Mathematica Slovaca

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Mathematica Slovaca, Vol. 34 (1984), No. 3, 281--294

Persistent URL: http://dml.cz/dmlcz/136361

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ON QUASIIDENTITIES OF TRANSITIVE QUASIGROUPS

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It is well known that a quasigroup $(Q, \cdot)$ is an isotope of an abelian group iff $(Q, \cdot)$ satisfies the condition of Thomsen $a\cdot b=d\cdot e$ and $a\cdot c=f\cdot e$ implies $d\cdot c=f\cdot b$ and $(Q, \cdot)$ is an isotope of a group iff $(Q, \cdot)$ satisfies the condition of Reidemeister $a\cdot b=c\cdot d$, $a\cdot e=c\cdot f$ and $x\cdot b=y\cdot d$ implies $x\cdot e=y\cdot f$. Similar to these conditions are the necessary and sufficient conditions we give for a quasigroup $(Q, \cdot)$ in order that the quasigroup be quasilinear. Such quasigroups are generalizations of linear ones (and also T-quasigroups) that were studied by J. Ježek and T. Kepka in [6] (P. Nemec and T. Kepka in [8, 10]).

This work was inspired by [4] where the author studied invariants of an isotopy $(\alpha, \beta, 1)$ of a group, where at least one α or β is a quasiautomorphism of the group.

1. Notations and preliminaries

If $(Q, \cdot)$ $(=Q$ if it does not lead to misunderstanding) is a quasigroup, then define $a\backslash b=c$ iff $a=c\cdot b$ iff $c\backslash a=b$. Then $(Q, /)$, $(Q, \backslash)$ are called the inverse quasigroups to $(Q, \cdot)$. For any $a\in Q$, L_a, R_a, T_a will be the translations by a, i.e.

$$L_ax=a\cdot x,\quad R_ax=x\cdot a,\quad T_ax=x\backslash a\quad \text{for all}\quad x \text{ in } Q.$$

Then

$$L_a^{-1}x=a\backslash x,\quad R_a^{-1}x=x/a,\quad T_a^{-1}x=a/x.$$

We shall use the following notations for a quasigroup $(Q, \cdot)$:

$$\mathcal{T}=\{L, R, T, L^{-1}, R^{-1}, T^{-1}\}^*;$$

$$Q({}^0X, {}^1X, \ldots, {}^nX)=\{{}^0X_{a_0}\,{}^1X_{a_1}\ldots{}^nX_{a_n}\colon a_i\in Q,\ i\in\{0, 1, \ldots, n\}\}$$

for some fixed ${}^iX\in\mathcal{T}$, $i\in\{0, 1, \ldots, n\}$ (thus ${}^iX_{a_i}$ is a translation of Q).

* The notations R_a, L_a, T_a, $\mathcal{T}$ etc. will be used only if the basic operation is written $(\cdot)$. We shall write R_a°, L_a°, T_a°, $\mathcal{T}^\circ$ etc. when another symbol (say $\circ$) is used.

Let $(Q, \circ)$ be a group and $\varphi(\psi)$ its automorphism (antiautomorphism). Then we shall say that $L_s^\circ\varphi(L_s^\circ\psi)$ is a quasiautomorphism (antiquasiautomorphism) of the group $(Q, \circ)$. In [1] it is shown that if $\gamma = L_s^\circ\varphi$ is a quasiautomorphism of $(Q, \circ)$, then there exists an automorphism ξ of $(Q, \circ)$ such that $\gamma = R_s^\circ\xi$. Analogously, for any antiautomorphism ψ there exists an antiautomorphism η such that $L_s^\circ\psi = R_s^\circ\eta$.

