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RIGHT COMPOSITIONS OF SEMIGROUPS

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Let S be a semigroup containing a minimal left ideal. Then S contains a kernel K which is a union of all the minimal left ideals of S. If a is any element of S, then $K \cdot a$ is a left ideal of S but not necessarily a minimal left ideal of S.

In connection with some questions concerning random walks on semigroups prof. L. Schmetterer asked me some years ago to characterize those semigroups for which $K \cdot a$ is a minimal left ideal of S for all $a \in S$.

In this paper we first show that such semigroups can be described as right compositions of a special type of semigroups (denoted in this paper as U_l-semigroups).

The converse problem is the following: Given a family of U_l-semigroups we have to decide whether they admit at least one right composition (which is then a semigroup of the desired type).

Though there is a general method how to proceed in concrete cases (see [3]), the solution of this question in reasonably simple terms seems hopeless. Hence we restrict our considerations to some special cases.

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For convenience we define:

Definition. *A semigroup S containing a minimal left ideal (hence a kernel K) is called a W_l-semigroup if for any $a \in S$ the product $K \cdot a$ is a minimal left ideal of S.*

Example 1,1. A simple semigroup containing a minimal left ideal is a W_l-semigroup.

Example 1,2. The semigroup $S = \{a, b, c, d\}$ with the multiplication table

	a	b	c	d
a	a	a	c	c
b	a	a	c	c
c	a	a	c	c
d	a	a	c	d

contains two minimal left ideals $L_1=\{a\}$, $L_2=\{c\}$. The kernel is $K=\{a, c\}$, and S is a W_l-semigroup.

Example 1,3. If S is a W_l-semigroup and E is a right zero semigroup, then the direct product $S\times E$ is again a W_l-semigroup.

Example 1,4. Recall that a left ideal of S is called universally minimal if it is contained in every left ideal of S. A semigroup containing a universally minimal left ideal is a W_l-semigroup.

Semigroups of the type mentioned in Example 1,4 will be of decisive importance in the whole of this paper. We define therefore:

Definition. *A semigroup containing a universally minimal left ideal will be called a U_l-semigroup.*

Note that the minimal left ideal L of a U_l-semigroup S is the kernel of S and L itself is a left simple semigroup.

We first give some necessary conditions which a W_l-semigroup must satisfy.

Let S be a W_l-semigroup and $K=\bigcup_{\nu} L_\nu$, where $\{L_\nu\}_{\nu\in M}$ is the set of all minimal left ideals of S. For a fixed $\alpha\in M$ denote $S_\alpha=\{x\mid x\in S, Kx=L_\alpha\}$, hence $KS_\alpha=L_\alpha$. Clearly $S=\bigcup_{\nu\in M}S_\nu$ and $S_\alpha\cap S_\beta=\emptyset$ for $\alpha\neq\beta$.

The set S_α is a left ideal of S. For, $K(SS_\alpha)=(K\,S)S_\alpha=K\,S_\alpha=L_\alpha$, hence $S\cdot S_\alpha\subset S_\alpha$. In particular, we have $S_\beta S_\alpha\subset S_\alpha$ for any pair α, β.

Clearly $L_\alpha\subset S_\alpha$ and L_α is the unique minimal left ideal of S contained in S_α. For any $a\in S_\alpha$, $L_\alpha a$ is a minimal left ideal of S contained in S_α, hence $L_\alpha a=L_\alpha$.

We finally show that L_α is the universally minimal left ideal of S_α. Suppose that L'_α is any left ideal of S_α and $a'\in L'_\alpha$. We then have: $L_\alpha=L_\alpha a'\subset L_\alpha L'_\alpha\subset S_\alpha L'_\alpha\subset L'_\alpha$, hence any left ideal of S_α contains L_α.

We have proved:

Lemma 1,1. *If S is a W_l semigroup, then S can be written as a union of disjoint U_l-semigroups: $S=\bigcup_{\alpha\in M}S_\alpha$, where $S_\alpha S_\beta\subset S_\beta$ for any pair α, $\beta\in M$.*

In Example 1,2 we have $S=S_1\cup S_2$, where $S_1=\{a, b\}$ and $S_2=\{c, d\}$.

