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STABILIZATION OF SOLUTIONS OF CERTAIN ONE-DIMENSIONAL DEGENERATE DIFFUSION EQUATIONS

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0. Introduction

Stimulated by paper [1] we are concerned with the asymptotic behaviour of nonnegative solutions of the initial-boundary value problem

$$\begin{aligned} &u_t = (u^m)_{xx} + f(u) && \text{in } (-L, L) \times R^+, \\ &u(\pm L, t) = 0 && \text{in } R^+, \\ &u(x, 0) = u_0(x) && \text{in } [-L, L], \end{aligned} \tag{0.1}$$

where $m > 1$, u_0 is bounded and nonnegative, f satisfies the following set of hypotheses:

(H1) $f \in C^1([0, \infty)), f(0) = 0, f'(0) < 0$;
(H2) $f(r) \leqslant cr^\gamma$ for some $c > 0$ and $m > \gamma > 1$;
(H3) there exists $r_0 > 0$ *such that* $\mathrm{f(r)} > 0$ for $r > r_0$ and $f(r) \leqslant 0$ for $0 \leqslant r \leqslant \mathrm{r}_0$;
(H4) there exists $r_1 > 0$ such that $f'(r) \geqslant 0$ for $r \geqslant r_1$,
(H5) there exists $r_2 > 0$ such that $\varphi(r) = r^m(mf(r) - rf'(r))$ is positive and nondecreasing for $r > r_2$ and $\varphi(r) \leqslant 0$ for $0 \leqslant r \leqslant r_2$.

These will be called the "hypotheses H". For example, we might consider $f(r) = ar^\gamma - br$ with $a, b > 0$ and $m > \gamma > 1$, which one may keep in mind as a model growth term.

It is well known that the equation in (0.1) appears in various physical, chemical and biological models. We mention only a model from biology, where (0.1) describes the growth and spread of a spatially distributed biological population, whose tendency to migrate is governed by the local population density [2]. The form of the function f in our example corresponds to an interesting case of the Verhulst law [2].

We begin by describing the set $E(L)$ of nonnegative stationary solutions of Problem (0.1). There are two critical values L_0 and L_1 such that

(i) $E(L) = \{0\}$ for $0 < L < L_0$,
(ii) $E(L_0)$ consists of $u = 0$ and one positive solution,
(iii) $E(L) = \{0, p, q\}$ for $L_0 < L \leqslant L_1$, where p and q are positive solutions with $p < q$ on $(-L, L)$,
(iv) for $L > L_1$, $E(L)$ consists of the trivial solution, one isolated positive solution q and continua of solutions generated by $p(.\,, L_1)$.

Aronson, Crandall and Peletier [1] studied $E(L)$ for Problem (0.1) with $f(u) = u(1-u)(u-a)$ and they find the structure of $E(L)$ to be the same, but there is a difference in the dependence of possible values of L on maxima of positive solutions u. This situation is indicated in the four diagrams in Figure 1.

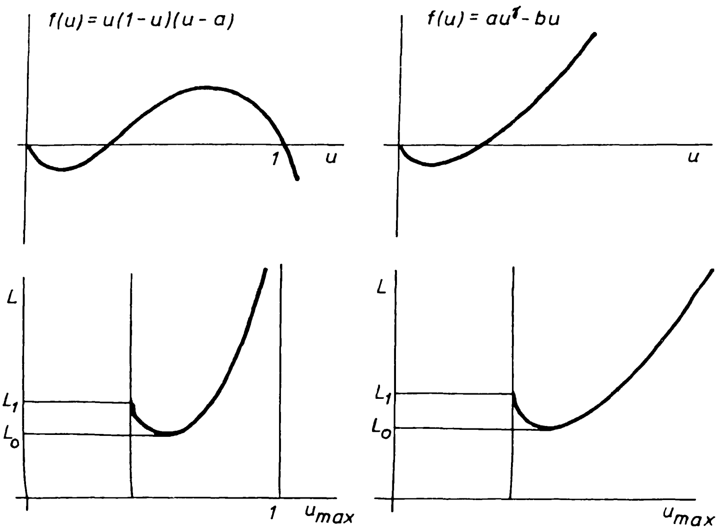

Fig. 1

Our stabilization result is a slight modification of the results of [1], it turns out that both the trivial solution and the large positive solution q are stable.

The existence of global solutions is proved by the Galerkin method and our proof is closely related to the work of Nakao [3].

1. Stationary solutions

A nonnegative function v is a stationary solution of Problem (0.1) when it is a solution of the problem

$$
\begin{aligned}
&(v^m)'' + f(v) = 0 \qquad \text{in } (-L, L),\\
&v(\pm L) = 0.
\end{aligned} \tag{1.1}
$$

v is called a solution of Problem (1.1) iff $w = v^m$ is a classical solution of $w'' + f(w^{1/m}) = 0$, $w(\pm L) = 0$.

