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ON THE TRIPARTITE CONJECTURE

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A complete tripartite graph $K(A, B, C)=K_{m,n,s}$, where m, n, s are positive integers, is a graph whose vertex set is the union of pairwise disjoint sets A, B, C (called parts of this graph) of cardinality m, n and s, respectively. Two vertices u and v of $K_{m,n,s}$ are adjacent if only if they belong to different parts.

An isomorphic factorisation of a graph $G=(V, E)$ is a partition $\{E_1, \dots, E_t\}$ of the edge set of G such that the spanning subgraphs $(V, E_1), \dots, (V, E_t)$ are all isomorphic to each other. Let G/t denote the set of graphs which occur as factors in an isomorphic factorisation of G into exactly t factors. We say G is divisible by t, written $t|G$, if G/t is not empty.

Harary, Robinson and Wormald in [2] investigated for which t a complete tripartite graph $K_{m,n,s}$ is divisible by t. They proved that if $t=2$ or $t=4$, then $K_{m,n,s}/t$ is not empty, and if $t>1$ (odd), $m\geqslant t(t+1)$ and t divides $2m+1$, then $K_{1,1,m}/t$ is empty.

The authors of [2] expressed the following conjecture.

Tripartite conjecture. Consider a complete tripartite graph $K_{m,n,s}$ and an integer $t>1$. If for all m, n and s the condition $t|mn+ms+ns$ implies the existence of a graph in $K_{m,n,s}/t$, then t is even, and conversely.

S. Quinn has just proved the tripartite conjecture for $t=6$. We shall prove that the conjecture holds if at least two parts have equal numbers of vertices. For the standard graph theoretic terminology we follow the book by Harary [1].

Theorem. *Let t be even and $t|m(m+2s)$. Then $K_{m,m,s}$ is divisible by t.*

Proof. Suppose $m(m+2s)$ is divisible by t. Since t is even, m must be even. At first we shall construct a graph in $K_{m,m,s}/4$.

Let A_1, A_2, B_1, B_2 and C be pairwise disjoint vertex sets such that A_1, A_2, B_1 and B_2 have cardinality $m/2$ each and C has cardinality s, and let $A=A_1\cup A_2$ and $B=B_1\cup B_2$. Define spanning subgraphs G_i $(i=1, 2, 3, 4)$ of $K(A, B, C)$ in the following way: $G_1=K(B_1, A_2\cup C)$, $G_2=K(B_2, A_1\cup C)$, $G_3=K(A_1, B_1\cup C)$ and $G_4=K(A_2, B_2\cup C)$. The edge sets of G_i partition the edge set of $K(A, B, C)$. Clearly G_i are all isomorphic to $K_{m/2, m/2+s}$ and hence the latter graph is in $K_{m,m,s}/4$. The graphs G_i are illustrated in the Figure. Here each letter represents a vertex set and each edge between two sets represents the inclusion of all edges joining the two sets. We consider two cases.

Case 1. Let $t \equiv 0 \pmod 4$, $t = 4 \cdot t_1$. Since by the hypothesis t divides $m(m+2s)$, then $(m/2)(m/2+s)$ must be divisible by t_1. As G_i is a complete bipartite graph, according to Theorem 1 from [2] there exists a graph G in G_i/t_1. Evidently, G is also in $K_{m,m,s}/t$.

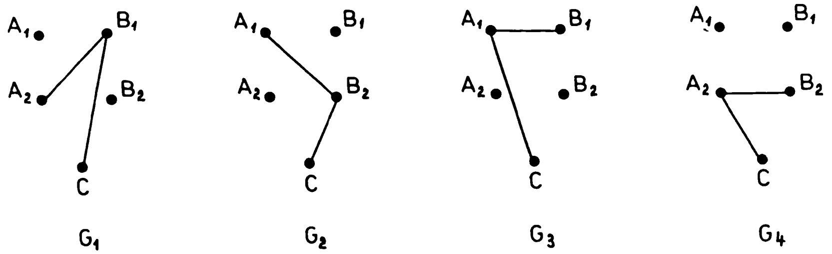

Fig. 1

Case 2. Let $t \equiv 2 \pmod 4$, $t = 2 \cdot t_2$, $t_2 \equiv 1 \pmod 2$. According to the assumption t divides $m(m+2s)$ so that t_2 divides $2(m/2)(m/2+s)$. As t_2 is odd, $(m/2)(m/2+s)$ must be divisible by t_2.