Conversely:

Lemma 1,2. *If a semigroup S can be written as a union of disjoint U_l-semigroups: $S=\bigcup_{\alpha\in M}T_\alpha$, and $T_\alpha T_\beta\subset T_\beta$ (for any pair α, $\beta\in M$), then S is a W_l-semigroup.*

Proof. Denote by L_α the kernel of T_α. We have $T_\alpha L_\alpha=L_\alpha$ and $SL_\alpha=\left\{\bigcup_{\nu}T_\nu\right\}T_\alpha L_\alpha\subset T_\alpha L_\alpha=L_\alpha$. Therefore L_α is a left ideal of S, hence a minimal left ideal of S (since it is minimal even in T_α).

The family $\{L_\nu\}_{\nu\in M}$ is exactly the set of all minimal left ideals of S. For, if L is a minimal left ideal of S, there exists some $\alpha\in M$ such that $T_\alpha\cap L\neq\emptyset$. Since $L\cap T_\alpha$

is a left ideal of S (and the more a left ideal of T_α) we have $L_\alpha \subset L \cap T_\alpha$, i.e. $L_\alpha \subset L$. Since both L and L_α are minimal left ideals of S, we conclude $L_\alpha = L$.

It follows that $K = \bigcup_{\nu \in M} L_\nu$ (the union of all minimal left ideals of S) is the kernel of S. For any $b \in S$, say $b \in T_\beta$, we have $Kb = \left(\bigcup_{\nu \in M} L_\nu\right)b = \bigcup_{\nu \in M} (L_\nu b)$. Since (for any $\nu \in M$) $L_\nu b$ is a minimal left ideal of S contained in T_β, we conclude $Kb = L_\beta$. Hence S is a W_l-semigroup.

Yoshida [4] and Petrich [3] introduced the following notion:

Definition. *Let $\{S_\nu\}_{\nu \in M}$ be a family of pairwise disjoint semigroups. We shall say that the family $\{S_\nu\}$ has a right composition if we can define on $S = \bigcup_{\nu \in M} S_\nu$ an associative multiplication (denoted by $*$) such that $S_\alpha * S_\beta \subset S_\beta$ for $\alpha \neq \beta$, while the multiplication in each S_α remains unaltered.*

S is then called a right composition of the family $\{S_\nu\}$. Given $\{S_\nu\}$ no right composition need exist or several right compositions may exist.

In this terminology Lemma 1,1 and Lemma 1,2 imply:

Theorem 1,1. *A semigroup S is a W_l-semigroup if and only if S is a right composition of U_l-semigroups.*

Remark. A U_l-semigroup S with the kernel L is right indecomposable, i.e. it cannot be written in the form of a union of two subsemigroups $S = T_1 \cup T_2$, $T_1 \cap T_2 = \emptyset$, where $T_1 T_2 \subset T_2$, $T_2 T_1 \subset T_1$. Since $ST_1 = (T_1 \cup T_2)T_1 = T_1^2 \cup T_2 T_1 \subset T_1$, and analogously $ST_2 \subset T_2$, both T_1, T_2 are left ideal of S. Since L is the minimal left ideal of S we have $L \subset T_1$, $L \subset T_2$, contrary to the assumption $T_1 \cap T_2 = \emptyset$.

The following follows directly from the proof of Lemma 1,2.

Lemma 1,3. *Let $\{S_\nu\}_{\nu \in M}$ be a family of disjoint U_l-semigroups and L_ν the kernel of S_ν. If $\{S_\nu\}$ has a right composition $S = \bigcup_{\nu \in M} S_\nu$, then each L_ν is a minimal left ideal of S and $K = \bigcup_{\nu \in M} L_\nu$ is the kernel of S.*

Suppose, as a special case, that one of the kernels L_ν in Lemma 1,3 contains an idempotent, hence L_ν is a left group. Then the kernel K of S, contains a minimal left ideal and an idempotent, hence it is completely simple. This implies that all L_ν, $\nu \in M$, are left groups, and all are isomorphic one to each other.

We state this explicitly:

Corollary 1,1. *If a family of U_l-semigroups $\{S_\nu\}_{\nu \in M}$ has a right composition and one of the kernels L_ν is a left group, then all L_ν are left groups and all are isomorphic one to the other.*

It follows, e.g., that two left groups which are not isomorphic cannot have a right composition.