Obviously $v \equiv 0$ is always a solution of Problem (1.1). If v is a positive solution of Problem (1.1), then there exists a $x_0 \in (-L, L)$ such that $0 < v(x) \leqslant v(x_0)$ for $x \in (-L, L)$ and $v'(x_0) = 0$. Conversely, let us seek conditions which guarantee that the solution of the initial value problem

$$
\begin{aligned}
&(v^m)'' + f(v) = 0,\\
&v(x_0) = \mu,\ v'(x_0) = 0
\end{aligned} \tag{1.2}
$$

is also a positive solution of Problem (1.1). Using standard manipulations we obtain from (1.2)

$$
\frac{1}{2}(v^m)'^2 + mF(v) = mF(\mu), \tag{1.3}
$$

where $F(v) = \displaystyle\int_0^v s^{m-1} f(s)\,\mathrm{d}s$.

The hypotheses (H3), (H4) imply that there is $\alpha > 0$ ($\alpha > r_0$) such that $F < 0$ on $(0, \alpha)$ and $F > 0$ on (α, ∞). For $\mu \in (0, \alpha)$, the values of the solutions of Problem (1.2) lie in some interval $[v_1, v_2]$, where $v_1, v_2 > 0$ and $v_i = \mu$ for some $i \in \{1, 2\}$. Thus in order that a solution of (1.2) represents a solution of Problem (1.1) it is necessary that $\mu \geqslant \alpha$. *If* $\mu > \alpha$, we can integrate (1.3) to obtain

$$
\sqrt{\frac{m}{2}} \int_{v(x)}^{\mu} \frac{s^{m-1}}{\sqrt{F(\mu) - F(s)}}\,\mathrm{d}s = |x_0 - x|. \tag{1.4}
$$

The integrand has a singular point at $s = \mu$, but by (H3) $F(\mu) - F(s) \geqslant \delta(\mu - s)$ for some $\delta > 0$ and s near μ, and so the singularity is integrable. For $\mu \geqslant \alpha$ we have $F(s) < F(\mu)$ if $s \in (0, \mu)$, thus we can define

$$
T(\mu) = \sqrt{\frac{m}{2}} \int_0^{\mu} \frac{s^{m-1}}{\sqrt{F(\mu) - F(s)}}\,\mathrm{d}s,\ \alpha \leqslant \mu. \tag{1.5}
$$

If $\mu = \alpha$, the integrand in (1.5) may have a second singularity at $s = 0$. However, from (H1) it follows that $-F(s) \geqslant ks^{m+1}$ for some $k > 0$ and $s > 0$ near 0, thus $s^{m-1}(-F(s))^{-1/2} \leqslant k^{-1/2} s^{(m-3)/2}$ near $s = 0$. Since $m > 1$, T is well defined on $[\alpha, \infty)$. For a positive solution of Problem (1.1), $v = 0$ only at $\pm L$. Therefore

$$
T(\mu) = |x_0 - L| = |x_0 + L|
$$

from which we conclude that $x_0 = 0$. We have proved

Lemma 1.1. *Let the hypotheses* (H1), (H3) *and* (H4) *hold. Then v is a positive solution of Problem* (1.1) *if and only if*

$$\sqrt{\frac{m}{2}} \int_{v(x)}^{\mu} \frac{s^{m-1}}{\sqrt{F(\mu) - F(s)}} \, \mathrm{d}s = |x|, \; |x| \leqslant L, \tag{1.6}$$

where $\mu \in [\alpha, \infty)$ and $L \in R^+$ are related by the equation $T(\mu) = L$ and α is the unique positive root of F.

This lemma yields that the number of positive solutions of (1.1) is determined by the number of roots of the equation $T(\mu) = L$. The following result describes the function T given by (1.5).

Lemma 1.2. *Let the hypotheses* H *hold, then*

(i) $T \in C([\alpha, \infty)) \cap C^1((\alpha, \infty))$,
(ii) $T(\mu) \to +\infty$ *as* $\mu \to +\infty$,
(iii) $T'(\mu) \to -\infty$ *as* $\mu \to \alpha$,
(iv) $T'(\mu)$ *has a unique root* $\mu_0 \in (\alpha, \infty)$.

Proof. Write

$$S(\mu) = \sqrt{\frac{2}{m}}\, T(\mu) = \int_0^{\mu} \frac{s^{m-1}}{\sqrt{F(\mu) - F(s)}} \, \mathrm{d}s.$$

If we set $\theta(s) = 2mF(s) - s^m f(s)$, we get

$$S'(\mu) = \frac{1}{2\mu} \int_0^{\mu} \frac{(\theta(\mu) - \theta(s)) s^{m-1}}{(F(\mu) - F(s))^{3/2}} \, \mathrm{d}s.$$

It means that $T \in C^1((\alpha, \infty))$. The proof of the facts that $S(\mu) \to S(\alpha)$ as $\mu \to \alpha$ and $S'(\mu) \to -\infty$ as $\mu \to \alpha$ employs the hypotheses (H1), (H3). The arguments are exactly the same as in the proof of analogous statements in [1]. In order to prove (ii) introduce the substitution $t = s/\mu$, which yields

$$S(\mu) = \mu^m \int_0^1 \frac{t^{m-1}}{\sqrt{F(\mu) - F(t\mu)}} \, \mathrm{d}t \geqslant \mu^m \int_{1/2}^1 \frac{t^{m-1}}{\sqrt{F(\mu) - F(t\mu)}} \, \mathrm{d}t = J$$

$F(\mu) - F(t\mu) \leqslant \frac{1}{2}\mu^m f(\mu)$ for $\frac{\mu}{2} \geqslant r_1$ (r_1 is from (H4)). This implies that $J \geqslant K\mu^{m/2}/\sqrt{f(\mu)}$ if μ is large enough, K is a positive constant. From (H2) we get $S(\mu) \to +\infty$ as $\mu \to +\infty$.

