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Vladimír Palko

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ON THE CONVERGENCE AND ABSOLUTE CONTINUITY OF SIGNED STATES ON A LOGIC

VLADIMÍR PALKO

Let $\mathscr{S}$ be a σ-algebra of subsets of a set X, μ a measure and v a signed measure on $\mathscr{S}$. v is said to be absolutely continuous with respect to μ (written $v \ll_\varepsilon \mu$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $|v(A)| < \varepsilon$ for all $A \in \mathscr{S}$. Let μ be a finite measure on $\mathscr{S}$ and v_n, $n = 1, 2, \ldots$, a sequence of finite signed measures on $\mathscr{S}$ such that $v_n \ll_\varepsilon \mu$ for all n and for every $A \in \mathscr{S}$ there exists a finite limit $\lim_{n \to \infty} v_n(A) = v(A)$. Then the well-known Vitali—Hahn—Saks theorem asserts that v is a signed measure, $v \ll_\varepsilon \mu$ as well and v_n are uniformly continuous with respect to μ, i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $|v_n(A)| < \varepsilon$, $n = 1, 2, \ldots$, for every $A \in \mathscr{S}$. The purpose of this paper is to investigate, whether this theorem holds for finite signed states on a logic. A counterexample is shown and a sufficient condition for the validity of this theorem is given. Finally, the case of finite signed states on the logic $\mathscr{L}(H)$ of all closed subspaces of a separable Hilbert space H is partially solved. A characterization of absolute continuity of nonnegative finite signed states on $\mathscr{L}(H)$ is given.

1. Preliminaries

We state some necessary definitions and known assertions which we shall use. Let $(\mathscr{L}, \leqq)$ be a partially ordered set with at least two elements, the largest element 1 and the smallest 0. Let $\mathscr{L}$ be equipped with an orthocomplementation $\perp : a \to a^\perp$, $a, a^\perp \in \mathscr{L}$, which satisfies

(i) $(a^\perp)^\perp = a$ for all $a \in \mathscr{L}$

(ii) if $a \leqq b$, then $b^\perp \leqq a^\perp$

(iii) $a \vee a^\perp = 1$, $a \wedge a^\perp = 0$ for all $a \in \mathscr{L}$.

We say that $a, b \in \mathscr{L}$ are orthogonal (written $a \perp b$) if $a \leqq b^\perp$. We shall assume that if $\{a_i\}_{i=1}^\infty$ is a sequence of pairwise orthogonal elements of $\mathscr{L}$, then $\bigvee_{i=1}^\infty a_i$ exists in $\mathscr{L}$. Finally, we insist that in $\mathscr{L}$ there holds the orthomodular identity: $a \leqq b$ implies

$b = a \vee (b \wedge a^{\perp})$. The set $\mathscr{L}$ satisfying all the conditions above will be called a *logic*. We note that a logic need not be a lattice. Two elements $a, b \in \mathscr{L}$ are said to be compatible if there exist pairwise orthogonal elements $a_1, b_1, c \in \mathscr{L}$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. A *Boolean sub-σ-algebra* $\mathscr{B}$ of a logic $\mathscr{L}$ ([6]) is a subset of $\mathscr{L}$ which satisfies the following conditions:

(i) $0, 1 \in \mathscr{B}$
(ii) $a, b \in \mathscr{B}$ implies that a, b are compatible, $a \vee b \in \mathscr{B}$, $a \wedge b \in \mathscr{B}$
(iii) if $a \in \mathscr{B}$, then $a^{\perp} \in \mathscr{B}$
(iv) if $\{a_i\}_{i=1}^{\infty}$ is a sequence of pairwise orthogonal elements of $\mathscr{B}$, then $\bigvee_{i=1}^{\infty} a_i \in \mathscr{B}$.