The situation is quite different if we replace the words "left groups" by "left simple semigroups". It is well known that there exist simple semigroups S containing a minimal left ideal in which the minimal left ideals are not isomorphic. (The first such example has been given by M. Teissier, see [1].) Any such semigroup is, of course, a W_l-semigroup.

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The foregoing considerations lead in a natural way to the following problem. Suppose that S_α, S_β are two disjoint semigroups (not necessarily U_l-semigroups). We have to find all right compositions of S_α and S_β (if such exist). This problem has been studied in [4] and in a modified presentation in [3]. The procedure roughly described is the following.

Denote by $\Lambda(S_\alpha)$ and $\Lambda(S_\beta)$ the semigroup of left translations of S_α and S_β respectively. Find a homomorphic mapping Φ of S_α into $\Lambda(S_\beta)$ and a homomorphic mapping Ψ of S_β into $\Lambda(S_\alpha)$ (if such exist). For $a \in S_\alpha$, $b \in S_\beta$ write explicitly Φ: $a \mapsto \varphi^a \in \Lambda(S_\beta)$ and Ψ: $b \mapsto \psi^b \in \Lambda(S_\alpha)$. To obtain a right composition $S = S_\alpha \cup S_\beta$ put

$$a * b = \varphi^a(b), \quad b * a = \psi^b(a) .$$

Unfortunately, owing to the necessary associativity of multiplication, Φ and Ψ cannot be arbitrary. They have to satisfy two rather complicated conditions concerning the (individual) elements φ^a, ψ^b (for any a, b). Any right composition is obtained if Φ and Ψ are chosen in accordance with these conditions.

This is a very complicated procedure. The special case of S_α, S_β being U_l-semigroups seems not to have much influence on simplifying the procedure just described.

Hence we do not choose this approach. We prefer to consider some classes of semigroups in which a construction in a reasonably simple manner is possible or the non-existence of a right composition can be easily verified. Hereby we shall be interested primarily in U_l-semigroups.

The following Lemma is known. (See [3], p. 68.) We sketch the proof since the notations introduced here will be used in the sequel.

Lemma 2,1. *Let $\{S_\nu\}_{\nu \in M}$ be a family of pairwise disjoint isomorphic semigroups. Then the family $\{S_\nu\}$ has at least one right composition.*

Proof. Suppose that $1 \in M$. For every $\nu \in M$ let φ_ν be a fixed chosen isomorphism of S_1 onto S_ν. Define the mapping $S_\alpha \to S_\beta$ by $\varphi_{\alpha\beta} = \varphi_\alpha^{-1}\varphi_\beta$, i.e. for $a \in S_\alpha$, we put $a\varphi_{\alpha\beta} = a\varphi_\alpha^{-1}\varphi_\beta = [a\varphi_\alpha^{-1}]\varphi_\beta \in S_\beta$. Then $\varphi_{\alpha\beta}$ is an isomorphism and for any $a \in S_\alpha$ we have

$$a\varphi_{\alpha\beta}\varphi_{\beta\gamma} = a\varphi_\alpha^{-1}\varphi_\beta\varphi_\beta^{-1}\varphi_\gamma = a\varphi_\alpha^{-1}\varphi_\gamma = a\varphi_{\alpha\gamma} .$$

(Hereby $\varphi_{\alpha\alpha}$ is the identity mapping of S_α onto S_α.) The set of mappings $\{\varphi_{\mu\nu}\}$

satisfies $\varphi_{\alpha\beta}\varphi_{\beta\gamma}=\varphi_{\alpha\gamma}$. With this set of functions $\{\varphi_{\mu\nu}\}$ we now define for any $a\in S_\alpha$, $b\in S_\beta$, (including the case $\alpha=\beta$)

$$a*b=(a\varphi_{\alpha\beta})b\ .$$

It is a routine matter to verify that this multiplication is associative. Hence with this multiplication $\bigcup_{\nu\in M} S_\nu$ is a right composition of the given family $\{S_\nu\}$.

Remark 1. It is easy to see that $\bigcup_{\nu\in M} S_\nu$ is isomorphic with the direct product $S_1\times E$, where E is a right zero semigroup and card $E=$card M.