Let $H_1 = G_1 \cup G_2$ and $H_2 = G_3 \cup G_4$. Then H_1 (as well as H_2) contains $2(m/2)(m/2+s)$ edges and since t_2 divides $(m/2)(m/2+s)$, we have $t_2 = a \cdot b$ for some a and b such that a divides $m/2$ and b divides $m/2+s$.

Let X_r, $Y_r (r=1, 2, \ldots, b)$ and U_j, B_j $(j=1, 2, \ldots, a)$ be vertex sets such that each X_r or Y_r has cardinality $(m/2+s)/b$ and each U_j or V_j has $m/(2a)$; let

$$A_2 \cup C = \bigcup_{r=1}^{b} X_r, \quad A_1 \cup C = \bigcup_{r=1}^{b} Y_r, \quad B_1 = \bigcup_{j=1}^{a} U_j, \quad B_2 = \bigcup_{j=1}^{a} V_j$$

and

$$Y_1 = X_b, \quad Y_2 = X_{b-1}, \ldots, Y_k = X_{b-k+1},$$

where k is the greatest integer such that $k(m/2+s)/b \leqslant s$. In the case of $k(m/2+s)/b < s$, let Y_{k+1} and X_{b-k} be such that $Y_{k+1} \cap X_{b-k} = \emptyset$, $\bigcup_{i=1}^{k+1} Y_i \supseteq C$ and $\bigcup_{i=b-k}^{b} X_i \supseteq C$.

We want for $b > 1$ to construct from sets X_r, Y_r set sequences $\{M_i\}_{i=1}^{b}$ and $\{N_i\}_{i=1}^{b}$ such that members of different sequences with equal indices will be disjoint. If $X_n \cap Y_n = \emptyset$, where $n = [(b+1)/2]$, put $M_i = X_i$ and $N_i = Y_i$ for every $i = 1, 2, \ldots, b$, and if $X_n \cap Y_n \neq \emptyset$, put $M_i = X_i$ $(i = 1, 2, \ldots, b)$ and

$$N_1 = Y_1, \quad N_2 = Y_2, \ldots, N_{n-1} = Y_{n-1}, \quad N_n = Y_{n+1}, \quad N_{n+1} = Y_n, \quad N_{n+2} = Y_{n+2}, \ldots, N_b = Y_b.$$

In the case of $b = 1$ put $M_1 = A_2 \cup C$ and $N_1 = A_1 \cup C$.

Define spanning subgraphs G_{ij} of H_1/t_2 as follows: for every ordered couple $(i, j) \in \{1, 2, \dots, b\} \times \{1, 2, \dots, a\}$ the graph $G_{ij} = K(M_i, U_j) \cup K(N_i, V_j)$. Graphs $K(M_i, U_j)$ [or $K(N_i, V_j)$] are complete bipartite graphs with parts M_i and U_j [or N_i and V_j, respectively]. It is clear that graphs G_{ij} are edge-disjoint and form a factorisation of H_1. Furthermore, each M_i or N_i has cardinality $(m/2+s)/b$ and U_j or V_j has cardinality $m/(2a)$. Clearly, G_{ij} are all isomorphic to $2K_{m/(2a),\,(m/2+s)/b}$. Hence the latter graph is in H_1/t_2. As H_1 is isomorphic to H_2 and $H_1 \cup H_2 = K(A, B, C)$, we have $t \mid K_{m,m,s}$.

REFERENCES

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О 3-ДОЛЬНОЙ ГИПОТЕЗЕ

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Резюме

В статье доказывается 3-дольная гипотеза при условии, если по крайней мере, две доли полного 3-дольного графа имеют одинаковое число вершин.