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*Mathematica Slovaca*, Vol. 35 (1985), No. 3, 217--222

Persistent URL: http://dml.cz/dmlcz/136393

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# THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS FOR FINSLER STRUCTURES

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## 1. Introduction

It is well known that the calculus of variations enables us to characterize many interesting properties of various geometrical structures; important examples are the variational theory of geodesics of connections in a Riemann or a Finsler space [3], [4], the theory of extremals in spaces of supporting vector densities [5], etc. With respect to the inverse problem, as to under what conditions the equations of geodesics of a given connection can be regarded as the equations of extremals of an integral variational functional, it seems that till now no explicit results have been obtained.

The present paper is concerned with the inverse problem for connections on the tangent bundle of a differential manifold. It is known that on a Finsler space there exists a connection whose geodesics coincide with the extremals of the Finsler structure, such that the covariant derivative of the metric tensor relative to this connection vanishes (the Cartan connection). Our contribution consists in showing that also the converse is true in the sense that if a connection on the tangent bundle is metrizable, it is precisely the Cartan connection of a Finsler structure. We also show that the equations of geodesics of a linear connection coincide with the Euler—Lagrange equations of a lagrangian if and only if the connection is metrizable (without positivity assumption).

## 2. Connections on the tangent bundle

Let $X$ be an $n$-dimensional smooth manifold. Recall the definition of the bundle of linear connections over $X$ [1]. Denote by $F^2X$ the principal $L^2_n$-bundle of 2-frames over $X$. The structure group $L^2_n$ of this bundle is the group of invertible 2-jets with source and target at the origin $0 \in R^n$ of the real, $n$-dimensional Euclidean space $R^n$. If $j^2_0\alpha \in L^2_n$, $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$, then the formulas $b^i_j(j^2_0\alpha) = D_j(\alpha^{-1})^i(0)$, $b^i_{jk}(j^2_0\alpha) = D_jD_k(\alpha^{-1})^i(0)$, $1 \leq i, j, k \leq n$, $j \leq k$, define a global coordinate system on $L^2_n$, and we set $a^i_j(j^2_0\alpha) = b^i_j(j^2_0\alpha^{-1})$ so that $a^i_jb^j_k = \delta^i_k$ (the Kronecker symbol). Put $Q = R^n \otimes (R^{n*} \odot R^{n*})$, where $R^n$ is considered with its natural vector space structure, $R^{n*}$ denotes the dual vector space, and $\odot$ is the symmetrized tensor product, and denote by $\Gamma^i_{jk}$, $1 \leq i, j, k \leq n$, $j \leq k$, the canonical coordinates on $Q$. Writing

$$\bar{\Gamma}^i_{jk} = a^i_p (b^q_j b^r_k \Gamma^p_{qr} + b^p_{jk}) \tag{2.1}$$

we obtain a left action of $L^2_n$ on $Q$ which defines a fiber bundle with type fiber $Q$, associated with $F^2X$. This fiber bundle is called the *bundle of linear connections over X*, and is denoted by $\Gamma X$. We note that in (2.1) as well as throughout this paper, the Einstein summation convention is used.

Let $TX$ be the tangent bundle of $X$. By a *connection* on $TX$ we mean a morphism $\Gamma$: $TX \to \Gamma X$ over $id_X$. A *geodesic* of a connection $\Gamma$ is a curve in $X$ satisfying, in each of the coordinates $x^i$ on $X$, the system of equations

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \tag{2.2}$$

where $\Gamma^i_{jk}$ are the components of $\Gamma$ relative to the coordinates $x^i$, and "dot" denotes differentiation with respect to parameter.

Denote by $T^r_s X$ the bundle of tensors over $X$, contravariant with respect to the first $r$ indices, and covariant with respect to the remaining $s$ indices. Given a connection $\Gamma$: $TX \to \Gamma X$, the *covariant derivative* $\nabla_\Gamma h$: $TX \to T^r_{s+1} X$ of a morphism $h$: $TX \to T^r_s X$ is defined in a standard manner. In particular, let $g$: $TX \to T^0_2 X$ be a morphism over $id_X$. Then $\nabla g$: $TX \to T^0_3 X$ is defined, in any coordinates $x^i$ on $X$, by

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial \dot{x}^s} \Gamma^s_{rk} \dot{x}^r - g_{im} \Gamma^m_{jk} - g_{jm} \Gamma^m_{ik}, \tag{2.3}$$

where $x^i$, $\dot{x}^i$ are the coordinates on $TX$ associated with $x^i$.

## 3. Variationality of a linear connection

Let $\Gamma$ be a *linear* connection on a manifold $X$, i.e., a section of the fiber bundle $\Gamma X$, $\Gamma^i_{jk}$ the components of $\Gamma$ with respect to some coordinates $x^i$ on $X$. Consider the equations of geodesics (2.2). For any *regular* tensor field $g$ of type (0, 2) on $X$ whose components with respect to $x^i$ are denoted by $g_{ij}$, i.e., such that $\det(g_{ij}) \neq 0$, (2.2) is equivalent with the equations

$$-\varepsilon_i = g_{im} (\ddot{x}^m + \Gamma^m_{pq} \dot{x}^p \dot{x}^q) = 0. \tag{3.1}$$

We shall say that the linear connection $\Gamma$ is *variational* if there exists a function $L$: $TX \to R$ (a *lagrangian* for (2.2)) and a regular tensor $g$ such that (3.1) are the Euler—Lagrange equations of $L$.