Remark 2. Suppose that in Lemma 2,1 the semigroups $\{S_\nu\}$ are (isomorphic) left groups. Then the right composition $S=\bigcup_\nu S_\nu$ constructed in Lemma 2,1 is a completely simple semigroup. This semigroup has a special property. If e_α is an idempotent of S_α, then $\varphi_{\alpha\beta}(e_\alpha)$ is necessarily an idempotent of S_β. If $e_\alpha=e_\alpha^2\in S_\alpha$, $e_\beta=e_\beta^2\in S_\beta$, then $e_\alpha*e_\beta=\varphi_{\alpha\beta}(e_\alpha)\cdot e_\beta=\varphi_{\alpha\beta}(e_\alpha)$. (We have used the fact that any idempotent of a left group L is a right identity of L.) Hence the product of two idempotents in S is an idempotent. It is well known that this need not be true for every completely simple semigroup. Hence the method used in Lemma 2,1 does not give all right compositions (even if φ_ν run over all possible isomorphisms $S_1\to S_\nu$). (This can be, of course, easily understood from the point of view of the Rees-matrix description of a completely simple semigroup. We shall not enter into a detailed description of this situation.)

Lemma 2,1 together with Corollary 1,1 implies:

Lemma 2,2. *A family of left groups has at least one right composition if and only if the members of the family are pairwise isomorphic.*

Remark 3. It should be once more emphasized that Lemma 2,2 does not hold if the words "left groups" are replaced by the words "left simple semigroups". At this writing I have no idea how to decide (in reasonably simple terms) under what conditions two non-isomorphic left simple semigroups without idempotents have a right composition.

Yoshida [4] has proved that a family of pairwise disjoint semigroups each with a right zero has at least one right composition.

This may lead to the suspicion that two U_l-semigroups with isomorphic kernels have at least one right composition. Example 2,1 below shows that this is not true.

We show this in a larger context inspired by a reasoning of Lallement—Petrich in [2].

Suppose that S_α, S_β are two disjoint semigroups containing an identity element ε_α and ε_β respectively. (Hence S_α, S_β are monoids.) Suppose that they have a right composition $S=S_\alpha\cup S_\beta$.

If $x\in S_\alpha$, the mapping $x\mapsto x\varepsilon_\beta$ is a homomorphism of S_α into S_β. For, if $x\in S_\alpha$,

$y \in S_\alpha$, then $x\varepsilon_\beta y\varepsilon_\beta = xy\varepsilon_\beta$. [This follows from the fact that $y\varepsilon_\beta \in S_\beta$, hence $\varepsilon_\beta y\varepsilon_\beta = y\varepsilon_\beta$.]

Also if $x \in S_\alpha$, the mapping $x \mapsto x\varepsilon_\beta\varepsilon_\alpha$ is a homomorphism of S_α into S_α. As a matter of fact if $x \in S_\alpha$, $y \in S_\alpha$, we have $x\varepsilon_\beta\varepsilon_\alpha \cdot y\varepsilon_\beta\varepsilon_\alpha = x\varepsilon_\beta(\varepsilon_\alpha y)\varepsilon_\beta\varepsilon_\alpha = x\varepsilon_\beta y\varepsilon_\beta\varepsilon_\alpha$. Since $y\varepsilon_\beta \in S_\beta$, we have $y\varepsilon_\beta = \varepsilon_\beta y\varepsilon_\beta$, so that $x\varepsilon_\beta\varepsilon_\alpha \cdot y\varepsilon_\beta\varepsilon_\alpha = xy\varepsilon_\beta\varepsilon_\alpha$.

We have

$$S_\alpha\varepsilon_\beta\varepsilon_\alpha \subset S_\beta\varepsilon_\alpha \subset S_\alpha \ , \tag{1}$$

and the inclusions here may be proper.

We now introduce the following class of monoids.

Definition. (Petrich [3].) *A monoid is called right unit-reductive if the identity map is the only (inner) right translation which is also a homomorphism.*

(In a monoid all right translations are inner. The kernel of such a semigroup cannot be a group.)