For the proof of (iv) we use (H5), from which we obtain the existence of $\mu_2 > r_2$ with $\theta(r) < 0$ for $r \in [r_2, \mu_2)$ and $\theta(r) > 0$ for $r > \mu_2$ ($r\theta'(r) = \varphi(r)$). If $\mu > \mu_2$, then $\theta(\mu) > \theta(s)$ for $0 \leqslant s < \mu$, hence $S'(\mu) > 0$ for $\mu > \mu_2$. Therefore S'

has at least one zero in $[r_2, \mu_2]$. In the same way as in [1] we could derive the inequality

$$S''(\mu) \geqslant \frac{1}{2\mu^2} \int_0^\mu \frac{\mu\theta'(\mu) - s\theta'(s)}{(F(\mu) - F(s))^{3/2}} s^{m-1} \, ds$$

for μ which are roots of S'. (H5) implies that $\mu\theta'(\mu) - s\theta'(s) \geqslant 0$ if $\mu \geqslant r_2$, $s \in [0, \mu]$ and on a set of positive measure this inequality is strong. Since $S'(\mu) = 0$ implies $S''(\mu) > 0$, there can be at most one zero of S'. Hence this completes the proof.

Set $L_0 = T(\mu_0)$ and $L_1 = T(\alpha)$. We distinguish four cases:

(i) $T(\mu) = L$ has no solutions for $0 < L < L_0$, hence there are no positive solutions of Problem (1.1) if $0 < L < L_0$.

(ii) $T(\mu) = L$ has one solution for $L = L_0$, therefore there is one positive solution $v(\cdot, \mu_0)$.

(iii) $T(\mu) = L$ has two solutions for $L_0 < L \leqslant L_1$, there are two positive solutions $p(\cdot, L)$, $q(\cdot, L)$ with $p < q$ on $(-L, L)$.

(iv) $T(\mu) = L$ has one solution μ_L for $L > L_1$, hence there is one positive solution $q(\cdot, L)$ (of Problem (1.1)) which corresponds to μ_L. In addition, there are families of nonnegative solutions of Problem (1.1) on $(-L, L)$. They are generated by $p(\cdot, L_1)$ because $F(\alpha) = 0$, so $(p^m)'(\pm L_1, L_1) = 0$ and it follows that $p(x, L_1)$ extended as 0 for $L \geqslant |x| \geqslant L_1$ is a solution for $L > L_1$. More generally, let N be a positive integer and $L \geqslant NL_1$. For each N-vector $z = (z_1, \ldots, z_N)$ which satisfies

$$-L \leqslant z_1 - L_1,\ z_i + L_1 \leqslant z_{i+1} - L_1,\ i = 1, \ldots, N-1,\ z_N + L_1 \leqslant L, \qquad (1.7)$$

the function

$$\varrho(x, z) = \begin{cases} p(x - z_i, L_1) & \text{for } |x - z_i| \leqslant L_1, \\ 0 & \text{if } |x - z_i| > L_1,\ i = 1, \ldots, N \end{cases}$$

is a nonnegative solution of Problem (1.1). We denote the collection of functions $\varrho(x, z)$ where $z \in R^N$ satisfies (1.7) as $P_N(L)$.

The complete description of the set of the stationary solutions (this set will be denoted by $E(L)$) is as follows:

Theorem 1.3. *Let the hypotheses H hold. Then*

$$E(L) = \begin{cases} \{0\} & \textit{for } 0 < L < L_0, \\ \{0, v(\cdot, \mu_0)\} & \textit{for } L = L_0, \\ \{0, p(\cdot, L), q(\cdot, L)\} & \textit{for } l_0 < L \leqslant L_1, \\ \{0, q(\cdot, L)\} \cup P_1(L) \cup \ldots \cup P_N(L) & \textit{for } L_1 < L, \\ N \textit{ is the maximal positive integer for which } NL_1 \leqslant L. & \end{cases}$$

In what follows the next lemma plays an important role.