Every Boolean sub-σ-algebra is also a Boolean σ-algebra in the usual sense. Every Boolean sub-σ-algebra is contained in a maximal ([6]). A very important case of logic is the logic $\mathscr{L}(H)$, whose elements are closed subspaces of a separable Hilbert space H over real or complex scalars. The partial ordering in $\mathscr{L}(H)$ is given by the usual set inclusion and for every $M \in \mathscr{L}(H)$ $M^{\perp}$ is the orthogonal complement of M. A collection $\mathscr{Q}$ of subsets of the set X is called a *σ-class* if $\mathscr{Q}$ satisfies:

(i) $\emptyset, X \in \mathscr{Q}$
(ii) $A \in \mathscr{Q}$ implies $X - A \in \mathscr{Q}$
(iii) if $\{A_i\}_{i=1}^{\infty}$ is a sequence of mutually disjoint sets from $\mathscr{Q}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{Q}$.

One can easily verify that $\mathscr{Q}$ is a logic, where the partial ordering is also given by the set inclusion and $A^{\perp}$ is defined as $X - A$. A σ-class $\mathscr{Q}$ is not a lattice in general. Let $\mathscr{Q}_t$ be the σ-class of subsets of the set X_t for every $t \in \Delta$, Δ is an index set. Let $X = \prod_{t \in \Delta} X_t$ be the cartesian product of X_t, $t \in \Delta$. A σ-class $\mathscr{Q}$ of subsets of X is called the *σ-class product* of $\mathscr{Q}_t$ $\left(\text{written } \mathscr{Q} = \prod_{t \in \Delta} \mathscr{Q}_t\right)$ if $\mathscr{Q}$ is the smallest σ-class containing sets $\Pi_t^{-1}(E)$ for every $t \in \Delta$, $E \in \mathscr{Q}_t$. Denote $\mathscr{A}_t = \{\Pi_t^{-1}(E), E \in \mathscr{Q}_t\}$. It is evident that if $\mathscr{Q}_t$ is a σ-algebra, then $\mathscr{A}_t$ is a maximal Boolean sub-σ-algebra of the logic $\mathscr{Q}$. A *signed state* on a logic $\mathscr{L}$ is a map m from $\mathscr{L}$ into $R \cup \{\infty\} \cup \{-\infty\}$ such that *(i)* $m(0) = 0$, *(ii)* $m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$ if a_i, $i = 1, 2, \ldots$, are mutually orthogonal elements of $\mathscr{L}$. A nonnegative signed state m is called a *state* if $m(1) = 1$. Let ω be a nonnegative signed state, m a signed state on $\mathscr{L}$; then m is said to be absolutely continuous with respect to ω if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(a) < \delta$ implies $|m(a)| < \varepsilon$ for every $a \in \mathscr{L}$. We say that m is dominated by ω (written $m \ll \omega$) if $\omega(a) = 0$ implies $m(a) = 0$ for all $a \in \mathscr{L}$.

2. The Vitali—Hahn—Saks theorem for signed states on a logic

Dvurečenskij ([1]) has given a simple proof that a finite limit of a sequence of finite signed states on a logic is also a signed state, but the following example shows that further assertions of the Vitali—Hahn—Saks theorem are not valid without some other assumptions.

Example 2.1. Let $\mathcal{Q}_i$ be a nontrivial σ-class of subsets of X_i, $i = 1, 2, \ldots$. Let us choose from every set X_i two points p_i^1, p_i^2 in such way that there exists a set $A_i \in \mathcal{Q}_i$, which separates p_i^1, p_i^2, i.e. $p_i^1 \in A_i$, $p_i^2 \in X_i - A_i$. It is possible owing to the nontriviality of $\mathcal{Q}_i$. Let $\mathcal{Q}$ be the σ-class product of $\mathcal{Q}_i$, $i = 1, 2, \ldots$. Define now the states ω, m, m_n, $n = 1, 2, \ldots$ on $\mathcal{Q}$ as follows:

$$\omega(\emptyset) = m_n(\emptyset) = m(\emptyset) = 0, \quad \omega(X) = m_n(X) = m(X) = 1, \quad n = 1, 2, \ldots.$$