Theorem 1.1. *Let $(Q, \circ)$ be a group with the identity e. Let δ, η be fixed permutations of Q and let y be an arbitrary element in Q. Then*

(i) *$R_y^\circ\delta = \delta R_{\eta y}^\circ$ (dually $L_y^\circ\delta = \delta L_{\eta y}^\circ$) implies that η is an automorphism and δ is a quasiautomorphism of $(Q, \circ)$ such that $\delta = L_{\delta e}^\circ\eta^{-1}$;*

(ii) *$R_y^\circ\delta = \delta L_{\eta y}^\circ$ implies that η is an antiautomorphism and δ is an antiquasiautomorphism of $(Q, \circ)$ such that $\delta = R_{\delta e}^\circ\eta^{-1}$.*

Proof. By the assumption, $\delta x \circ y = \delta(x \circ \eta y)$ for all x, y in Q. If $x = e$ (e is the identity of $(Q, \circ)$), then $\delta e \circ y = \delta\eta y$, i.e. $L_{\delta e}^\circ = \delta\eta$, whence $\delta = L_{\delta e}^\circ\eta^{-1}$. Further, for every x, y, z in Q we have

$$\delta(x \circ \eta(y \circ z)) = \delta x \circ (y \circ z) = (\delta x \circ y) \circ z = \delta(x \circ \eta y) \circ z = \delta(x \circ \eta y \circ \eta z).$$

If again $x = e$, then $\eta(y \circ z) = \eta y \circ \eta z$, i.e. η is an automorphism of $(Q, \circ)$. (ii) The proof is similar.

Let a group $(Q, \circ)$, with the identity e, be an isotope of a quasigroup $(Q, \cdot)$ by the rule $x \cdot y = \alpha x \circ \beta y$. It is easy to verify that for every x, y, $z \in Q$,

$$(1.1) \qquad L_x = L_{\alpha x}^\circ\beta, \quad R_y = R_{\beta y}^\circ\alpha, \quad T_z = \beta^{-1}IL_z^{\circ -1}\alpha, \quad IR_a = L_a^{\circ -1}I,$$

where the map $I: Q \to Q$ is defined by $x \circ Ix = e$ and $z^{-1} = Iz$.

Definition 1.1. *Let $(Q, \circ)$ be a group and let α, β be permutations of Q. A quasigroup $(Q, \cdot)$, where*

$$(1.2) \qquad (\cdot) = (\circ)^{(\alpha, \beta, 1)},$$

is called quasilinear if at least one of α, β, $\alpha\beta^{-1}$, $\alpha^{-1}\beta$ is an quasiautomorphism or antiquasiautomorphism of $(Q, \circ)$. If $\alpha(\beta)$ is a quasiautomorphism of $(Q, \ )$, then $(Q, \cdot)$ is called the left-hand (right-hand) linear. A quasigroup that is both, the left-hand and the right-hand linear is called linear. If moreover $(Q, \circ)$ is an abelian group, then $(Q, \cdot)$ is called T-quasigroup.

2. Quasiidentities of transitive quasigroups

Definition 2.1. *Let $m \geq 1$ be any integer and let $\{\delta\}$, A_0, A_1, ..., A_m be sets of mappings. We shall say that $\mathscr{A} = (A_0, A_1, ..., A_m)$ has the property $\delta(n)$ (or $\mathscr{A}$ is $\delta(n)$) $(1 \leq n \leq m)$ if for an arbitrary integer t and for every $\varphi_{t+0} \in A_{t+0}$, $\varphi_{t+1} \in A_{t+1}$, ..., $\varphi_{t+n-1} \in A_{t+n-1}$ there exist $\varphi_{t+n} \in A_{t+n}$, ..., $\varphi_{t+m} \in A_{t+m}$ (operation +*