Lemma 1.4. *Let $u_0 \in L^\infty(-L, L)$, $0 < L < \infty$. Then there exists $L' \geqslant L$, such that $u_0 \leqslant q(\cdot, L')$.*

Proof. Denote $q_n = q(\cdot, L_n)$, where $L_n \geqslant L$, $n = 1, 2, \ldots$, $L_n \to \infty$. as $n \to \infty$, $\eta_n = \inf\limits_{x \in [-L, L]} q_n(x) = q_n(\pm L)$, $\mu_n = q_n(0)$. We show by contradiction that $\eta_n \to \infty$ as $n \to \infty$. Let there exist $K > 0$ such that $\eta_n \leqslant K$ for all positive integers n. From (1.6) we obtain

$$\sqrt{\frac{m}{2}} \int_{\eta_n}^{\mu_n} \frac{s^{m\;1}}{\sqrt{F(\mu_n) - F(s)}} \, \mathrm{d}s = L,$$

further

$$\sqrt{\frac{2}{m}} L \geqslant \int_K^{\mu_n} \frac{s^{m\;1}}{\sqrt{F(\mu_n)}} \, \mathrm{d}s \geqslant \frac{1}{m} (c\mu_n^{m+\gamma})^{-1\;2} (\mu_n^m - K^m), \tag{1.8}$$

since $F(\mu_n) - F(s) \leqslant F(\mu_n) \leqslant c\mu_n^{m+\gamma}$ for K large enough. The last expression in (1.8) tends to infinity as $n \to \infty$, which is the contradiction and we obtain the conclusion.

2. Existence theorem for smooth initial data

We shall write $D = (-L, L)$, $Q_T = D \times (0, T)$, $Q = D \times R^+$. The function spaces we use are almost familiar and we omit the definitions.

Theorem 2.1. *Let $T > 0$ be arbitrarily fixed. Suppose that f is locally Lipschitz continuous and satisfies* (H2). *If $u_0 \geqslant 0$, $u_0 \in H_0^1(D)$, then Problem (0.1) admits a nonnegative solution $u(x, t)$ such that*

$$\begin{aligned} &(u^{(m+1)\;2})_t(t) \in L^2([0, T]; L^2(D)), \\ &u^m(t) \in L^\infty([0, \infty); H_0^1(D)) \cap H^1([0, T]; L^2(D)), \\ &u(t) \in C([0, \infty); L^2(D)) \end{aligned}$$

and the equation is satisfied in the sense that

$$\int_D u(t)\varphi(t) \, \mathrm{d}x + \iint_{Q_t} (-u\varphi_t + (u^m)_x \varphi_x - f(u)\varphi) \, \mathrm{d}x \, \mathrm{d}s = \int_D u_0 \varphi(0) \, \mathrm{d}x, \tag{2.1}$$

$0 \leqslant t \leqslant T$, for all $\varphi \in C^{2,1}(\bar{Q}_T)$ such that $\varphi = 0$ at $x = \pm L$, $0 \leqslant t \leqslant T$. Moreover the following estimate holds:

$$\frac{4m}{(m+1)^2} \int_0^t \|(u^{(m+1)\;2})_t(s)\|_{L^2(D)}^2 \, \mathrm{d}s + V(u(t)) \leqslant V(u_0) \tag{2.2}$$

for $0 \leqslant t \leqslant T$, *where*

$$V(\xi) = \int_D \left(\frac{1}{2}(\xi^m)_x^2 - mF(\xi)\right)\mathrm{d}x,\ F(r) = \int_0^r s^{m-1}f(s)\,\mathrm{d}s.$$

Remark 2.2. In the proof we proceed similarly as Nakao proceeds in [3], but we essentially employ the fact that we consider only one space-variable $(D = (-L, L))$, hence $\|v\|_{L^\infty(D)} \leqslant \sqrt{2L}\left(\int_D |v_x|^2\,\mathrm{d}x\right)^{1/2}$.

Proof of Theorem 2.1. We extend first f as 0 in R^-. Setting $v = |u|^m \operatorname{sgn}(u)$, $\beta(v) = |v|^{1/m}\operatorname{sgn}(v)$, $\psi(v) = f(\beta(v))$ we rewrite Problem (0.1):

$$\begin{aligned} &(\beta(v))_t - v_{xx} - \psi(v) = 0 && \text{in } D \times R^+,\\ &v(x,0) = v_0(x)\,(= u_0^m(x)) && \text{in } \bar{D},\\ &v(\pm L, t) = 0 && \text{in } R^+. \end{aligned} \tag{2.3}$$

It is easy to see that for nonnegative v Problems (2.3) and (0.1) are equivalent. In what follows it is more convenient for us to consider Problem (2.3). We use the Galerkin method, for which we regularize the equation in (2.3) because the leading term $(\beta(v))_t = \beta'(v)v_t$ has singularities at $v = 0$, $v = \infty$ and $\psi(v)$ need not be locally Lipschitz continuous at $v = 0$. Thus we first consider the modified problem

$$\begin{aligned} &(\beta_\varepsilon(v) + \varepsilon v)_t - v_{xx} - \psi_\varepsilon(v) = 0,\ \varepsilon > 0,\\ &v(x,0) = v_0(x),\ v(\pm L, t) = 0, \end{aligned} \tag{2.4}$$

where $\beta_\varepsilon(v) = \int_0^v \beta_\varepsilon'(s)\,\mathrm{d}s$, $\beta_\varepsilon'(s) = m^{-1}(|s| + \varepsilon)^{(1-m)/m}$,

$$\psi_\varepsilon(v) = \begin{cases} \psi(v+\varepsilon) - \psi(\varepsilon) & \text{for } v \geqslant 0,\\ 0 & \text{for } v < 0. \end{cases}$$