If $E \in \mathcal{Q}$, $\emptyset \neq E \neq X$, then there exists a unique positive integer i and set $A \in \mathcal{Q}_i$ so that $E = \Pi_i^{-1}(A)$. Then we define

$$\omega(E) = \sum_{p_i^j \in A} a_i^j, \quad \text{where} \quad a_i^1 = \frac{1}{2i}, \quad a_i^2 = 1 - \frac{1}{2i}$$

$$m(E) = \sum_{p_i^j \in A} \frac{1}{2}$$

$$m_n(E) = m(E) \quad \text{if} \quad n \geqq i$$

$$m_n(E) = \omega(E) \quad \text{if} \quad n < i, \quad n = 1, 2, \ldots.$$

It may be verified without difficulty that $m_n \ll_\varepsilon \omega$, $n = 1, 2, \ldots$ and $\lim_{n \to \infty} m_n(E) = m(E)$ for every $E \in \mathcal{Q}$, but m_n are not uniformly continuous with respect to ω and m is not absolutely continuous with respect to ω.

Remark 2.1. It is evident that such counterexample may be constructed also for the product $\mathcal{Q} = \prod_{t \in \Delta} \mathcal{Q}_t$, where Δ is not countable.

Remark 2.2. From the counterexample above it immediately follows that the absolute continuity of finite signed states does not coincide with the notion of dominancy in general.

The following type of the Vitali—Hahn—Saks theorem on a logic $\mathcal{L}$ is true.

Theorem 2.1. *Let $\mathcal{L}$ be a logic containing only a finite number of maximal Boolean sub-σ-algebras. Let ω be a nonnegative finite signed state, m_n a sequence of finite signed states on $\mathcal{L}$ such that $m_n \ll_\varepsilon \omega$, $n = 1, 2, \ldots$, and for every $a \in \mathcal{L}$ there exists a finite limit $\lim_{n \to \infty} m_n(a) = m(a)$. Then m_n are uniformly continuous with respect to ω and m is a signed state, $m \ll_\varepsilon \omega$.*

Proof: By assumption there exist maximal Boolean sub-σ-algebras $\mathcal{B}_i$ of $\mathcal{S}$, $i=1, 2, \ldots, k$ such that $\mathcal{S}=\bigcup_{i=1}^{k}\mathcal{B}_i$. The Loomiss theorem ([4]) of the representation of Boolean σ-algebras asserts that every Boolean σ-algebra $\mathcal{B}$ is a σ-homomorphic image of a σ-algebra $\mathcal{S}$ of subsets of a set X. Using this theorem one gets sets X_i, σ-algebras $\mathcal{S}_i$ of subsets of X_i and maps h_i: $\mathcal{S}_i \to \mathcal{B}_i$, which are surjective σ-homomorphisms, $i=1, 2, \ldots, k$. Now we define on $\mathcal{S}_i$ real functions μ_i, ν_{ni}, ν_i, $n=1, 2, \ldots$, $i=1, 2, \ldots, k$, in the following way: $\nu_i(A)=m(h_i(A))$, $\nu_{ni}(A)=m_n(h_i(A))$, $\mu_i(A)=\omega(h_i(A))$, $n=1, 2, \ldots$, $A \in \mathcal{S}_i$, $i=1, 2, \ldots, k$. Evidently μ_i is a finite measure on $\mathcal{S}_i$, ν_{ni} is a sequence of finite signed measures on $\mathcal{S}_i$, $\nu_{ni} \ll_\varepsilon \mu_i$, $\lim_{n\to\infty} \nu_{ni}(A)=\nu_i(A)$ for every $A \in \mathcal{S}_i$, $i=1, 2, \ldots, k$. Using the classical Vitali—Hahn—Saks theorem we get immediately that ν_{ni} are uniformly continuous with respect to μ_i, ν_i is a signed measure, $\nu_i \ll_\varepsilon \mu_i$, $i=1, 2, \ldots, k$. If $\varepsilon>0$ is given, then there exists $\delta_i>0$ so that $A \in \mathcal{S}_i$, $\mu_i(A)<\delta_i$ implies $|\nu_{ni}(A)|<\varepsilon$, $n=1, 2, \ldots$. Denote $\delta=\min\{\delta_1, \ldots, \delta_k\}$. Let $\omega(a)<\delta$. A positive integer j and a set $A \in \mathcal{S}_j$ exist so that $a=h_j(A)$. Then $\omega(a)=\omega(h_j(A))=\mu_j(A)<\delta \leqq \delta_j$. Thus $|m_n(a)|=|\nu_{nj}(A)|<\varepsilon$, $n=1, 2, \ldots$. We have shown that m_n are uniformly continuous with respect to ω. If $|m_n(a)|<\varepsilon$, then also $|m(a)|\leqq\varepsilon$. Hence $m \ll_\varepsilon \omega$. The theorem is proved.