Lemma 2,3. *If S_α and S_β are right unit-reductive monoids, then a right composition $S_\alpha \cup S_\beta$ exists if and only if S_α, S_β are isomorphic monoids.*

Proof. With respect to Lemma 2,1, it is sufficient to prove the necessity. For $a \in S_\alpha$, the mapping $a \mapsto a\varepsilon_\beta\varepsilon_\alpha$ is a homomorphism of S_α into S_α. By suppostition $\varepsilon_\beta\varepsilon_\alpha = \varepsilon_\alpha$. The relation (1) implies $S_\beta\varepsilon_\alpha = S_\alpha$. Analogously we obtain $S_\alpha\varepsilon_\beta = S_\beta$. Let $a \in S_\alpha$, $b \in S_\beta$. The homomorphism $\Psi_{\alpha\beta}\colon S_\alpha \to S_\beta$ defined by $a \mapsto a\varepsilon_\beta$ and the homomorphism $\psi_{\beta\alpha}\colon S_\beta \to S_\alpha$ defined by $b \mapsto b\varepsilon_\alpha$ are mutually inverse one-to-one mappings since

$$a \overset{\Psi_{\alpha\beta}}{\mapsto} a\varepsilon_\beta \overset{\Psi_{\beta\alpha}}{\mapsto} a\varepsilon_\beta\varepsilon_\alpha = a\varepsilon_\alpha = a \ .$$

Hence S_α, S_β are isomorphic semigroups.

Example 2,1. Consider the semigroups $S_1 = \{e, a, b\}$ and $S_2 = \{E, A, B, C\}$ with the following multiplication tables:

	e	a	b
e	e	a	b
a	a	a	a
b	b	b	b

	E	A	B	C
E	E	A	B	C
A	A	A	A	A
B	B	B	B	B
C	C	B	B	B

Both are U_l-semigroups with a unit element and a kernel isomorphic to the two-element left zero semigroup. S_1 is right unit-reductive since the right translations ϱ_a, ϱ_b are not homomorphisms. We have, e.g., $ea \cdot ba \neq (eb)a$ and $eb \cdot ab \neq (ea)b$ · S_2 is right unit-reductive since the right translations ϱ_A, ϱ_B, ϱ_C are not homomorphisms. We have $EA \cdot BA \neq (EB)A$, $EB \cdot AB \neq (EA)B$ and $EC \cdot AC \neq (EA)C$.

Since S_1 and S_2 are not isomorphic, S_1 and S_2 cannot have a right composition.

Remark 4. Suppose that S_α and S_β are left simple semigroups without idempotents. Adjoin an identity element ε_α, ε_β to S_α and S_β respectively. Then S_α^1, S_β^1 are right unit-reductive semigroups. The semigroups S_α^1, S_β^1 have a right composition if and only if S_α^1, S_β^1 are isomorphic, hence if S_α, S_β are isomorphic.

Comparing with Remark 3 we see that a rather trivial modification (adjunction of an identity element) substantially changes the situation.

Remark 5. In the general theory of right compositions as developed in [3] the constructions simplify considerably if we suppose that the semigroups S_ν, $\nu \in M$, are right cancellative. For U_l-semigroups this condition is rather uninteresting since the following assertion holds:

Assertion. A right cancellative U_l-semigroup is a left group.

Proof. Let S be a U_l-semigroup with kernel L. The semigroup L is left simple and right cancellative. It is well known (see [1]) that this implies that L is a left group. Denote by e an idempotent of L. Then $L = Se$. Suppose for an indirect proof that $S - L \neq \emptyset$ and let $x \in S - L$. Then $xe \in L$ and since e is a right unit of L we have $xe \cdot e = xe$. By supposition this implies $xe = x$, hence $x \in L$, a contradiction. Therefore $S = L$.

3

Let $\{S_\nu\}_{\nu \in M}$ be a family of pairwise disjoint U_l-semigroups. We denote by L_ν the kernel of S_ν and we suppose that all L_ν, $\nu \in M$, are isomorphic left groups.

In this section we give a "reasonably simple" sufficient condition under which the family $\{S_\nu\}$ has at least one right composition. (See Theorem 3,1 below.)

If $e_\alpha = e_\alpha^2 \in L_\alpha$, then the mapping $S_\alpha \to L_\alpha$ defined by $a \mapsto ae_\alpha$ ($a \in S_\alpha$) is a mapping of S_α onto L_α which leaves the elements of L_α fixed.