Take a basis $\{w_j\}_{j=1}^\infty$ in $H_0^1(D)$ (it can be chosen arbitrarily smooth since $D = (-L, L)$) and construct approximate solutions

$$v_{n,\varepsilon}(t) = \sum_{j=1}^n y_j^n(t)w_j,\ n = 1, 2, \ldots$$

through the system of ordinary differential equations

$$\begin{aligned} &((\beta_\varepsilon'(v_{n,\varepsilon}(t)) + \varepsilon)(v_{n,\varepsilon}(t))_t, w_j)_0 + ((v_{n,\varepsilon}(t))_x, (w_j)_x)_0 -\\ &\quad - (\psi_\varepsilon(v_{n,\varepsilon}(t)), w_j)_0 = 0 \qquad (j = 1, 2, \ldots, n),\\ &\quad v_{n,\varepsilon}(0) = v_{0,n} \in [w_1, \ldots, w_n], \end{aligned} \tag{2.5}$$

where the initial data are chosen in such a way that

$$v_{0,n} \to v_0 \qquad \text{in } H_0^1(D) \text{ as } n \to \infty, \tag{2.6}$$

and $(\cdot, \cdot)_0$ denotes the inner product in $L^2(D)$.

The above system (2.5) with respect to $y_j^n(t)$ has a solution (which is unique) on some interval, say, $[0, T_{n,\varepsilon}]$, because no singularity appears in (2.5) and ψ_ε is locally Lipschitz continuous.

Now we derive a priori estimates for $v_{n,\varepsilon}(t)$. Multiplying (2.5) by $(y_j^n(t))_t$, summing up over j and integrating we get

$$\int_0^t \int_D (\beta_\varepsilon'(v_{n,\varepsilon}) + \varepsilon)|(v_{n,\varepsilon})_t|^2 \,\mathrm{d}x\,\mathrm{d}s + V_\varepsilon(v_{n,\varepsilon}(t)) = V_\varepsilon(v_{0,n}) \tag{2.7}$$

for $0 \leqslant t \leqslant T_{n,\varepsilon}$, where we set $V_\varepsilon(v) = \dfrac{1}{2}\displaystyle\int_D |v_x|^2 \,\mathrm{d}x - \int_D G_\varepsilon(v)\,\mathrm{d}x$ and $G_\varepsilon(v) = \displaystyle\int_0^v \psi_\varepsilon(s)\,\mathrm{d}s$. From the assumption (H2) it follows that we can estimate

$$\int_D G_\varepsilon(v)\,\mathrm{d}x \leqslant c\int_D |v|^{\frac{\gamma+m}{m}}\,\mathrm{d}x \leqslant C_1\left(\int_D |v|^2\,\mathrm{d}x\right)^{\frac{\gamma+m}{2m}} \leqslant \frac{1}{4}\int_D |v_x|^2\,\mathrm{d}x + C_2$$

for ε sufficiently small, where the constants C_1, C_2 do not depend on n, ε. Therefore we have

$$V_\varepsilon(v) \geqslant \frac{1}{4}\int_D |v_x|^2\,\mathrm{d}x - C_2$$

which (together with (2.7)) gives

$$\int_0^t \int_D (\beta_\varepsilon'(v_{n,\varepsilon}) + \varepsilon)|(v_{n,\varepsilon})_t|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{4}\int_D |(v_{n,\varepsilon}(t))_x|^2\,\mathrm{d}x \leqslant C < \infty \tag{2.8}$$

for $0 \leqslant t \leqslant T_{n,\varepsilon}$, which implies that we can take $T_{n,\varepsilon} = T$, since the constant C does not depend on n, ε. Moreover (2.8) yields the following estimates

$$\int_0^T \int_D \left\{\left(\int_0^{v_{n,\varepsilon}} \sqrt{\beta_\varepsilon'(s)}\,\mathrm{d}s\right)_t\right\}^2 \mathrm{d}x\,\mathrm{d}t \leqslant C, \tag{2.9}$$

$$\|v_{n,\varepsilon}\|_{L^\infty([0,T];H_0^1(D))} \leqslant C^*, \tag{2.10}$$

which implies

$$\|v_{n,\varepsilon}\|_{L^\infty(Q_T)} \leqslant (2L)^{1/2}C^*. \tag{2.11}$$

By a simple calculation we obtain from (2.9) and (2.11)

$$\|(v_{n,\varepsilon})_t\|^2_{L^2([0,T];L^2(D))} \leqslant m(\|v_{n,\varepsilon}\|_{L^\infty(Q_T)} + \varepsilon)^{\frac{m-1}{m}} C. \tag{2.12}$$

We remark that C, C^* do not depend on T.