Recall that the expressions $\varepsilon_i = \varepsilon_i(x^j, \dot{x}^j, \ddot{x}^j)$ are the Euler—Lagrange expressions of a lagrangian depending, in general, on $x^j, \dot{x}^j, \ddot{x}^j$, if and only if

$$\frac{\partial \varepsilon_i}{\partial x^k} - \frac{\partial \varepsilon_k}{\partial x^i} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^k} - \frac{\partial \varepsilon_k}{\partial \dot{x}^i}\right) = 0, \tag{3.2}$$

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^k} + \frac{\partial \varepsilon_k}{\partial \dot{x}^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^k} + \frac{\partial \varepsilon_k}{\partial \ddot{x}^i}\right) = 0, \tag{3.3}$$

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^k} - \frac{\partial \varepsilon_k}{\partial \ddot{x}^i} = 0 \tag{3.4}$$

(see [2]), [6]).

**Theorem 1.** *A necessary and sufficient condition that the linear connection $\Gamma$ be variational is that there exists a regular tensor $f$ of type* (0, 2) *on $X$ such that in any coordinates $x^i$ on $X$,*

$$g_{ij} = g_{ji}, \tag{3.5}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right). \tag{3.6}$$

Proof. Assume that the equations (2.2), where $\Gamma^i_{jk}$ are components of a linear connection, are variational, and take a tensor $g$ such that $\varepsilon_i$ (3.1) is the Euler—Lagrange expression of a lagrangian. Then the relations (3.2)—(3.4) hold; (3.4) gives

$$g_{ij} = g_{ji}, \tag{3.7}$$

i.e., $g$ is a symmetric tensor; (3.3) implies

$$g_{ij}\Gamma^j_{pk} + g_{kj}\Gamma^j_{pi} - \frac{\partial g_{ik}}{\partial x^i} = 0 \tag{3.8}$$

from which (3.6) follows. It is readily verified that because of these two relations, (3.2) is satisfied identically.

Conversely, if $g$ is symmetric and (3.6) holds, we set

$$L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j, \tag{3.9}$$

which defines a lagrangian for (3.1); that is, (2.2) is variational. This completes the proof.

We note that the lagrangian (3.9) can be obtained from (3.1) by the standard Tonti construction in the normal coordinates of $g$.

## 4. Variationality of a connection on the tangent bundle

Let us briefly recall the notion and basic properties of the *Cartan connection* associated with a Finsler structure on $TX$, defined by a metric function $L\colon TX \to R$. Put in any coordinates $x^i$ on $X$

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial \dot{x}^i \partial \dot{x}^j} \,. \tag{4.1}$$

$g_{ij}$ are the components of a morphism $g\colon TX \to T^0_2 X$ over $id_X$ which is called the *metric tensor* of $L$. By the well-known properties of $L$, $\det(g_{ij}) \neq 0$, that is, $g$ is *regular*, and

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{jk}}{\partial \dot{x}^i} = \frac{\partial g_{ki}}{\partial \dot{x}^j}, \qquad \frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k = 0. \tag{4.2}$$

Denoting by $g^{ij}$ the elements of the inverse matrix of $(g_{ij})$ we further put

$$\gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right) . \tag{4.3}$$

The Euler—Lagrange equations of $L$ are then expressed by

$$g_{im}(\ddot{x}^m + \gamma^m_{pq} \dot{x}^p \dot{x}^q) = 0. \tag{4.4}$$

The *Cartan connection* associated with $L$ is a connection $\Gamma\colon TX \to \Gamma X$ defined by

$$\Gamma^m_{jk} = g^{mi} \Gamma_{i,jk}, \tag{4.5}$$

$$\Gamma_{i,jk} = g_{im}\gamma^m_{jk} - \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial \dot{x}^s} \gamma^s_{lk} + \frac{\partial g_{ik}}{\partial \dot{x}^s} \gamma^s_{lj} - \frac{\partial g_{jk}}{\partial \dot{x}^s} \gamma^s_{li} \right) \dot{x}^l$$
$$+ \frac{1}{4} g^{st} \left( \frac{\partial g_{tj}}{\partial \dot{x}^s} \frac{\partial g_{ik}}{\partial \dot{x}^m} + \frac{\partial g_{tk}}{\partial \dot{x}^s} \frac{\partial g_{ij}}{\partial \dot{x}^m} - \frac{\partial g_{kj}}{\partial \dot{x}^s} \frac{\partial g_{ti}}{\partial \dot{x}^m} \right) \gamma^m_{pq} \dot{x}^p \dot{x}^q .$$

This is a unique connection on $TX$ for which $\nabla_\Gamma g = 0$ (see (2.3)). Moreover, by (4.2),

$$\Gamma^m_{pq} \dot{x}^p \dot{x}^q = \gamma^m_{pq} \dot{x}^p \dot{x}^q . \tag{4.6}$$

Hence the geodesics of $\Gamma$ are precisely the extremals of the Finsler structure $L$.