3. Convergence and absolute continuity of signed states on $\mathcal{L}(H)$

Throughout this section H denotes a separable Hilbert space over complex or real scalars. All the following theorems will be proved for the complex case. Their proofs for the real case are the same or simpler. The deep theorem of Gleason ([3]) asserts that every finite nonnegative signed state m on $\mathcal{L}(H)$, where $\dim H \geqq 3$, is of the form $m(M)=\operatorname{tr} TP^M$, $M \in \mathcal{L}(H)$, where T is a hermitean positive operator of the trace class and P^M is the projector corresponding to M. A hermitean operator T on H is an operator of the trace class if there exists an orthonormal basis $\{\varphi_i\}$ such that $\sum_i |(T\varphi_i, \varphi_i)|<\infty$. Then the sum $\operatorname{tr} T=\sum_i (T\varphi_i, \varphi_i)$ is called a trace of T and it is independent of the used basis ([5]). Dvurečenskij ([2]) generalized Gleason's theorem and proved that every finite signed state on $\mathcal{L}(H)$ is of the form $m(M)=\operatorname{tr} TP^M$, where T is a hermitean (not necessarily positive) operator of the trace class. We prove now that if H is finite dimensional, $\dim H \neq 2$, then the Vitali—Hahn—Saks theorem is valid on $\mathcal{L}(H)$.

Theorem 3.1. *Let H be finite dimensional, $\dim H \neq 2$. Let ω be a non-negative finite signed state, $\{m_n\}_{n=1}^{\infty}$ a sequence of finite signed states such that $m_n \ll_\varepsilon \omega$,*

$n = 1, 2, \ldots,$ *and for every* $M \in \mathscr{L}(H)$ *there exists a finite limit* $\lim_{n\to\infty} m_n(M) = m(M)$. *Then* m_n *are uniformly continuous with respect to* ω *and* m *is a signed state,* $m \ll_F \omega$.

Proof: Since the case $\dim H = 1$ is trivial, suppose $\dim H = p \geqq 3$. Let s be a finite signed state on $\mathscr{L}(H)$; then there exists a hermitean operator T such that $s(M) = \sum_{q=1}^{p} (TP^M\varphi_q, \varphi_q)$, where $\{\varphi_q\}_{q=1}^{p}$ is an orthonormal basis in H. Suppose that φ_q are eigenvectors of T and γ_q the corresponding eigenvalues; then s is of the form $s(M) = \sum_{q=1}^{p} \gamma_q (P^M\varphi_q, \varphi_q)$. Let $\{e_i\}_{i=1}^{p}$ be any fixed orthonormal basis of H; then all φ_q are linear combinations of $e_1, \ldots, e_p$, i.e. $\varphi_q = \sum_{i=1}^{p} \lambda_i^q e_i$, $q = 1, \ldots, p$. Applying that expression one gets

$$s(M) = \sum_{q=1}^{p} \gamma_q \left(P^M \sum_{i=1}^{p} \lambda_i^q e_i, \sum_{j=1}^{p} \lambda_j^q e_j\right) =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{q=1}^{p} \gamma_q \lambda_i^q \bar{\lambda}_j^q (P^M e_i, e_j), \quad M \in \mathscr{L}(H).$$

Denote $\alpha_{ij} = \sum_{q=1}^{p} \gamma_q \lambda_i^q \bar{\lambda}_j^q$; then

$$s(M) = \sum_{i=1}^{p} \sum_{j=1}^{p} \alpha_{ij} (P^M e_i, e_j), \quad \alpha_{ij} = \bar{\alpha}_{ji}, \quad i, j = 1, \ldots, p.$$