From (2.10)—(2.12) we get using the standard arguments

$$v_{n,\varepsilon} \to v_\varepsilon \quad \text{weakly* in } L^\infty([0,T]; H^1_0(D)), \tag{2.13}$$

$$v_{n,\varepsilon} \to v_\varepsilon \quad \text{weakly in } H^1([0,T]; L^2(D)), \tag{2.14}$$

$$v_{n,\varepsilon} \to v_\varepsilon \quad \text{strongly in } L^2(Q_T), \tag{2.15}$$

$$v_{n,\varepsilon} \to v_\varepsilon \quad \text{a.e. in } Q_T, \tag{2.16}$$

$$v_{n,\varepsilon} \to v_\varepsilon \quad \text{in } C([0,T]; L^2(D)), \tag{2.17}$$

along a subsequence as $n \to \infty$. From (2.10)—(2.17) we also have

$$\beta_\varepsilon(v_{n,\varepsilon}) \to \beta_\varepsilon(v_\varepsilon) \quad \text{strongly in } L^p(Q_T),\ 1 \leqslant p < \infty,$$
$$\beta_\varepsilon(v_{n,\varepsilon}) \to \beta_\varepsilon(v_\varepsilon) \quad \text{in } C([0,T]; L^2(D)).$$

Now from (2.5) it easily follows that

$$\int_0^T \int_D (-\beta_\varepsilon(v_\varepsilon)\varphi_t + (v_\varepsilon)_x\varphi_x - \psi_\varepsilon(v_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_D \beta_\varepsilon(v_\varepsilon(T))\varphi(T)\,\mathrm{d}x = \\ = \int_D \beta_\varepsilon(v_0)\varphi(0)\,\mathrm{d}x \qquad \text{for } \varphi \in C^{2,1}(\bar{Q}_T). \tag{2.18}$$

Since the estimates (2.9)—(2.11) do not depend on ε, they hold for v_ε as well. Taking $\varepsilon \to 0$, we obtain (2.13)—(2.17) for v_ε, v instead of $v_{n,\varepsilon}$, v_ε respectively. Using the obvious inequality $|\beta_\varepsilon(a) - \beta_\varepsilon(b)| \leqslant \beta_\varepsilon(|a-b|)$ for $a, b \in R$, $ab \geqslant 0$, one can show that

$$\beta_\varepsilon(v_\varepsilon) \to \beta(v) \qquad \text{in } C([0,T]; L^2(D)).$$

From the definition of ψ_ε we have $\|\psi_\varepsilon(v_\varepsilon) - \psi(v)\|_{L^p(Q_T)} \leqslant \|\psi(v_\varepsilon + \varepsilon) - \psi(v)\|_{L^p(Q_T)} + \|f(\varepsilon^{1/m})\|_{L^p(Q_T)}$, thus

$$\psi_\varepsilon(v_\varepsilon) \to \psi(v) \text{in } L^p(Q_T),\ 1 \leqslant p < \infty.$$

The identity (2.1) now follows from (2.18) and we conclude that v is the desired solution. A comparison theorem in the next section implies its nonnegativity.

In order to derive the inequality (2.2) from (2.7) we apply standard arguments. From (2.9) we get

$$(g_\varepsilon(v_{n,\varepsilon}))_t = \left(\int_0^{v_{n,\varepsilon}} \sqrt{\beta_\varepsilon'(s)}\,\mathrm{d}s\right)_t \to \chi_\varepsilon \text{ weakly in } L^2(Q_T).$$

Since $v_{n,\varepsilon} \to v_\varepsilon$ a.e. ((2.16)) and $v_{n,\varepsilon}$ is L^∞-bounded ((2.11)), Lebesgue's theorem gives

$$g_\varepsilon(v_{n,\varepsilon}) \to g_\varepsilon(v_\varepsilon) \qquad \text{in } L^p(Q_T),\ 1 \leqslant p < \infty,$$

hence $\chi_\varepsilon = (g_\varepsilon(v_\varepsilon))_t$ a.e.. Similarly, one can show that

$$g_\varepsilon(v_\varepsilon) \to g(v) = \int_0^v \sqrt{\beta'(s)}\,\mathrm{d}s = v^{\frac{m+1}{2m}} \text{ and } (g_\varepsilon(v_\varepsilon))_t \to \left(v^{\frac{m+1}{2m}}\right)_t.$$

3. Existence, continuous dependence, comparison, regularization and stabilization of solutions with bounded initial data

Definition 3.1. [1] *A solution u of Problem* (0.1) *on* $[0, T]$ *is a function u with the following properties*:
(i) $u \in C([0, T]; L^1(D)) \cap L^\infty(Q_T)$,
(ii) $\int_D u(t)\varphi(t)\,\mathrm{d}x - \iint_{Q_t} (u\varphi_t + u^m\varphi_{xx} + f(u)\varphi)\,\mathrm{d}x\,\mathrm{d}s = \int_D u_0\varphi(0)\,\mathrm{d}x,\ 0 < t \leqslant T$

for all $\varphi \in C^{2,1}(\bar{Q}_T)$ such that $\varphi \geqslant 0$, $\varphi = 0$ at $x = \pm L$ and $0 \leqslant t \leqslant T$. A solution on $[0, \infty)$ *means a solution on each* $[0, T]$, *a subsolution* (*supersolution*) *is defined by* (i) *and* (ii) *with equality replaced by* $\leqslant$ ($\geqslant$).

Remark 3.2. Clearly, the solution from Theorem 2.1. is also a solution in the sense of the above definition.

