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# ON POINTS OF LOWER AND UPPER QUASI—CONTINUITY OF MULTIVALUED MAPS

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A subset $A$ of a topological space $X$ is called [1, 2, 7]:
— semi-open if $A \subset \overline{\mathrm{Int}\, A}$,
— semi-closed if $X \setminus A$ is semi-open.

If for some $x \in X$ and a semi-open set $A \subset X$ we have $x \in A$, we say that $A$ is a semi-neighbourhood of $x$.

The union of all semi-open sets contained in $A$ is called the semi-interior of $A$. We denote it by s-Int $A$. The intersection of all semi-closed sets containing $A$ is called the semi-closure of $A$ and is denoted by $\underline{A}$.

In the sequel we will use the following properties of semi-open and semi-closed sets.

**Lemma 1.**
(a) *The intersection of an open set and a semi-open set is semi-open.*
(b) $\mathrm{Int}\, A \subset \text{s-Int}\, A \subset A \subset \underline{A} \subset \bar{A}$.
(c) $\mathrm{Int}\, \bar{A} \subset \underline{A}$.
(d) *A point $x$ belongs to $\underline{A}$ if and only if $U \cap A \neq \emptyset$ for every semi-neighbourhood $U$ of $x$.*
(e) *A set $A$ is semi-closed if and only if $\underline{A} = A$.*
(f) *The boundary of a semi-open (semi-closed) set is nowhere dense.*
(g) s-Int $A \setminus \mathrm{Int}\, A$ *is a nowhere dense set.*

Proof. Proofs of (a)—(f) are in papers [1, 2, 5, 7, 10]. Now we will show (g). Let $x \in B = \text{s-Int}\, A \setminus \mathrm{Int}\, A$ and let $U$ be an open neighbourhood of $x$. Since $U \cap \text{s-Int}\, A$ is a non-empty semi-open set, the set $V = \mathrm{Int}(U \cap \text{s-Int}\, A)$ is non-empty. However $V \subset \mathrm{Int}\, A$, so $V \cap B = \emptyset$ and $B$ is nowhere dense.

Let $(Y, d)$ be a metric space. For any $y \in Y$, $A \subset Y$ and $\varepsilon > 0$ we denote

$$K(y, \varepsilon) = \{x \in Y: d(x, y) < \varepsilon\},$$
$$K(A, \varepsilon) = \cup \{K(y, \varepsilon): y \in A\}.$$

Moreover we use the symbol $\mathscr{K}(Y)$ to denote the class of all non-empty compact subsets of $Y$.

For any multivalued map $F: X \to \mathscr{X}(Y)$ and a set $A \subset Y$ we will write

$$F^+(A) = \{x \in X: F(x) \subset A\},$$
$$F^-(A) = \{x \in X: F(x) \cap A \neq \emptyset\}.$$

A multivalued map $F: X \to \mathscr{X}(Y)$ is said to be:
— upper quasi-continuous at a point $x_0 \in X$ if for each $\varepsilon > 0$ there exists a semi-neighbourhood $U$ of $x_0$ such that $F(x) \subset K(F(x_0), \varepsilon)$ for $x \in U$,
— lower quasi-continuous at $x_0$ if for each $\varepsilon > 0$ and $y \in F(x_0)$ there exists a semi-neighbourhood $U$ of $x_0$ such that $F(x) \cap K(y, \varepsilon) \neq \emptyset$ for $x \in U$.
By $E_u(F)$ and $E_l(F)$ we denote the set of all points at which $F$ is upper or lower quasi-continuous respectively. A map $F$ is upper (lower) quasi-continuous if $E_u(F) = X$ (resp. $E_l(F) = X$), [4, 5, 8, 11]. Moreover by $C_u(F)$ and $C_l(F)$ we denote the set of all points at which $F$ is upper or lower semicontinuous, respectively. The symbol $Q_+$ is used to denote the set of all positive rational numbers.
Using some modification of the Fort method [6] we give the characterization of the set $E_l(F)$ (resp. $E_u(F)$) for upper (lower) quasi-continuous maps.

