'_1)_v$. Again $[v] = [v_1]$ because of $[a]$, $[b'_1] \leqslant [v_1] \leqslant [v]$, $[v] \in [a] \vee [b'_1]$. Choose any $b_1 \in (v_1 \wedge b)_{b'_1}$. Then $b\Theta b_1$. We have $v_1 \geqslant a \geqslant u_1$, $v_1 \geqslant b_1 \geqslant b'_1 \geqslant u_1$, $v_1 \in a \vee b_1$, $v_1 \in a \vee b'_1$, $u_1 \in a \wedge b_1$, $u_1 \in a \wedge b'_1$, hence $b_1 = b'_1$. Then $[b] = [b_1] = [b'_1] = [b']$ and the proof is complete.

5.5. Theorem. *The class of all distributive directed multilattices is a variety.*

Proof. It is sufficient to prove that the class of all distributive directed multilattices is closed under the construction of factor multilattices. Let M be a distributive directed multilattice, $\Theta \in \operatorname{Con} M$. Assume that $[v] \geqslant [a]$, $[b]$, $[b'] \geqslant [u]$, $[v] \in [a] \vee [b]$, $[v] \in [a] \vee [b']$, $[u] \in [a] \wedge [b]$, $[u] \in [a] \wedge [b']$. There exist $v_1, v_2 \in [v]$, $u_1, u_2 \in [u]$ such that $v_1 \in a \vee b$, $v_2 \in a \vee b'$, $u_1 \in a \wedge b$, $u_2 \in a \wedge b'$. Take $r \in (v_1 \wedge v_2)_a$, $s \in (u_1 \vee u_2)_a$. Evidently $[r] = [v]$, $[s] = [u]$. Now let $b_1 \in (b \wedge r)_{u_1}$, $b'_1 \in (b' \wedge r)_{u_2}$. Then $u_1 \in b_1 \wedge a$, $u_2 \in b'_1 \wedge a$. We will prove that $r \in a \vee b_1$. Let $t \in (a \vee b_1)_r$. Then $v_1 \geqslant b \geqslant b_1$, $v_1 \geqslant r \geqslant t \geqslant b_1$, $v_1 \in b \vee r$, $v_1 \in b \vee t$, $b_1 \in b \wedge r$, $b_1 \in b \wedge t$, therefore $r = t$. Analogously $r \in a \vee b'_1$. Further choose $b_2 \in (b_1 \vee s)_r$, $b'_2 \in (b'_1 \vee s)_r$. To prove $s \in a \wedge b_2$ let us suppose that $z \in (a \wedge b_2)_s$. Then $b_2 \geqslant b_1 \geqslant u_1$, $b_2 \geqslant z \geqslant s \geqslant u_1$, $b_2 \in b_1 \vee z$, $b_2 \in b_1 \vee s$, $u_1 \in b_1 \wedge z$, $u_1 \in b_1 \wedge s$, hence $z = s$. Analogously $s \in a \wedge b'_2$. Now we have $r \geqslant a$, b_2, $b'_2 \geqslant s$, $r \in a \vee b_2$, $r \in a \vee b'_2$, $s \in a \wedge b_2$, $s \in a \wedge b'_2$, which implies $b_2 = b'_2$. The relation $u_1 \Theta s$ implies $b_1 \Theta b_2$ and the relation $v_1 \Theta r$ gives $b \Theta b_1$. Hence $[b] = [b_2]$. Analogously it can be proved that $[b'] = [b'_2]$. It follows that $[b] = [b_2] = [b'_2] = [b']$ and this completes the proof.