Theorem 3.3. *Let f be locally Lipschitz continuous.*
(i) *Let u, $\hat{u}$ be solutions of Problem* (0.1) *on* $[0, T]$ *with initial data u_0 and $\hat{u}_0$, respectively. Let K be a Lipschitz constant for f on* $[-M, M]$, *where* $M = \max(\|u\|_{L^\infty(Q_T)}, \|\hat{u}\|_{L^\infty(Q_T)})$. *Then*

$$\|u(t) - \hat{u}(t)\|_{L^1(D)} \leqslant e^{Kt}\|u_0 - \hat{u}_0\|_{L^1(D)},\ 0 \leqslant t \leqslant T.$$

(ii) *Let u be a subsolution and $\hat{u}$ a supersolution of Problem* (0.1) *with initial data u_0 and $\hat{u}_0$. Then if $u_0 \leqslant \hat{u}_0$, it follows that $u \leqslant \hat{u}$.*

Theorem 3.3. has been proved in [1]. In what follows we shall assume that f satisfies the hypotheses H.

Corollary 3.4 *Problem* (0.1) *has a nonnegative global solution u if $u_0 \in L^\infty(D)$, $u_0 \geqslant 0$.*

Proof. Lemma 1.4. yields that there is $q(\cdot, L')$, $L' \geqslant L$ such that $0 \leqslant u_0 \leqslant q(\cdot, L')$ in D. Take a sequence $\{u_{0n}\} \subset H_0^1(D)$ such that $0 \leqslant u_{0n} \leqslant q$ and $\|u_0 - u_{0n}\|_{L^1(D)} \to 0$ as $n \to \infty$. Let u_n be the solution of Problem (0.1) (in the sense of Definition 3.1.) with the initial function u_{0n}. From Theorem 3.3. we have

$$\sup_{0 \leqslant t \leqslant T} \|u_j(t) - u_k(t)\|_{L^1(D)} \leqslant e^{KT}\|u_{0j} - u_{0k}\|_{L^1(D)},$$

hence there is $u \in C([0, T]; L^1(D))$, $0 \leqslant u \leqslant q$ such that $u_n \to u$ in $C([0, T]; L^1(D))$ and u is the solution of Problem (0.1) with the initial function u_0, hence the conclusion.

In order to show the regularization and the stabilization of solutions we need the notion of sub- and supersolutions of Problem (1.1). A subsolution of Problem (1.1) is a function $v \in C([-L, L])$, $v(\pm L) \leqslant 0$ for which $\int_D (\varphi'' v^m + \varphi f(v))\,\mathrm{d}x \geqslant 0$ for all $\varphi \in C^2(\bar{D})$, $\varphi \geqslant 0$, $\varphi(\pm L) = 0$.
A supersolution is defined by reversing the inequality and requiring $v(\pm L) \geqslant 0$.

Theorem 3.5. *For each $\tau > 0$ and each supersolution $\bar{v}$ (of Problem (1.1)) there is a constant $M(\tau, \bar{v})$ such that for the solution u with $u_0 \leqslant \bar{v}$ the following assertions hold*:
(i) $(u^m)_x(t) \in L^\infty(D)$ *for* $t \geqslant \tau$,
(ii) $\|(u^m)_x(t)\|_{L^\infty(D)} \leqslant M(\tau, \bar{v})$ *and*

$$\textit{essential variation } (u^m)_x(t) \leqslant M(\tau, \bar{v}) \textit{ for } t \geqslant \tau.$$

This regularizing property of the equation and all other results in this section can be shown in a similar way as the analogous results in [1], with the only difference that we have no universal supersolution (of Problem (1.1)) like $\bar{v} = 1$ in [1]. But according to Lemma 1.4. for every $u_0 \in L^\infty(D)$ there is a number L' such that $u_0 \leqslant q(\cdot, L')$.

Let $u_0 \geqslant 0$, $u_0 \in L^\infty(D)$ and $u = u(t, u_0)$ be the solution of Problem (0.1) emanating from u_0. For each $\tau > 0$ define the semiorbit $\gamma_\tau(u_0) = \{u(t, u_0): t \geqslant \tau\}$. The theorem above yields that $\gamma_\tau(u_0) \subset X(\tau, \bar{v})$ if $u_0 \leqslant \bar{v}$, where $X(\tau, \bar{v})$ is the complete metric space consisting of those $w \in L^\infty(D)$ such that $0 \leqslant w \leqslant \bar{v}$, $(w^m)_x \in L^\infty(D)$, $\|(w^m)_x\|_{L^\infty(D)} \leqslant M(\tau, \bar{v})$, ess var $(w^m)_x \leqslant M(\tau, \bar{v})$, equipped with the metric

$$d(u, v) = \|u - v\|_{L^1(D)} + \|(u^m - v^m)_x\|_{L^2(D)}.$$

We also set $X(\bar{v}) = \{u \in L^\infty(D): 0 \leqslant u \leqslant \bar{v}, (u^m)_x \in L^2(D)\}$
equipped with the same metric. The compactness of $X(\tau, \bar{v})$ follows from the fact that a set which is bounded in L^∞ and in variation is precompact in L^1.