**Theorem 1.** *Let $X$ be a topological space and let $(Y, d)$ be a metric one. If $F$: $X \to \mathscr{X}(Y)$ is an upper quasi-continuous multivalued map, then the set $X \setminus E_l(F)$ is of the first category.*

Proof. Let $N(Fx), \varepsilon) = inf\{n \geq 1$: there exist points $y_1, y_2, \ldots y_n$ such that $F(x) \subset \bigcup_{i=1}^{n} K(y_i, \varepsilon)\}$. By $H(n, \varepsilon)$ we denote the set of all points $x \in X$ which satisfy the following conditions (1) and (2):

(1) $N(F(x), \varepsilon) \geq n$

(2) for each $\varepsilon' \in (0, 3\varepsilon)$ and for each semi-neighbourhood $U$ of $x$ there exists $x' \in U$ such that $F(x) \not\subset K(F(x'), \varepsilon')$.

Let $x \in X$ and $m = N(F(x), \varepsilon)$. Then there exist points $y_1, y_2, \ldots\ y_m \in Y$ such that $F(x) \subset \bigcup_{i=1}^{m} K(y_i, \varepsilon)$. By the upper quasi-continuity of $F$ there exists a semi-neighbourhood $U$ of $x$ such that $F(x') \subset \bigcup_{i=1}^{m} K(y_i, \varepsilon)$ for $x' \in U$. Hence

(3) for any $x \in X$ there exists a semi-neighbourhood $U$ of $x$ such that $N(F(x'), \varepsilon) \leq N(F(x), \varepsilon)$ for $x' \in U$.

If $N(F(x), \varepsilon) < n$, then according to (3) there exists a semineighbourhood $U$ of $x$ such that $N(Fx'), \varepsilon) \leq N(F(x), \varepsilon) < n$ for $x' \in U$. Hence $U \cap H(n, \varepsilon) = \emptyset$ and it follows from Lemma 1 (d) that $x \notin \underline{H}(n\, \varepsilon)$. Thus we have shown

(4) if $x \in \underline{H}(n, \varepsilon)$, then $N(F(x), \varepsilon) \geq n$.

Let $x \in H(n, \varepsilon)$, $n = N(F(x), \varepsilon)$ and let $U$ be an open neighbourhood of $x$. Then we can choose points $y_1, y_2, \ldots, y_n \in Y$ such that $F(x) \subset \bigcup_{i=1}^{n} K(y_i, \varepsilon)$. The upper quasi-continuity of $F$ at $x$ implies that there exists a semi-neighbourhood $W$ of $x$ such that $F(x') \subset \bigcup_{i=1}^{n} K(y_i, \varepsilon)$ for $x' \in W$. it follows from Lemma 1 that $U \cap W$ is a semi-neighbourhood of $x$. According to (2) there exists a point $x_1 \in U \cap W$ for which $F(x) \not\subset K(F(x_1), 2\varepsilon)$ holds. Let $y \in F(x) \backslash\backslash K(F(x_1), 2\varepsilon)$. Then $y \in K(y_j, \varepsilon)$ for some $j \leqslant n$ and it is easy to verify that $F(x_1) \cap K(y_j, \varepsilon) = \emptyset$. Hence $F(x_1) \subset \cup \{K(y_i, \varepsilon)\colon i \leqslant n,\ i \neq j\}$, which implies $N(F(x_1), \varepsilon) \leqslant n - 1$. Therefore — by virtue of (4) — we have $U \not\subset H(n, \varepsilon)$. Thus

$$\text{(5)} \qquad \text{if } N(F(x), \varepsilon) = n, \text{ then } x \in \operatorname{Fr} \underline{H}(n, \varepsilon).$$

For any $x \in H(n, \varepsilon)$ we have $(FN((x), \varepsilon) = n + k$ for some $k = 0, 1, \ldots$. Using analogous arguments as in the proof of (5) we can prove by the induction with respect to $k$ that $H(n, \varepsilon) \subset \operatorname{Fr} \underline{H}(x, \varepsilon)$. Since the boundary of semi-closed set is nowhere dense (Lemma 1), the set $H(n, \varepsilon)$ is nowhere dense. Take $\varepsilon_0 \in Q_+$, a point $x_0 \notin \bigcup_{\varepsilon \in Q_+} \bigcup_{n=1}^{\infty} H(n, \varepsilon)$ and $y_0 \in F(x_0)$. As for some $n \geqslant 1$ we have $N\left(F(x_0), \frac{1}{3}\varepsilon_0\right) = n$ and $x_0 \notin H\left(n, \frac{1}{3}\varepsilon_0\right)$, there exists a semineighbourhood $U$ of $x_0$ and $\varepsilon' < \varepsilon_0$ such that $F(x_0) \subset K(F(x'), \varepsilon')$ for $x' \notin U$. It implies $F(x') \cap K(y_0, \varepsilon_0) \neq \emptyset$ for $x' \in U$, i.e. $F$ is lower quasi-continuous at $x_0$. Thus we have shown the inclusion

$$\text{(6)} \qquad X \backslash E_1(F) \subset \bigcup_{\varepsilon \in Q_+} \bigcup_{n=1}^{\infty} H(n, \varepsilon).$$

Since $H(n, \varepsilon)$ is nowhere dense the set $X \backslash E_1(F)$ is of the first category, which finishes the proof.