6. The lattice of varieties

The family of all varieties of directed multilattices ordered by inclusion has the least element — the variety of all one-element lattices and the greatest element — the variety of all directed multilattices. As the intersection of a family of varieties is again a variety, we shall speak of the lattice of varieties of directed multilattices (leaving out the fact that elements of this lattice are classes). Obviously every variety of directed multilattices different from the least one contains the variety $\mathscr{D}$ of all distributive lattices. If $\mathscr{K}$ is a variety of distributive directed multilattices and $\mathscr{V}$ a variety of lattices, both different from the variety of all one-element lattices, then $\mathscr{K} \cap \mathscr{V} = \mathscr{D}$. We will show that there exist infinitely many varieties of distributive directed multilattices and infinitely many of them cover $\mathscr{D}$.

Given a class $\mathscr{K}$ of directed multilattices, let $P\mathscr{K}$, $S\mathscr{K}$, and $H\mathscr{K}$ denote, respectively, the clase of all those directed multilattices that are isomorphic with direct products of multilattices in $\mathscr{K}$, the class of all directed multilattices isomorphic with subalgebras of multilattices in $\mathscr{K}$, and the class of all homomorphic images of multilattices in $\mathscr{K}$. The fact that a class $\mathscr{K}$ of directed multilattices is a variety can be then expressed by requiring that $\mathscr{K} = P\mathscr{K} = S\mathscr{K} = H\mathscr{K}$. In what follows we shall apply the following theorem, which can be proved essentially in the same way as for algebras (for the proof for lattices see [7]). It can be generalized for arbitrary multialgebras.

6.1. Theorem. *Let $\mathscr{K}$ be a nonempty class of directed multilattices. Then $HSP\mathscr{K}$ is the smallest variety containing $\mathscr{K}$.*

Given a set I and a filter $\mathscr{F}$ on I, i.e. a nonempty set of subsets of I satisfying (1) $A \cap B \in \mathscr{F}$ whenever $A, B \in \mathscr{F}$, and (2) $A \in \mathscr{F}$ whenever $A \supseteq B$, $B \in \mathscr{F}$, we can define a binary relation $\Theta(\mathscr{F})$ on the direct product $\Pi(M_i\colon i \in I)$ of directed multilattices M_i by the rule:

$$f\,\Theta(\mathscr{F})\,g \text{ if and only if } I(f,g) = \{i \in I\colon f(i) = g(i)\} \in \mathscr{F}.$$

It can be proved easily that $\Theta(\mathscr{F})$ is a congruence relation on $\Pi(M_i\colon i \in I)$ and we can construct the factor multilattice $\Pi(M_i\colon i \in I)/\Theta(\mathscr{F})$. This factor multilattice will be called a *filter product* of $(M_i\colon i \in I)$, more specifically the filter product of $(M_i\colon i \in I)$ by $\mathscr{F}$, and it will be denoted by $\Pi_{\mathscr{F}}(M_i\colon i \in I)$. By an *ultraproduct* of $(M_i\colon i \in I)$ a filter product of $(M_i\colon i \in I)$ by an ultrafilter is meant.

We will use the following two lemmas, which are generalizations of analogous lemmas for lattices (cf. [7]). Their proofs are essentially the same as those for lattices.

6.2. Lemma. *Let I be a nonempty set, $\mathscr{F}$ a filter on I. If $\mathscr{F} = \{K \subseteq I\colon K \supseteq J\}$ for a fixed $J \subseteq I$, then for any family $(M_i\colon i \in I)$ of directed multilattices the filter* *product* $\Pi_{\mathscr{F}}(M_i: i \in I)$ *is isomorphic to* $\Pi(M_i: i \in J)$; *in particular if* $\mathscr{F} = \{K \subseteq I: j \in K\}$ *for a fixed* $j \in I$, *then* $\Pi_{\mathscr{F}}(M_i: i \in I)$ *is isomorphic to* M_j.

6.3. Lemma. *Let* $M_0, \ldots, M_{n-1}$ *be finite directed multilattices and let* $(M_i: i \in I)$ *be a nonempty family of directed multilattices such that for every* $i \in I M_i$ *is one of* $M_0, \ldots, M_{n-1}$. *If* $\mathscr{U}$ *is an ultrafilter on* I, *then* $\Pi_{\mathscr{U}}(M_i: i \in I)$ *is isomorphic to* M_j *for some* $j \in \{0, \ldots, n-1\}$.