Thus all signed states m_n may be expressed as follows:

$$m_n(M) = \sum_{i=1}^{p} \sum_{j=1}^{p} \beta_{ij}^n (P^M e_i, e_j), \quad \beta_{ij}^n = \bar{\beta}_{ji}^n.$$

The limit function m is also a signed state, thus m is also of the form

$$m(M) = \sum_{i=1}^{p} \sum_{j=1}^{p} \beta_{ij} (P^M e_i, e_j), \quad \beta_{ij} = \bar{\beta}_{ji}.$$

Let M_k be the one-dimensional subspace of H generated by the vector e_k. Evidently $m_n(M_k) = \beta_{kk}^n$, $n = 1, 2, \ldots,$ and $m(M_k) = \beta_{kk}$. Hence

$$\lim_{n\to\infty} \beta_{kk}^n = \beta_{kk}, \quad k = 1, \ldots, p. \tag{1}$$

Suppose M_{kq} to be the subspace generated by the vector $e_k + e_q$. Then

$$m_n(M_{kq}) = \beta_{kq}^n (P^{M_{kq}} e_k, e_q) + \beta_{qk}^n (P^{M_{kq}} e_q, e_k) + \beta_{kk}^n (P^{M_{kq}} e_k, e_k) + \beta_{qq}^n (P^{M_{kq}} e_q, e_q) \tag{2}$$

and

$$m(M_{kq}) = \beta_{kq}(P^{M_{kq}}e_k, e_q) + \beta_{qk}(P^{M_{kq}}e_q, e_k) + \beta_{kk}(P^{M_{kq}}e_k, e_k) + \beta_{qq}(P^{M_{kq}}e_q, e_q). \tag{3}$$

We can easily verify that $P^{M_{kq}}e_k = P^{M_{kq}}e_q = \frac{1}{2}(e_k + e_q)$. The last and (1), (2), (3) imply

$$\lim_{n\to\infty} \beta^n_{kq} + \beta^n_{qk} = \beta_{kq} + \beta_{qk}. \tag{4}$$

Similarly, using the one-dimensional subspace N_{kq} generated by the vector $ie_k + e_q$ (i is the imaginar unit) one gets

$$\lim_{n\to\infty} \beta^n_{kq} - \beta^n_{qk} = \beta_{kq} - \beta_{qk}. \tag{5}$$

As an immediate consequence of (4) and (5) one obtains

$$\lim_{n\to\infty} \beta^n_{kq} = \beta_{kq}, \quad k, q = 1, \ldots, p.$$

Let us prove now the uniform continuity of m_n with respect to ω. If $\varepsilon > 0$ is given, then there exists a positive integer r such that $|\beta^r_{ij} - \beta^n_{ij}| < \frac{\varepsilon}{2p^2}$ for $n > r$, $i, j = 1, 2, \ldots, p$, and there exist δ_n such that $\omega(M) < \delta_n$ implies $|m_n(M)| < \frac{\varepsilon}{2}$, $n = 1, \ldots, r$. Denote $\delta = \min\{\delta_1, \ldots, \delta_r\}$. Suppose $\omega(M) < \delta$. If $n \leqq r$, $|m_n(M)| < \frac{\varepsilon}{2} < \varepsilon$, and if $n > r$, then

$$|m_n(M)| \leqq |m_r(M)| + |m_n(M) - m_r(M)| < \frac{\varepsilon}{2} + \sum_{i=1}^{p} \sum_{j-1}^{p} |\beta^n_{ij} - \beta^r_{ij}| < \varepsilon.$$

Thus $|m_n(M)| < \varepsilon$ for all n, hence also $|m(M)| \leqq \varepsilon$. The uniform continuity of m_n and the absolute continuity of m with respect to ω are proved.

Remark 3.1. If $\dim H = 2$, then the Vitali—Hahn—Saks theorem is not valid. It follows from Remark 2.1. and from the fact that in this case $\mathscr{S}(H)$ is isomorphic with the σ-class product $\prod_{t \in \langle 0, 1 \rangle} \mathscr{B}_t$, where $\mathscr{B}_t$ is the potence set of any two-element set.