**Theorem 2.** *Let $X$ be a topological space and let $Y$ be a metric one. If $Y$ is separable, then for each multivalued map $F\colon X \to \mathscr{Z}(Y)$ the sets $E_u(F) \backslash C_u(F)$ and $E_1(F) \backslash C_1(F)$ are of the first category.*

Proof. Let $\{y_n\colon n \geqslant 1\}$ be a dense subset of $Y$. We use the symbol $\alpha$ to denote the set of all finite one-to-one sequences of natural numbers. Then $\alpha = \{(n_{k,1}, n_{k,2}, \ldots, n_{k,j(k)})\colon k \geqslant 1\}$. Let $L_k = \{y_{n_{k,1}}, y_{n_{k,2}}, \ldots, y_{n_{k,j(k)}}\}$. If $x_0 \in E_u(F) \backslash \backslash C_u(F)$, then exists $\varepsilon \in Q_+$ such that $x_0 \in \operatorname{Int} F^+(KFx_0), 2\varepsilon))$. Since $F(x_0)$ is compact we can choose $L_k$ such that $F(x_0) \subset K(L_k, \varepsilon) \subset K(F(x_0), 2\varepsilon)$. Hence $x_0 \in \text{s-Int}\, F^+(KL_k, \varepsilon)) \backslash \operatorname{Int} F^+(KL_k, \varepsilon))$. Lemma 1 implies that the set $\text{s-Int}\, F^+(K(L_k, \varepsilon) \backslash \operatorname{Int} F^+(K(L_k, \varepsilon))$ is nowhere dense. Therefore the conclusion follows from the inclusion

$$E_u(F)\setminus C_u(F) \subset \bigcup_{k=1}^{\infty} \bigcup_{\varepsilon\in Q_+} [\text{s-Int}\, F^+(K(L_k,\varepsilon))\setminus \text{Int}\, F^+(KL_k,\varepsilon))].$$

The proof of the second part is analogous. It suffices to see

$$E_1(F)\setminus C_1(F) \subset \bigcup_{j=1}^{\infty} \bigcup_{\varepsilon\in Q_+} [\text{s-Int}\, F^-(K(y_j,\varepsilon))\setminus \text{Int}\, F^-(K(y_j,\varepsilon))].$$

**Corollary 1** [4]. *Let $X$ be a topological space and let $Y$ be a metric one. If $Y$ is separable and $F\colon X\to \mathscr{X}(Y)$ is an upper quasi-continuous map, then $X\setminus C_u(F)$ and $X\setminus C_1(F)$ are of the first category sets.*

**Proposition.** *If a multivalued map $F\colon X\to \mathscr{X}(Y)$ satisfies at a point $y\in X$ the following condition:*

(*) *for each $\varepsilon>0$ there exists a semi-neighbourhood $U$ of $x$ such that $F(x)\subset KF(x'),\varepsilon$ for $x'\in U$;*

*then $F$ is lower quasi-continuous at $x$.*

The simple proof is omitted.

The lower quasi-continuity does not imply the property (*).

Example 1. Let $X=Y$ be the space of real numbers with the natural topology. The multivalued map given by the formula:

$$F(x)=\begin{cases} [1,2) & \text{for}\, x<1 \\ [1,3] & \text{for}\, x=1 \\ [2,3] & \text{for}\, x>1 \end{cases}$$

is lower quasi-continuous but it does not satisfy (*) at $x=1$.

A topological space $X$ is called extremally disconnected if for every open set $U\subset X$ the closure $\bar{U}$ is open in $X$ [3, p. 452].