The proof of the following lemma is clear.

6.4. Lemma. *If* M *is a directed multilattice which is not a lattice, then* M *has a four-element subset represented in Fig. 3.*

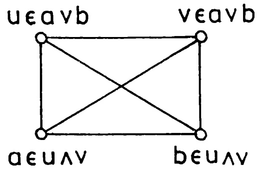

Fig. 3

6.5. Lemma. *Let* M_1, M_2 *be directed multilattices,* φ *a homomorphism of* M_1 *onto* M_2. *If* $a, b, u, v \in M_2$ *are as in Fig. 3, then there exist* $a', b', u', v' \in M_1$ *as in Fig. 3 such that* $\varphi(a') = a$, $\varphi(b') = b$, $\varphi(u') = u$, $\varphi(v') = v$.

Proof. Take any $a_1, b_1 \in M_1$ such that $\varphi(a_1) = a$, $\varphi(b_1) = b$. Since $u, v \in \in \varphi(a_1) \vee \varphi(b_1)$, there exist $u', v' \in M_1$ satisfying $\varphi(u') = u$, $\varphi(v') = v$, u', $v' \in a_1 \vee b_1$. Now let $a' \in (u' \wedge v')_{a_1}$, $b' \in (u' \wedge v')_{b_1}$. Then $\varphi(a')$, $\varphi(b') \in u \wedge v$ and since φ is order-preserving, there must be $\varphi(a') = a$, $\varphi(b') = b$. Evidently u', $v' \in a' \vee b'$.

Let M be a directed multilattice, U a subset of M. Define the sets $U^{(k)}$ for nonnegative integers k as follows: $U^{(0)} = U$; if $U^{(l)}$ is defined for some nonnegative integer l, we set $U^{(l+1)} = \cup\{x \vee y: x, y \in U^{(l)}\}$ for l even and $U^{(l+1)} = \cup\{x \wedge y: x, y \in U^{(l)}\}$ for l odd. It is easy to verify that $\cup\{U^{(k)}: k \geqslant 0\}$ is the smallest subalgebra of M containing U.

Given a nonempty class $\mathscr{L}$ of directed multilattices, denote by $S^-\mathscr{L}$ the class of all members of $S\mathscr{L}$ that are generated by a four-element subset represented in Fig. 3.

6.6. Lemma. *Let* $\mathscr{K}$ *be a class of directed multilattices. Suppose that there exists a positive integer* k *such that if* $A \in S^-\mathscr{K}$ *and* U *is a four-element subset of* A *shown in Fig. 3 generating* A, *then* $A = U^{(k)}$. *Then* $S^-P\mathscr{K} = PS^-\mathscr{K}$.

Proof. Let $A \in S^-P\mathscr{K}$. Then there exist $M_i \in \mathscr{K}$ $(i \in I)$ and four elements $u, v, a, b \in A$ as in Fig. 3 such that A is a subalgebra of $\Pi(M_i: i \in I)$ generated by