A question arises as to what the situation is if H is not finite dimensional. We shall prove only that if m_n are non-negative, $m_n \ll_\varepsilon \omega$, then also $m \ll_\varepsilon \omega$. The proof is based on a characterization of absolute continuity of non-negative signed states. The following theorem gives such a characterization.

Theorem 3.2. *Let ω, m be non-negative finite signed states on $\mathscr{L}(H)$, $\dim H \neq 2$, $\omega(M) = \sum_i \lambda_i (P^M \varphi_i, \varphi_i)$ and $m(M) = \sum_j \xi_j (P^M \psi_j, \psi_j)$, where $\{\varphi_i\}$, $\{\psi_j\}$ and $\{\lambda_i\}$, $\{\xi_j\}$ are orthonormal systems (not necessarily complete) of eigenvectors and sequences of (positive) eigenvalues of operators corresponding to ω and m, respectively. The following statements are equivalent:*

(i) $m \ll_\varepsilon \omega$

(ii) $m \ll \omega$

(iii) For every j ψ_j is an element of the subspace M_1 generated by the vectors $\{\varphi_i\}$

Proof: We shall assume that the number of φ_i and ψ_j is infinite. Of course, the proof in the case of finite number of φ_i or ψ_j is more simple. We can also assume that the sequence $\{\lambda_i\}_{i=1}^\infty$ is decreasing.

The proof of *(i)* $\Rightarrow$ *(ii)* is trivial.

(ii) $\Rightarrow$ *(iii)* Let N_1 be the orthogonal complement of M_1. Evidently $\omega(N_1) = 0$, thus $m(N_1) = \sum_{j=1}^\infty \xi_j (P^{N_1}\psi_j, \psi_j) = 0$. Hence $(P^{N_1}\psi_j, \psi_j) = \|P^{N_1}\psi_j\|^2 = 0$, $j = 1, 2, \ldots$. Thus ψ_j is orthogonal to N_1 and hence $\psi_j \in M_1$, $j = 1, 2, \ldots$.

(iii) $\Rightarrow$ *(i)* Choose any positive integer k, denote $m_k(M) = (P^M \psi_k, \psi_k)$, $M \in \mathscr{L}(H)$. We shall prove $m_k \ll_\varepsilon \omega$. $\{\varphi_i\}_{i=1}^\infty$ is a complete orthonormal system in M_1, hence $\psi_k = \sum_{i=1}^\infty (\psi_k, \varphi_i)\varphi_i$. Denote $\psi_{ik} = \sum_{j=i+1}^\infty (\psi_k, \varphi_j)\varphi_j$. Then $\lim_{i\to\infty} \|\psi_{ik}\| = 0$. If $\varepsilon > 0$ is given, then i_1 exists such that $\|\psi_{i_1 k}\| < \frac{\sqrt{\varepsilon}}{2}$. Denote $K = \max\{|(\psi_k, \varphi_1)|, \ldots, |(\psi_k, \varphi_{i_1})|\}$, and $K_1 = i_1^2 \frac{K^2}{\sqrt{\lambda_{i_1}}} + 2i_1 \frac{K}{\sqrt{\lambda_{i_1}}}$. Choose $\delta = \frac{\varepsilon^2}{4K_1^2}$. If $\omega(M) < \delta$, then $\lambda_i \|P^M \varphi_i\|^2 < \frac{\varepsilon^2}{4K_1^2}$, $i = 1, 2, \ldots$. The last implies

$$\|P^M \varphi_i\|^2 < \frac{\varepsilon^2}{4K_1^2 \lambda_{i_1}}, \quad i = 1, \ldots, i_1$$

and hence

$$\|P^M \varphi_i\| < \frac{\varepsilon}{2K_1\sqrt{\lambda_{i_1}}}, \quad i = 1, 2, \ldots, i_1.$$

We estimate now $m_k(M)$ using known properties of the scalar product, the Schwarz inequality and the fact that the projector P^M is linear and hermitean.