To study the asymptotic behaviour of $u(t, u_0)$ we introduce its ω-limit set:

$$\omega(u_0) = \{w \in X(\bar{v}): u(t_n, u_0) \to w \text{ in } X(\bar{v}) \text{ for some sequence } \{t_n\} \text{ with } t_n \to \infty \text{ as } n \to \infty\}.$$

The basic observations are collected in the next lemma.

Lemma 3.6. *Let $u_0 \leqslant \bar{v}$. Then*

(i) *$\gamma_\tau(u_0)$ is a precompact subset of $X(\bar{v})$ for $\tau > 0$.*
(ii) *$u(\cdot, u_0) \in C((0, \infty); X(\bar{v}))$.*
(iii) *$\omega(u_0)$ is nonempty and connected in $X(\bar{v})$.*
(iv) *If $w \in \omega(u_0)$, then $u(t, w) \in \omega(u_0)$ for $t \geqslant 0$.*

Theorem 3.7. *Let the hypotheses H hold. Then $\omega(u_0) \subset E(L)$.* This result can be derived from the inequality (2.2) in a similar way as in [1].

Lemma 3.8. *Let $u_0 \in [\underline{v}, \bar{v}] = \{w \in L^\infty(D) : \underline{v} \leqslant w \leqslant \bar{v}$ a.e. on D, $\underline{v}$, $\bar{v}$ are sub- and supersolution of Problem (1.1)$\}$. Then*

(i) *$u(t, u_0) \in [\underline{v}, \bar{v}]$ for $t \geqslant 0$,*
(ii) *$\omega(u_0) \subset [\underline{v}, \bar{v}] \cap X(\bar{v})$.*

Corollary 3.9. *If $u_0 \in L^\infty(D)$, $u_0 \in [\underline{v}, \bar{v}]$ and $[\underline{v}, \bar{v}] \cap E = \{g\}$, then $u(t, u_0) \to g$ in $X(\bar{v})$ as $t \to \infty$.*

This Corollary can be used to determine domains of attraction of 0 and $q(\cdot, L)$. These equilibria are stable and all the other elements of $E(L)$ are unstable. As an example we will construct a domain of attraction of $q(\cdot, L)$ for $L \in (L_0, L_1]$. Choose $l \in [L_0, L)$ and $y \in (-L, L)$ such that $-L \leqslant y - l$, $y + l \leqslant L$. Set

$$\underline{v}(x) = \begin{cases} p(x - y, l) & \text{for } x \in [y - l, y + l], \\ 0 & \text{if } x \notin [y - l, y + l]. \end{cases}$$

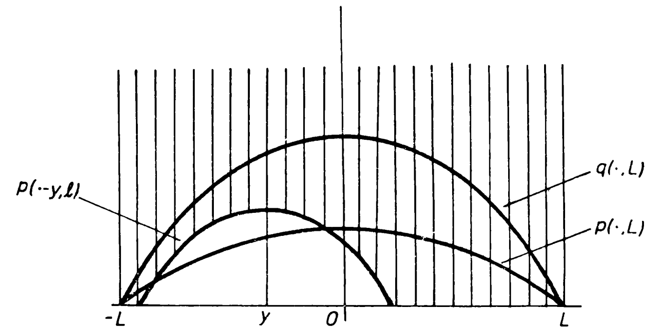

Fig. 2. Domain of attraction for $q(\cdot, L)$ where $L \in (L_0, L_1]$.

Then $\underline{v}$ is a subsolution of Problem (1.1). Since $l \in [L_0, L)$, we have

$$\underline{v}(y) = p(0, l) > p(0, L) \geqslant p(y, L)$$

If $u_0 \geqslant \underline{v}$ a.e. in D, then $u(t, u_0) \to q(\cdot, L)$ in $X(q(\cdot, L'))$ for some L' because

$$[\underline{v}, q(\cdot, L')] \cap E(L) = \{q(\cdot, L)\}.$$

We remark that $\bar{v} \equiv r_0$ is always a supersolution and $[0, r_0] \cap E(L) = \{0\}$, hence $u(t, u_0) \to 0$ if $u_0 \leqslant r_0$.

REFERENCES

[1] ARONSON, D. G., CRANDALL, M. G., PELETIER, L. A.: Stabilization of solutions of a degenerate nonlinear diffusion problem, Nonlinear Analysis 6, 1982, 1001—1022.

[2] GURTIN, M. E., MACCAMY, R. C.: On the diffusion of biological population, Mathematical Biosciences 33, 1977, 35—49.

[3] NAKAO, M.: Existence, nonexistence and some asymptotic behaviour of global solutions of a nonlinear degenerate parabolic equation, Math. Rep. College General Edc., Kyushu Univ. 14, 1983, 1—21.

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СТАБИЛИЗАЦИЯ РЕШЕНИЙ НЕКОТОРЫХ ВЫРОЖДАЮЩИХСЯ ДИФФУЗНЫХ УРАВНЕНИЙ В ОДНОЙ ПРОСТРАНСТВЕННОЙ ПЕРЕМЕННОЙ

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Резюме

В статье исследуется асимптотическое поведение решений возмущенного уравнения типа нестационарной фильтрации с одной пространственной переменной с однородными условиями Дирихле (0.1) при выполнении условый (H1)—(H5).