$U = \{u, v, a, b\}$. For $f \in \Pi(M_i\colon i \in I)$ let $f(i)$ denote the projection of f into M_i. Set $U_i = \{u(i), v(i), a(i), b(i)\}$ for each $i \in I$. Then U_i is either a four-element set as in Fig. 3 or $u(i) = v(i) = a(i) = b(i)$. Denote by N_i the subalgebra of M_i generated by U_i. Let $I_1 = \{i \in I\colon \text{card}\, U_i = 4\}$. If $i \notin I_1$, then N_i is evidently the one-element set u_i, while for $i \in I_1$ we have $N_i \in S^- \mathcal{K}$. We will show that the mapping $A \to \Pi(N_i\colon i \in I_1)$ assigning to $f \in A$ its restriction to I_1, denoted by f/I_1, is an isomorphism of A onto $\Pi(N_i\colon i \in I_1)$. This mapping is evidently a one-to-one homomorphism. It remains to show that it is onto. We have $\Pi(N_i\colon i \in I_1) = \Pi(U_i^{(k)}\colon i \in I_1) = \Pi(U_i^{(0)}\colon i \in I_1) \cup \Pi(U_i^{(1)}\colon i \in I_1) \cup \ldots \cup \Pi(U_i^{(k)}\colon i \in I_1)$. First we will show that every element of $\Pi(U_i^{(0)}\colon i \in I_1)$ has a pre-image in A. Let $f' \in \Pi(U_i^{(0)}\colon i \in I_1)$. Introduce the denotation $I_1^u = \{i \in I_1 : f'(i) = u(i)\}$, $I_1^v = \{i \in I_1 : f'(i) = v(i)\}$, $I_1^a = \{i \in I_1 : f'(i) = a(i)\}$, $I_1^b = \{i \in I_1 : f'(i) = b(i)\}$. Define $g', h' \in \Pi(N_i\colon i \in I_1)$ in such a way that $g'(i) = h'(i) = a(i)$ for every $i \in I_1^a$, $g'(i) = h'(i) = b(i)$ for every $i \in I_1^b$, $g'(i) = a(i)$ for every $i \in I_1^u \cup I_1^v$, $h'(i) = b(i)$ for every $i \in I_1^u \cup I_1^v$. Then $f' \in g' \vee h'$, $g', h' \in u/I_1 \wedge v/I_1$ and hence f' has a pre-image in A. Assume that l is a nonnegative integer less than k and every element of $\Pi(U_i^{(l)}\colon i \in I_1)$ has a pre-image in A. We will show that then every element of $\Pi(U_i^{(l+1)}\colon i \in I_1)$ has a pre-image in A. We can suppose, e.g., that l is even. Let $f' \in \Pi(U_i^{(l+1)}\colon i \in I_1)$. Then for each $i \in I_1$ it is $f'(i) \in x_i \vee y_i$ for some $x_i, y_i \in U_i^{(l)}$. Define $g', h' \in \Pi(U_i^{(l)}\colon i \in I_1)$ as follows: $g'(i) = x_i$, $h'(i) = y_i$ for every $i \in I_1$. Then $f' \in g' \vee h'$ and since by assumption g', h' have a pre-image in A, f' has a pre-image too. We have proved that $S^- P\mathcal{K} \subseteq PS^- \mathcal{K}$.
To prove $PS^- \mathcal{K} \subseteq S^- P\mathcal{K}$, take $\Pi(A_i\colon i \in I)$, where A_i is a subalgebra of a directed multilattice $M_i \in \mathcal{K}$ generated by a set $U_i = \{u_i, v_i, a_i, b_i\}$ as in Fig. 3. Define $u, v, a, b \in \Pi(A_i\colon i \in I)$ in such a way that for every $i \in I$ $u(i) = u_i$, $v(i) = v_i$, $a(i) = a_i$, $b(i) = b_i$. Then u, v, a, b are as in Fig. 3. Let A be the subalgebra of $\Pi(M_i\colon i \in I)$ generated by $\{u, v, a, b\}$. By the first part of the proof $A = \Pi(A_i\colon i \in I)$ and since evidently $\Pi(A_i\colon i \in I)$ is a subalgebra of $\Pi(M_i\colon i \in I)$, we have $\Pi(A_i\colon i \in I) \in S^- P\mathcal{K}$.