**Lemma 2.** [9, p. 966]. *A topological space $X$ is extremally disconnected if and only if the intersection of two semi-open sets is semi-open.*

**Theorem 3.** *A topological space $X$ is extremally disconnected if and only if for each metric space $Y$ and for each lower quasi-continuous map $F\colon X\to\mathscr{X}(Y)$ the condition (*) holds at every point $x\in X$.*

Proof. Susppose that $X$ is extremally disconnected and $F\colon X\to\mathscr{X}(Y)$ is a lower quasi-continuous map. We establish $x\in X$ and $\varepsilon>0$. Since $F(x)$ is compact we can choose points $y_1,y_2,\ldots,y_m\in F(x)$ such that $F(x)\subset \bigcup_{j=1}^{m} K\left(y_j,\frac{1}{2}\varepsilon\right)$.

There exist semi-neighbourhoods $U_j$ of $x$ such that $F(x') \cap K\left(y_j, \frac{1}{2}\varepsilon\right) \neq \emptyset$ for $x' \in U_j, j = 1, 2, \ldots, m$. Then the set $U = \bigcap_{j=1}^{m} U_j$ is a semi-neighbourhood of $x$ and $F(x) \subset K(F(x'), \varepsilon)$ for $x' \in U$. Thus $(*)$ is satisfied.

Conversely, suppose that $X$ is not extremally disconnected. Then there exists an open set $U \subset X$ such that $\bar{U}$ is not open. Hence $\emptyset \neq U \neq X$. Let $Y$ be the space of real numbers with the natural topology. We define the map $F\colon X \to \mathscr{L}(Y)$ by

$$F(x) = \begin{cases} [1,2] & \text{for } x \in X \setminus \bar{U} \\ [1,3] & \text{for } x \in \operatorname{Fr} \bar{U} \\ [2,3] & \text{for } x \in \operatorname{Int} \bar{U}\,. \end{cases}$$

This map is lower quasi-continuous but is has not the property $(*)$ at every point $x \in \operatorname{Fr} \bar{U}$.

**Theorem 4.** *Let $X$ be an extremally disconnected space and let $Y$ be a metric one. If $F\colon X \to \mathscr{L}(Y)$ is a lower quasi-continuous map, then $X \setminus E_u(F)$ is of the first category.*

Proof. Let $M(F(x)), \varepsilon) = \sup\{m \geqslant 1\colon$ then there exist points $y_1, y_2, \ldots, y_m \in F(x)$ such that $d(y_1, y_j) > \varepsilon$ for $i, h \geqslant m, i \neq j\}$. By $G(n, \varepsilon)$ we denote the set of all points $x \in X$ at which the following conditions (7) and (8) are satisfied:

(7) $$M(F(x), \varepsilon) \geqslant n$$

(8) for each $\varepsilon' \in (0, 3\varepsilon)$ and for each semi-neighbourhood $U$ of $x$ there exists $x' \in U$ such that $F(x') \not\subset K(F(x), \varepsilon')$.

Let $x \in X$, $\varepsilon \in Q_+$ and $m = M(F(x), \varepsilon)$. We can choose points $y_1, y_2, \ldots, y_m \in F(x)$ and $r > 0$ such that $d(y_i, y_j) > \varepsilon + 2r$ for $i, j \leqslant m$, $i \neq j$. The map $F$ is lower quasi-continuous at $x$, so there exist semi-neighbourhoods $U_1, U_2, \ldots, U_m$ of $x$ such that $F(x') \cap K(y_i, r)\, i \neq \emptyset$ for $x' \in U_i$, $i \leqslant m$. Since $X$ is extremally disconnected, it follows from Lemma 2 that $U = \bigcap_{i=1}^{m} U_i$ is a semineighbourhood of $x$. Moreover $f(x') \cap K(y_i, r) \neq \emptyset$ for $x' \in U$, $i \leqslant m$. Hence for $y_i' \in F(x') \cap K(y_j, r)$ we have $d(y_i, y_j) \leqslant 2r + d(y_i', y_j')$, which implies $d(y_i', y_j') > \varepsilon$ for $i \neq j$. Consequently $M(F(x'), \varepsilon) \geqslant m$. Thus we have shown

(9) for each $x \in X$ there exists a semi-neighbourhood $U$ of $x$ such that $M(F(x'), \varepsilon) \geqslant M(F(x), \varepsilon)$ for $x' \in U$.

If for $x \in X$ we have $M(F(x), \varepsilon) > n$ then by (9) there exists a semi-neighbourhood $U$ of $x$ such that $M(F(x'), \varepsilon) > n$ for $x' \in U$. Thus $U \cap G(n, \varepsilon) = \emptyset$ and by virtue of Lemma 1, $x \notin \underline{G}(n, \varepsilon)$. Thus we obtain

(10) if $x \in \underline{G}(n, \varepsilon)$, then $M(F(x), \varepsilon) \leqslant n$.