$$m_k(M) = \left(P^M\left(\sum_{i=1}^{i_1}(\psi_k, \varphi_i)\varphi_i + \psi_{i_1 k}\right), \sum_{i=1}^{i_1}(\psi_k, \varphi_i)\varphi_i + \psi_{i_1 k}\right)$$

$$\leqq \sum_{i=1}^{i_1}\sum_{j=1}^{i_1} |(\psi_k, \varphi_i)|\,|(\psi_k, \varphi_j)|\,\|P^M\varphi_i\| + 2\sum_{i=1}^{i_1}|(\psi_k, \varphi_i)|\,|(\psi_{i_1 k}, P^M\varphi_i)| +$$

$$+\|P^M\psi_{i_1k}\|^2 < i_1^2K^2\frac{\varepsilon}{2K_1\sqrt{\lambda_{i_1}}} + 2i_1K\frac{\varepsilon}{2K_1\sqrt{\lambda_{i_1}}} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $m_k \ll_\varepsilon \omega$ for every positive integer k. Then also $\sum_{j=1}^{n}\xi_j m_j \ll_\varepsilon \omega$ for every n. If $\varepsilon > 0$ is given, there exist $\delta > 0$ and a positive integer n_1 such that $\sum_{j=n_1+1}^{\infty}\xi_j < \frac{\varepsilon}{2}$ and $\omega(M) < \delta$ implies $\sum_{j=1}^{n_1}\xi_j(P^M\psi_j, \psi_j) < \frac{\varepsilon}{2}$. Hence

$$m(M) = \sum_{j=1}^{n_1}\xi_j(P^M\psi_j, \psi_j) + \sum_{j=n_1+1}^{\infty}\xi_j(P^M\psi_j, \psi_j) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The absolute continuity of m with respect to ω is shown.

Now it is easy to prove the following theorem.

Theorem 3.3. *Let ω be a non-negative finite signed state, $\{m_n\}_{n=1}^{\infty}$ a sequence of non-negative finite signed states on $\mathscr{L}(H)$, $\dim H \neq 2$, $m_n \ll_\varepsilon \omega$ for every n. Let there for every $M \in \mathscr{L}(H)$ exists a finite limit $\lim_{n\to\infty} m_n(M) = m(M)$. Then $m \ll_\varepsilon \omega$.*

Proof: It is sufficient to prove $m \ll_\varepsilon \omega$. If $\omega(M) = 0$, then $m_n(M) = 0$ and so $m(M) = \lim_{n\to\infty} m_n(M) = 0$. Thus $m \ll \omega$.

Remark 3.2. Evidently the above theorem is also valid if m_n are non-positive.

Remark 3.3. Theorem 3.2. does not hold in general if m is not non-negative. In the following example m is neither non-negative nor non-positive and the implication *(ii) $\Rightarrow$ (iii)* is not valid.

Example 3.1. Denote $H = R^3$, $\varphi = (-1, 1, 0)$, $\psi_1 = (1, 0, 0)$, $\psi_2 = (0, 1, 0)$, $\omega(M) = (P^M\varphi, \varphi)$, $m(M) = (P^M\psi_1, \psi_1) - (P^M\psi_2, \psi_2)$, $M \in \mathscr{L}(H)$. Evidently $m \ll \omega$, but ψ_1, ψ_2 are not elements of the one-dimensional subspace generated by φ.

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Katedra matematiky
Elektrotechnickej fakulty
Slovenskej vysokej školy technickej
Gottwaldovo nám. 19
812 19 Bratislava

О СХОДИМОСТИ И АБСОЛЮТНОЙ НЕПРЕРЫВНОСТИ ОБОБЩЕННЫХ СОСТОЯНИЙ НА ЛОГИКЕ

Vladimír Palko

Резюме

В этой работе мы изучаем абсолютную непрерывность предела сходимой последовательности обобщенных состояний на логике и равномерную абсолютную непрерывность членов этой последовательности. Мы показываем, что для обобщенных состояний на логике не верна теорема Витали, Хана и Сакса, и даём достаточное условие для верности этой теоремы. Дальше изучается абсолютная непрерывность обобщенных состояний на логике $\mathscr{L}(H)$, которая состоит из замкнутых подпространств сепарабельного пространства Гильберта. Дана характеризация абсолютной непрерывности конечных неотрицательных обобщенных состояний на этой логике.