Let be $a \in S_\alpha$, $b \in S_\beta$, $\alpha \neq \beta$, $e_\alpha = e_\alpha^2 \in L_\alpha$, $e_\beta = e_\beta^2 \in L_\beta$. The following is a natural way how to try to define a product $a * b$. We first project a into L_α, b into L_β (i.e. we consider $ae_\alpha \in L_\alpha$, $be_\beta \in L_\beta$). Next we introduce for the family of isomorphic semigroups $\{L_\nu\}_{\nu \in M}$ the set of isomorphisms $\{\varphi_{\mu\nu}\}$ defined in Lemma 2,1. Hereafter we define

$$a * b = (ae_\alpha)\varphi_{\alpha\beta} \cdot be_\beta \ .$$

Since $(ae_\alpha)\varphi_{\alpha\beta}$ is contained in L_β, further $L_\beta \cdot b = L_\beta$, and e_β is a right unit of L_β, this is equivalent to define

$$a * b = (ae_\alpha)\varphi_{\alpha\beta} \cdot b \tag{2}$$

We have to check the associativity.

If $c \in S_\gamma$ and $\alpha \neq \beta$, $\beta \neq \gamma$, we have

$$(a * b) * c = [(ae_\alpha)\varphi_{\alpha\beta} \cdot b] * c = [(ae_\alpha)\varphi_{\alpha\beta} \cdot b]\varphi_{\beta\gamma} \cdot c = \\ = (ae_\alpha)\varphi_{\alpha\gamma} \cdot (be_\beta)\varphi_{\beta\gamma} \cdot c ;$$

$$a * (b * c) = a * [(be_\beta)\varphi_{\beta\gamma} \cdot c] = (ae_\alpha)\varphi_{\alpha\gamma} \cdot (be_\beta)\varphi_{\beta\gamma} \cdot c .$$

Hence $(a * b) * c = a * (b * c)$.

The same is true if $\beta = \gamma$. In this case (with $b' \in S_\beta$) we have

$$a * (b * b') = (ae_\alpha)\varphi_{\alpha\beta} \cdot bb' ,$$

$$(a * b) * b' = [(ae_\alpha)\varphi_{\alpha\beta} be_\beta] * b' = (ae_\alpha)\varphi_{\alpha\beta} be_\beta b' . \tag{3}$$

Since $(ae_\alpha)\varphi_{\alpha\beta} \cdot b \in L_\beta$ we have $(ae_\alpha)\varphi_{\alpha\beta} be_\beta = (ae_\alpha)\varphi_{\alpha\beta} \cdot b$, and the term on the right hand of (3) is $(ae_\alpha)\varphi_{\alpha\beta} bb'$.

Unfortunately if $\alpha = \beta$ and $a, a' \in S_\alpha$, we have $(a * a') * b = (aa'e_\alpha)\varphi_{\alpha\beta} \cdot b$, $a * (a' * b) = a * [(a'e_\alpha)\varphi_{\alpha\beta} \cdot b] = (ae_\alpha)\varphi_{\alpha\beta} \cdot (a'e_\alpha)\varphi_{\alpha\beta} \cdot b = (ae_\alpha a'e_\alpha)\varphi_{\alpha\beta} \cdot b = (ae_\alpha a')\varphi_{\alpha\beta} \cdot b$. (The equality $ae_\alpha a'e_\alpha = ae_\alpha a'$ holds since $ae_\alpha a' \in L_\alpha$.)

Hence the associativity law for the multiplication holds if for any $\alpha, \beta \in M$

$$(aa'e_\alpha)\varphi_{\alpha\beta} \cdot b = (ae_\alpha a')\varphi_{\alpha\beta} \cdot b$$

$(a, a' \in S_\alpha, b \in S_\beta)$.

In particular putting $b = e_\beta$ we must have $(aa'e_\alpha)\varphi_{\alpha\beta} = (ae_\alpha a')\varphi_{\alpha\beta}$. Since $\varphi_{\alpha\beta}$ is an isomorphism of L_α onto L_β this implies $aa'e_\alpha = ae_\alpha a'$ for any $a, a' \in S_\alpha$, $e_\alpha \in L_\alpha$.

Conversely, if $aa'e_\alpha = ae_\alpha a'$ holds, then $(a * a') * b = a * (a' * b)$.

Clearly the mapping $x \mapsto xe_\nu (x \in S_\nu, e_\nu \in L_\nu)$ leaves the elements of L_ν fixed and it is an endomorphism of S_ν if and only if $xye_\nu = xe_\nu ye_\nu = xe_\nu y$ for any $x, y \in S_\nu$.