These remarks serve as a motivation for the following definitions. Let $\Gamma$ be a connection on $TX$, $x^i$ any coordinates on $X$, $\Gamma^i_{jk}$ the components of $\Gamma$ with respect to these coordinates. We say that $\Gamma$ is (locally) *variational* if the following condition holds: there exists a system of functions $\gamma^i_{jk}$ of $x^p$, $\dot{x}^p$ such that

$$\gamma^i_{jk} \dot{x}^j \dot{x}^k = \Gamma^i_{jk} \dot{x}^j \dot{x}^k , \tag{4.7}$$

and a regular mapping $g\colon TX \to T_2^0X$ over $id_X$, whose components are denoted by $g_{ij}$, such that the functions

$$\varepsilon_i = -g_{im}(\ddot{x}^m + \gamma^m_{pq}\dot{x}^p\dot{x}^q) \tag{4.8}$$

are the Euler—Lagrange expressions of a lagrangian $L = L(x^i, \dot{x}^i, \dot{x}^i)$. We say that $\Gamma$ is *metrizable* if there exists a regular, positive-definite mapping $g\colon TX \to T_2^0X$ over $id_X$ such that (1) $g_{ij} = g_{ji}$, (2) $(\partial g_{ij}/\partial \dot{x}^k)\cdot \dot{x}^j = 0$, and (3) $\nabla_\Gamma g = 0$. We note that the properties of a metrizable connection reflect the properties of the Cartan connection.

Let us denote

$$\gamma_i = g_{im}\Gamma^m_{pq}\dot{x}^p\dot{x}^q. \tag{4.9}$$

**Theorem 2.** *A necessary and sufficient condition that $\Gamma$ be variational is that there exists a regular mapping $g\colon TX \to T_2^0X$ over $id_X$ such that in any coordinates $x^i$ on $X$*

$$g_{ij} - g_{ji} = 0, \tag{4.10}$$

$$\frac{\partial \gamma_i}{\partial \dot{x}^k} + \frac{\partial \gamma_k}{\partial \dot{x}^i} - 2\frac{\partial g_{ik}}{\partial x^l}\dot{x}^l = 0, \tag{4.11}$$

$$\frac{\partial g_{im}}{\partial \dot{x}^k} - \frac{\partial g_{km}}{\partial \dot{x}^i} = 0, \tag{4.12}$$

$$\frac{\partial \gamma_i}{\partial x^k} - \frac{\partial \gamma_k}{\partial x^i} - \frac{1}{2}\frac{\partial}{\partial x^m}\left(\frac{\partial \gamma_i}{\partial \dot{x}^k} - \frac{\partial \gamma_k}{\partial \dot{x}^i}\right)\dot{x}^m = 0. \tag{4.13}$$

Proof. Substituting (3.1) and (4.9) in (3.2)—(3.4) and omitting the dependent relations one immediately obtains (4.10)—(4.13).

**Theorem 3.** *Each metrizable connection is variational. More precisely, a metrizable connection is the Cartan connection of a Finsler structure.*

Proof. Let $\Gamma$ be a metrizable connection on $TX$, $g\colon TX \to T_2^0X$ a morphism satisfying the requirements (1)—(3) (see the definition of a metrizable connection). Put

$$\gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right). \tag{4.14}$$

Using (3) in the form (2.3) we obtain by means of cyclic permutations

$$2\gamma_{i,jk} - 2g_{im}\Gamma^m_{jk} - \frac{\partial g_{ij}}{\partial \dot{x}^s}\Gamma^s_{rk}\dot{x}^r - \frac{\partial g_{ik}}{\partial \dot{x}^s}\Gamma^s_{rj}\dot{x}^r + \frac{\partial g_{jk}}{\partial \dot{x}^s}\Gamma^s_{ri}\dot{x}^r = 0. \tag{4.15}$$

By (2),

$$(\gamma_{i,pq} - g_{im}\Gamma^m_{pq})\dot{x}^p\dot{x}^q = 0. \tag{4.16}$$

Hence the left hand side expressions of the equations of geodesics of $\Gamma$ can be expressed in the form

$$\varepsilon_i = -g_{im}(\ddot{x}^m + \Gamma^m_{pq}\dot{x}^p\dot{x}^q) = -g_{im}\ddot{x}^m - \gamma_{i,pq}\dot{x}^p\dot{x}^q. \tag{4.17}$$

It is readily verified that $\varepsilon_i$ are the Euler—Lagrange expressions of the lagrangian $L = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j$.

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Received February 7, 1983



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## ОБРАТНАЯ ВАРИАЦИОННАЯ ЗАДАЧА ДЛЯ ПРОСТРАНСТВ ФИНСЛЕРА


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### Резюме


В работе показывается, что всякая метризуемая связность на касательном пространстве является связностью Картана некоторой структуры Финслера и что линейная связность на многообразии вариационная тогда и только тогда, когда она метризуемая.