6.7. Lemma. *Let $(M_i\colon i \in I)$ be a family of directed multilattices such that the supremum of the lengths of $M_i (i \in I)$ is finite. Suppose that there exists a positive integer k such that if x/y is weakly projective into z/t in some M_i, then x/y is weakly projective into z/t in no more than k steps. If $a, b \in M = \Pi(M_i\colon i \in I)$, $a \geqslant b$, then $u\Theta(a, b)\, v$ if and only if $u(i)\, \Theta(a(i), b(i))\, v(i)$ for all $i \in I$.*

Proof. It is easy to see that if $u\Theta(a, b)\, v$, then $u(i)\, \Theta(a(i), b(i))\, v(i)$ for all $i \in I$. Now let $u(i)\, \Theta(a(i), b(i))\, v(i)$ for all $i \in I$. Then for every $i \in I$ there exists a finite chain $e_0^i \geqslant e_1^i \geqslant \ldots \geqslant e_{n_i}^i$ in M_i such that $e_0^i \in u(i) \vee v(i)$, $e_{n_i}^i \in u(i) \wedge v(i)$ and all e_{j-1}^i / e_j^i are weakly projective into $a(i)/b(i)$. Since the set $\{n_i\colon i \in I\}$ is bounded and one-element quotient is weakly projective into any quotient, we can suppose that $n_i = n$ for every $i \in I$. Further we can suppose that for each $i \in I$ and

$j \in \{1, \ldots, n\}$ there exists a sequence of quotients $e^i_{j-1}/e^i_j = x^{ij}_0/y^{ij}_0,\ x^{ij}_1/y^{ij}_1, \ldots$ $\ldots, x^{ij}_{l_{ij}}/y^{ij}_{l_{ij}} = a(i)/b(i)$ such that x^{ij}_0/y^{ij}_0 is a lower transpose of a subquotient of x^{ij}_1/y^{ij}_1, x^{ij}_1/y^{ij}_1 is an upper transpose of a subquotient of x^{ij}_2/y^{ij}_2 etc., because every quotient is both an upper and a lower transpose of itself. Finally we can suppose that $l_{ij} = l$ does not depend on i,j. Now define $e_0, e_1, \ldots, e_n \in M$ in such a way that $e_j(i) = e^i_j$ for every $i \in I, j \in \{0, \ldots, n\}$. Consider the sequence of quotients $e_{j-1}/e_j = x^j_0/y^j_0,\ x^j_1/y^j_1, \ldots, x^j_l/y^j_l = a/b$ such that $x^j_r(i) = x^{ij}_r,\ y^j_r(i) = y^{ij}_r$ for every $i \in I$, $j \in \{1, \ldots, n\}$, $r \in \{0, \ldots, l\}$. Then evidently x^j_0/y^j_0 is a lower transpose of a subquotient of x^j_1/y^j_1, x^j_1/y^j_1 is an upper transpose of a subquotient of x^j_2/y^j_2 etc. Hence all e_{j-1}/e_j are weakly projective into a/b. We have proved that $u\Theta(a, b)\,v$.

6.8. Lemma. *Let the assumptions of the preceding lemma be fulfilled and let* card Con $M_i = 2$ *for each* $i \in I$. *If* $a, b \in M$, $a \geqslant b$ *and* $\mathscr{F}$ *is the filter on* I *generated by* $I(a, b) = \{i \in I\colon a(i) = b(i)\}$, *then* $\Theta(a, b) = \Theta(\mathscr{F})$.

Proof. It is sufficient to prove that $u\Theta(a, b)v$ if and only if $I(u, v) \supseteq I(a, b)$. Let $u\Theta(a, b)\,v$. Then $u(i)\,\Theta(a(i), b(i))\,v(i)$ for each $i \in I$. Let $i \in I(a, b)$. Then $a(i) = b(i)$, which implies that $\Theta(a(i), b(i))$ is the least congruence relation on M_i. It follows that $u(i) = v(i)$, hence $i \in I(u, v)$. Conversely let $I(u, v) \supseteq I(a, b)$. Take $i \in I$. We will show that $u(i)\,\Theta(a(i), b(i))\,v(i)$. If $i \notin I(a, b)$, then $\Theta(a(i), b(i))$ is the greatest congruence relation on M_i and hence $u(i)\,\Theta(a(i), b(i))\,v(i)$ holds true. If $i \in I(a, b)$, then $i \in I(u, v)$ and again $u(i)\,\Theta(a(i), b(i))\,v(i)$.