We have proved:

Lemma 3,1. *Under the suppositions introduced above the multiplication on* $\bigcup_{\nu \in M} S_\nu$ *defined by (2) is associative if and only if for each* $\nu \in M$, *the mapping* $x \mapsto xe_\nu (x \in S_\nu, e_\nu = e_\nu^2 \in L_\nu)$ *is an endomorphism of* S_ν *onto* L_ν.

Lemma 3,2. *If for some idempotent* $e \in L_\nu$ *the mapping* $x \mapsto xe$ *is an endomorphism, then the same is true for any idempotent* $e' \in L_\nu$.

Proof. Let be $x, y \in S_\nu$. The equality $xye = xey$ implies (putting $y = e'$) $xe'e = xee'$. Since e, e' are right units of L_ν, we have $xe = xe'$ for any $x \in S_\nu$. Hence $(xe')y = (xe)y = (xy)e = (xy)e'$.

Definition. *Let S be a* U_l*-semigroup with the kernel L. An endomorphism h of S onto L is called an L-endomorphism if h leaves the elements of L fixed.*

Lemma 3,3. *Let S be a* U_l*-semigroup the kernel of which is a left group L. Any L-endomorphism of S is of the form* $x \mapsto xe$, $x \in S$, $e = e^2 \in L$.

Proof. Let there be $x \in S$, $e = e^2 \in L$, and h an L-endomorphism. Then $xe \in L$, hence $h(xe) = xe$. This implies $h(x)h(e) = xe$ and since $h(e) = e$ and $h(x) \in L$, we have $h(x)h(e) = h(x)$, hence $h(x) = x \cdot e$.

Example 3,1. The mapping $x \mapsto xe$ need not be an endomorphism. Consider, e.g., the U_l-semigroup $S = \{e, a, b\}$ with the multiplication table

	e	a	b
e	e	a	b
a	a	a	a
b	b	b	b

None of the right translations ϱ_a, ϱ_b is an endomorphism. We have, e.g., $\varrho_a(eb) = \varrho_a(b) = ba = b$, while $\varrho_a(e)\varrho_a(b) = ea \cdot ba = a$.

Lemma 3,1 may be formulated as follows:

Lemma 3,4. *The multiplication on* $\bigcup_{v \in M} S_v$ *defined by* (2) *is associative if and only if each* S_v *has an* L_v*-endomorphism.*

This implies:

Theorem 3,1. *Let* $\{S_v\}_{v \in M}$ *be a family of* U_l*-semigroups, whereby the kernels of all* S_v *are isomorphic left groups. Suppose that each* S_v *has an* L_v*-endomorphism. Then there exists at least one right composition of this family.*

As a special case consider the case of each L_v being a group with the identity element e_v. Then (for $x \in S_v$) the mapping $x \mapsto xe_v$ is an L_v-homomorphism since (for any $x, y \in S_v$) we have $ye_v = e_v ye_v$, whence $xye_v = x(e_v ye_v) = (xe_v)(ye_v)$. This implies:

Theorem 3,2. *Let* $\{S_v\}_{v \in M}$ *be a family of* U_l*-semigroups. Suppose that the kernel of each* S_v *is a group. Then there exists at least one right composition of this family if and only if all the kernels are isomorphic groups.*

Remark 1. The semigroups S_v in Theorem 3,1 are exactly those semigroups which are ideal extensions of a left group L determined by a partial homomorphism.

The usefulness of Theorem 3,1 is underlined by the fact that there is a very simple method to decide whether a U_l-semigroup with a completely simple kernel has an L-endomorphism.

Theorem 3,3. *Let S be a* U_l*-semigroup the kernel of which is a left group L Denote by E the set of all idempotents of L. Then S has an L-endomorphism iff for every* $x \in S$ *we have* card $(xE) = 1$.