Now we will show that $G(n, \varepsilon)$ is nowhere dense. Let $x_0 \in G(n, \varepsilon)$, $m = M(F(x_0), \varepsilon)$ and let $U$ be an open neighbourhood of $x_0$. We can take points $y_1, y_2, \ldots, y_m \in \in F(x_0)$ and $r \in \left(0, \frac{1}{2}\varepsilon\right)$ such that $d(y_i, y_j) > \varepsilon + 2r$ for $i, j \leqslant m$, $i \neq j$. The lower quasi-continuity of $F$ at $x_0$, Theorem 3 and (8) imply the existence of a point $x_1 \in U$ such that $F(x_0) \subset K(F(x_1), r)$ and $F(x_1) \not\subset K(F(x_0), 2\varepsilon)$. So we can choose $z_1, z_1, \ldots, z_m \in F(x_1)$ such that $d(z_i, y_i) < r$ for $i \leqslant m$, and a point $z_{m+1} \in F(x_1) \setminus \setminus K(F(x_0), 2\varepsilon)$. Then $d(z_i, z_j) > \varepsilon$ for $i, j = 1, 2, \ldots, m+1$, $i \neq j$, and consequently $M(F(x_1), \varepsilon) \geqslant m + 1$. From the last inequality and (10) it follows that $U \subset \overline{G(n, \varepsilon)}$. Thus we have $G(n, \varepsilon) \subset \overline{G(n, \varepsilon)} \setminus \operatorname{Int} \overline{G(n, \varepsilon)} \subset \operatorname{Fr} G(n, \varepsilon)$. Since the boundary of each semi-closed set is nowhere dense (Lemma 1), the set $G(n, \varepsilon)$ is nowhere dense. Let $x \notin \bigcup_{n=1}^{\infty} \bigcup_{\varepsilon \in Q_+} G(n, \varepsilon)$. For an established $\varepsilon \in Q_+$ we have $M\left(F(x), \frac{1}{3}\varepsilon\right) = n$. Then (8) is not satisfied. Hence there exists a semi-neighbourhood $U$ of $x$ such that $F(x') \subset K(F(x), \varepsilon)$ for $x' \in U$, i.e. the map $F$ is upper quasi-continuous at $x$. So we have

$$X \setminus E_u(F) \subset \bigcup_{n=1}^{\infty} \bigcup_{\varepsilon \in Q_+} G(n, \varepsilon), \tag{11}$$

and the proof is completed.

Results obtained in Theorem 1 and 4 can be extended to the case when $Y$ is a uniform space (all notions concerning uniform spaces are used as in [3]). The proofs are similar to the previousones; it suffices to consider a suitable family of pseudometrics instead of a metric $d$ on $X$. So we have:

**Theorem A.** *Let $X$ be a topological space and let $(Y, \mathfrak{U})$ be a uniform one. If $F$: $X \to \mathscr{X}(Y)$ is an upper quasi-continuous multivalued map, then $X \setminus E_l(F) = \cup \{A_p : p \in P\}$, where $A_p$ are of the first category sets and* card $P = w(\mathfrak{U})$, *and $w(\mathfrak{U})$ is the weight of the uniformity $\mathfrak{U}$.*

**Theorem B.** *Let $x$ be an extremally disconnected space and let $(Y, \mathfrak{U})$ be a uniform one. If $F$: $X \to \mathscr{X}(Y)$ is a lower quasi-continuous map, then $X \setminus E_u(F) = \cup \{D_p : p \in P\}$, where $D_p$ are of the first category sets and* card $P = w(\mathfrak{U})$.

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## ТОЧКИ КВАЗИ-НЕПРЕРЫВНОСТИ СВЕРХУ И СНИЗУ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ

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Резюме

Пусть $F$ будет квази-непрерывное сверху (снизу) многозначное отображение определённое на топологическом (экстремально несвязным) пространстве из компактными значениями в равномерном пространстве $(y, \mathfrak{U})$. Тогда множество всех точек, в которых $F$ не яавляется квази — непрерывным снизу (сверху) есть объединение некоторого семейства множеств первой категории. Мощность этого семейства равна весу равномерной структуры $\mathfrak{U}$. Более того если пространство $y$ есть сепарабельно, то такой же самый вид имеет множество точек разрыва отображения $F$.