Using 6.8 and 6.2 we get:

6.9. Lemma. *Under the assumptions and denotations as in the preceding lemma the factor multilattice* $\Pi(M_i\colon i \in I)/\Theta(a, b) = \Pi(M_i\colon i \in I)/\Theta(\mathscr{F})$ *is isomorphic to* $\Pi(M_i\colon i \in I(a, b))$.

6.10. Lemma. *Let the assumptions of* 6.7 *be fulfilled and let* card Con $M_i = 2$ *for each* $i \in I$. *For any congruence relation* Θ *on* $M = \Pi(M_i\colon i \in I)$ *there exists a filter* $\mathscr{F}$ *on* I *such that* $\Theta = \Theta(\mathscr{F})$.

Proof. Let $\Theta \in \operatorname{Con} M$. Then $\Theta = \sup\{\Theta(a_\lambda, b_\lambda)\colon \lambda \in \Lambda\}$, where $\{(a_\lambda, b_\lambda)\colon \lambda \in \Lambda\} = \{(a, b) \in M \times M\colon a \geqslant b,\ a\Theta b\}$. Let $\mathscr{F}$ be the filter generated by $\{I(a_\lambda, b_\lambda)\colon \lambda \in \Lambda\}$. To prove $\Theta = \Theta(\mathscr{F})$, we need to show that $u\Theta v$ is equivalent to $I(u, v) \in \mathscr{F}$. We can suppose that u, v are comparable elements, e.g. $u \geqslant v$. If $u\Theta v$, then $u = a_\lambda$, $v = b_\lambda$ for some $\lambda \in \Lambda$. Hence $I(u, v) = I(a_\lambda, b_\lambda) \in \mathscr{F}$. Let conversely $I(u, v) \in \mathscr{F}$. Then there exist $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $I(u, v) \supseteq I(a_{\lambda_1}, b_{\lambda_1}) \cap \ldots \cap I(a_{\lambda_r}, b_{\lambda_r})$. Define $u_0, u_1, \ldots, u_r \in M$ as follows:

$$u_0 = v;$$

$$u_1(i) = \begin{cases} u(i) & \text{if } i \notin I(a_{\lambda_1}, b_{\lambda_1}), \\ u_0(i) & \text{if } i \in I(a_{\lambda_1}, b_{\lambda_1}); \end{cases}$$

$$u_2(i) = \begin{cases} u(i) & \text{if } i \notin I(a_{\lambda_1}, b_{\lambda_1}) \cap I(a_{\lambda_2}, b_{\lambda_2}), \\ u_1(i) & \text{if } i \in I(a_{\lambda_1}, b_{\lambda_1}) \cap I(a_{\lambda_2}, b_{\lambda_2}); \end{cases}$$

$$\vdots$$

$$u_r(i) = \begin{cases} u(i) & \text{if } i \notin I(a_{\lambda_1}, b_{\lambda_1}) \cap \ldots \cap I(a_{\lambda_r}, b_{\lambda_r}), \\ u_{r-1}(i) & \text{if } i \in I(a_{\lambda_1}, b_{\lambda_1}) \cap \ldots \cap I(a_{\lambda_r}, b_{\lambda_r}). \end{cases}$$

Then $v = u_0 \Theta(a_{\lambda_1}, b_{\lambda_1}) u_1 \Theta(a_{\lambda_2}, b_{\lambda_2}) u_2 \ldots u_{r-1} \Theta(a_{\lambda_r}, b_{\lambda_r}) u_r = u$, hence $u \Theta v$.