Proof. L can be written as a union of disjoint groups: $L = \bigcup_{\alpha \in A} T_\alpha$. Denote by ε_α the identity element of T_α, so that $E = \{\varepsilon_\alpha \mid \alpha \in A\}$.

a) Necessity. By the proof of Lemma 3,2 if $x \mapsto x\varepsilon_v (v \in A)$ is an L-endomorphism, and $x \in S$, we have $x\varepsilon_\alpha = x\varepsilon_\nu$ for all ε_α, $\alpha \in A$. Hence xE is a one-point set (depending, of course, on x).

b) Sufficiency. Suppose that the condition is satisfied. Let $x, y \in S$ and ε_β any element of E. Consider the product $xe_\beta ye_\beta$. The element ye_β is contained in a subgroup of L, say, $ye_\beta \in T_\gamma$. Hence $\varepsilon_\gamma ye_\beta = ye_\beta$. By supposition $x\varepsilon_\beta = x\varepsilon_\gamma$. Hence $x\varepsilon_\beta y\varepsilon_\beta = x\varepsilon_\gamma y\varepsilon_\beta = xy\varepsilon_\beta$. This shows that $x \mapsto x\varepsilon_\beta$ is an L-endomorphism of S. Theorem 3,3 is proved.

Example 3,2. Consider the following two U_l-semigroups S_1 and S_2:

	a	b	c
a	a	a	a
b	b	b	b
c	a	a	c

	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

Here (in both cases) $L = E = \{a, b\} \cdot S_1$ has an L-endomorphism since card $(cE) = 1$, S_2 has not an L-endomorphism since card $(cE) = 2$.

Remark 2. If a U_l-semigroup S contains a left (or two-sided) identity element, then S does not have an L-endomorphism unless L is a group.

Remark 3. If S is, e.g., a regular semigroup to find card (xS) it is not necessary to consider all $x \in S - L$. It is sufficient to check only the idempotents contained in $S - L$. For, any $x \in S$ has an idempotent right identity: $x = xe_x$, and $xE = x \cdot (e_xE)$. If card $(e_xL) = 1$, then card $(xE) = 1$. If card $(e_xL) > 1$, an L-endomorphism does not exist.

We conclude with one example using Theorem 3,1 and the multiplication (2).

Example 3,3. Consider the semigroups S_1 and S_2 given by the multiplication tables:

	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	a	a	c	c
d	a	a	d	d

	A	B	C
A	A	A	A
B	B	B	B
C	A	A	C

Here $L_1 = \{a, b\}$, $L_2 = \{A, B\}$. Both semigroups have an L-endomorphism. Choose the isomorphisms φ_{12} and φ_{21}, as $\varphi_{12} = \{a \mapsto A, b \mapsto B\}$ and $\varphi_{21} = \{A \mapsto a, B \mapsto b\}$. Next put in (2) $e_1 = a$, $e_2 = A$. We then have, e.g.,

$$d * C = (d \cdot a)\varphi_{12} \cdot C = a\varphi_{12} \cdot C = AC = A\ ,$$
$$C * d = (C \cdot A)\varphi_{21} \cdot d = A\varphi_{21} \cdot d = a \cdot d = a\ .$$

In this manner we obtain a right composition $S = S_1 \cup S_2$ described by the multiplication table:

	a	b	c	d	A	B	C
a	a	a	a	a	A	A	A
b	b	b	b	b	B	B	B
c	a	a	c	c	A	A	A
d	a	a	d	d	A	A	A
A	a	a	a	a	A	A	A
B	b	b	b	b	B	B	B
C	a	a	a	a	A	A	C

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ПРАВЫЕ КОМПОЗИЦИИ ПОЛУГРУПП

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Резюме

Пусть S — полугруппа, содержащая минимальный левый идеал, следовательно — ядро K. Изучается строение S в случае, когда для каждого $a \in S$ левый идеал $K \cdot a$ — минимальный левый идеал S. В этом случае S — объединение непересекающихся полугрупп:

$$S = \bigcup_{\nu} S_{\nu}, \quad \nu \in M\,.$$

При этом $S_{\alpha}S_{\beta} \subset S_{\beta}$ гля всяких $\alpha, \beta \in M$ и ядро полугруппы S_{ν} есть простая слева полугруппа.

Рассматриваются тоже частные случаи довольно сложной обратной задачи. Задана система полугрупп $\{S_{\nu}\}$, $\nu \in M$, с некоторыми естественными ограничениями. В множестве

$$\bigcup_{\nu \in M} S_{\nu} = S$$

требуется определить умножение (не меняя умножение в S_{ν}) так, чтобы S являлась полугруппой, в которой имеет место $S_{\alpha}S_{\beta} \subset S_{\beta}$ для всяких $\alpha, \beta \in M$.