6.11. Lemma. *Let $M_0, \ldots, M_{n-1}$ be finite directed multilattices with two-element lattices of congruence relations and let $(M_i\colon i \in I)$ be a nonempty family of directed multilattices such that for every $i \in I$ M_i is one of $M_0, \ldots, M_{n-1}$. If $A \in H\{\Pi(M_i\colon i \in I)\}$, then there is $j \in \{0, \ldots, n-1\}$ such that $M_j \in H\{A\}$.*

Proof. Denote $M = \Pi(M_i\colon i \in I)$. If $A \in H\{M\}$, then by 4.5 A is isomorphic to M/Θ for some $\Theta \in \operatorname{Con} M$. Since the assumptions of 6.10 are obviously fulfilled, there exists a filter $\mathscr{F}$ on I such that $\Theta = \Theta(\mathscr{F})$. Let $\mathscr{U}$ be an ultrafilter on I with $\mathscr{U} \supseteq \mathscr{F}$. Then $\Theta(\mathscr{U}) \supseteq \Theta(\mathscr{F})$ and using 4.6 we obtain that $\Pi_{\mathscr{U}}(M_i\colon i \in I) = M/\Theta(\mathscr{U})$ is isomorphic to $(M/\Theta(\mathscr{F}))/(\Theta(\mathscr{U})/\Theta(\mathscr{F}))$. By 6.3 $\Pi_{\mathscr{U}}(M_i\colon i \in I)$ is isomorphic to M_j for some $j \in \{0, \ldots, n-1\}$. We conclude that M_j is isomorphic to $A/(\Theta(\mathscr{U})/\Theta(\mathscr{F}))$, which means $M_j \in H\{A\}$.

6.12. Theorem. *Let $M_0, \ldots, M_n$ be finite directed multilattices pairwise nonisomorphic that are not lattices, have only trivial congruence relations and do not have proper subalgebras which are not lattices. Then the variety $\mathscr{V}_1$ generated by $\{M_0, \ldots, M_{n-1}\}$ is a proper subclass of the variety $\mathscr{V}_2$ generated by $\{M_0, \ldots, M_{n-1}, M_n\}$.*

Proof. It is obvious that $\mathscr{V}_1 \subseteq \mathscr{V}_2$. We will show that $M_n \in \mathscr{V}_2 - \mathscr{V}_1$. Assume that $M_n \in \mathscr{V}_1$. Then $M_n \in HSP\{M_0, \ldots, M_{n-1}\}$, i.e. M_n is a homomorphic image of a subalgebra A of a direct product $\Pi(M_i\colon i \in I)$, where for every $i \in I$ $M_i \in \{M_0, \ldots, M_{n-1}\}$. In view of 6.4 and 6.5 there are distinct elements u, v, a, $b \in A$ as in Fig. 3, whose images (under the given homomorphism φ of A onto M_n) are also distinct and as in Fig. 3. Let B be the subalgebra of $\Pi(M_i\colon i \in I)$ generated by $\{u, v, a, b\}$. By 6.6 B is isomorphic to $\Pi(M_j\colon j \in J)$, where $M_j \in \{M_0, \ldots, M_{n-1}\}$ for every $j \in J$. Since $\varphi(B)$ is a subalgebra of M_n that is not a lattice, we have $\varphi(B) = M_n$. Using the preceding lemma we obtain $M_l \in H\{M_n\}$ for some $l \in \{0, \ldots, n-1\}$, a contradiction.

In Fig. 4 and Fig. 5 there are shown multilattices fulfilling conditions of the preceding theorem. Multilattices in Fig. 4 are distributive, while those in Fig. 5 are modular but not distributive.

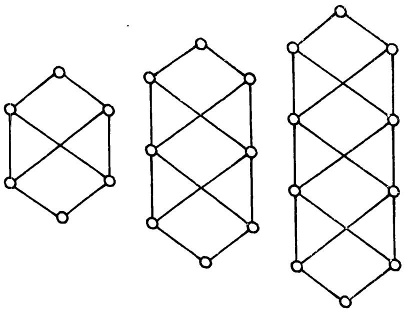

Fig. 4

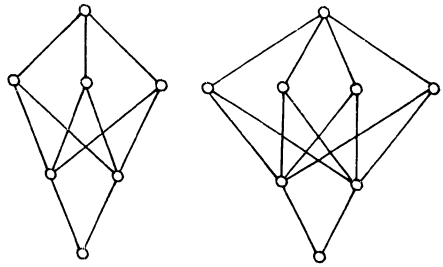

Fig. 5

Using distributive multilattices shown in Fig. 4, by 6.12 we can construct an infinite chain of varieties of distributive directed multilattices. Hence we have:

6.13. Corollary. *In the lattice of varieties of directed multilattices there are infinite chains of distributive varieties.*

6.14. Theorem. *Let M be a finite distributive directed multilattice that is not a lattice, has only trivial congruence relations and does not have proper subalgebras which are not lattices. Then the smallest variety containing M covers the variety $\mathscr{D}$ of all distributive lattices in the lattice of varieties of directed multilattices.*

Proof. Let M fulfil the stated conditions. Denote by $\mathscr{V}$ the smallest variety containing M. Evidently $\mathscr{D} \subset \mathscr{V}$. Suppose that $\mathscr{V}_1$ is a variety of directed multilattices satisfying $\mathscr{D} \subset \mathscr{V}_1 \subseteq \mathscr{V}$. We will show that $\mathscr{V}_1 = \mathscr{V}$. Take any $C \in \mathscr{V}_1 - \mathscr{D}$. Then C is not a lattice, because all lattices of $\mathscr{V}_1$ are distributive and hence belong to $\mathscr{D}$. If we show $M \in HSP\{C\}$, then $M \in \mathscr{V}_1$ and $\mathscr{V} \subseteq \mathscr{V}_1$. To prove $M \in HSP\{C\}$ we reason analogously as in the proof of 6.12. Since $C \in \mathscr{V}_1 \subseteq \mathscr{V} = HSP\{M\}$, there exists a homomorphism φ of a subalgebra A of a direct product $\Pi(M_i\colon i \in I)$ onto C, with $M_i = M$ for every $i \in I$. Lemmas 6.4 and 6.5 ensure the existence of distinct elements $u, v, a, b \in A$ as in Fig. 3, whose images $\varphi(u)$, $\varphi(v)$, $\varphi(a)$, $\varphi(b)$ are also distinct and as in Fig. 3. Let B be the subalgebra of $\Pi(M_i\colon i \in I)$ generated by $\{u, v, a, b\}$. By 6.6 B is isomorphic to $\Pi M_j\colon j \in J)$ with $M_j = M$ for every $j \in J$. In view of 6.11 we have $M \in H\{\varphi(B)\}$ and since $\varphi(B)$ is a subalgebra of C, we conclude $M \in HS\{C\}$.

There are two varieties of lattices that cover the variety of all distributive lattices in the lattice of varieties of lattices. In contrast with this we have:

6.15. Corollary. *There are infinitely many varieties of distributive directed multilattices covering the variety of all distributive lattices in the lattice of varieties of directed multilattices.*

Proof. To verify this it is sufficient to show that if M_1, M_2 are any non-isomorphic multilattices of Fig. 4, then $HSP\{M_1\} \neq HSP\{M_2\}$. This follows immediately from 6.12.

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КОНГРУЭНЦИИ И МНОГООБРАЗЯ МУЛЬТИРЕШЕТОК

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Резюме

В статье изучаются конгруэнции на направленных мультирешетках. Полученные результаты используются при исследовании некоторых вопросов, касающихся многообразий направленных мультирешеток.