takes $\mathrm{mod}(m+1)$) *such that* $\varphi_0\varphi_1\ldots\varphi_m=\delta$. *If* $\{\delta\}$, $A_0, A_1, \ldots, A_m$, *are sets of the mappings* $Q\rightarrow Q$, *then we shall say that the set* $\mathscr{B}=Q(A_0, A_1, \ldots, A_m)=\{\varphi_0\varphi_1\ldots\varphi_m\colon \varphi_i\in A_i, 0\leqslant i\leqslant m\}$ *has the property* $\delta(n)$ (*or* $\mathscr{B}$ *is* $\delta(n)$) *if* $(A_0, A_1, \ldots, A_m)$ *has the property* $\delta(n)$. *If* $\mathscr{A}$ (*resp.* $\mathscr{B}$) *is* $\delta(n)$ *and* δ *is the identity map, then we write* $\mathscr{A}$ (*or* $\mathscr{B}$, *resp.*) *is* $\mathrm{id}(n)$.

Lemma 2.1. *If a set A is* $\delta(n)$, *then* $\mathscr{A}$ *is* $\delta(s)$ *for any* $1\leqslant s\leqslant n$.
Proof. Obvious.

Definition 2.2. *Let A be any (non-empty) index set and let* $\{\varphi_i\colon i\in A\}$ *be a collection of mappings* φ_i. *The collection will be called disjoint if* $\varphi_i(a)=\varphi_j(a)$ *implies* $i=j$.

Lemma 2.2. *Let* $(Q, \cdot)$ *be a quasigroup,* $X, Y\in\mathscr{T}$ *and let k be a fixed point in Q. Then the following are equivalent*

(i) $Q(X, Y)$ *is disjoint;*
(ii) $Q(X, Y, Y^{-1}, X^{-1})$ *is* $\mathrm{id}(3)$;
(iii) $Q(X, Y)=\{X_kY_r\colon r\in Q\}$;
(iv) $Q(X, Y)=\{X_rY_k\colon r\in Q\}$.

Proof. (i)→(ii)→(iii) and (ii)→(iv) are obvious. (iii)→(i). Let $X_aY_br=X_cY_dr$. There exist $p, s\in Q$ such that $X_aY_b=X_kY_p$ and $X_cY_d=X_kY_s$. Then $X_kY_pr=X_kY_sr$, whence $p=s$. The proof of (iv)→(i) is dual.

Lemma 2.3. *Let* δ *be a fixed permutation of a quasigroup* $(Q, \cdot)$. *Denote* $\mathscr{A}=Q({}^0X, {}^1X, \ldots, {}^nX)$ *for some* ${}^iX\in\mathscr{T}$, $i\in\{0, 1, 2, \ldots, n\}$, $n\geqslant 2$. *Then*

(i) $\mathscr{A}$ *is* $\delta(n)$ *implies that* $Q({}^iX, {}^{i+1}X)$ *is disjoint for all* $i\in\{0, 1, 2, \ldots, n-1\}$;
(ii) $\mathscr{A}$ *is* $\mathrm{id}(n)$ *implies that* $Q({}^tX, {}^{t+1}X, \ldots, {}^{t+n}X)$ *is* $\mathrm{id}(n)$ *for every* $t\in\{0, 1, \ldots, n\}$;
(iii) $\mathscr{A}$ *is* $\mathrm{id}(n)$ *implies that* $Q({}^iX, {}^{i+1}X)$ *and* $Q({}^{i+1}X^{-1}, {}^iX^{-1})$** *are both disjoint for all* $i\in\{0, 1, \ldots, n\}$ (*the operation* + *in the above indices takes* $\mathrm{mod}(n+1)$)*.*

Proof. (i) Without loss of generality assume $i=0$. Let $k\in Q$ be fixed. For any a, $b\in Q$ there exist $c_2, c_3, \ldots, c_n, y\in Q$ such that

$${}^0X_a\,{}^1X_b\,{}^2X_{c_2}\,{}^3X_{c_3}\ldots{}^nX_{c_n}=\delta={}^0X_k\,{}^1X_y\,{}^2X_{c_2}\,{}^3X_{c_3}\ldots{}^nX_{c_n}.$$

Thus ${}^0X_a\,{}^1X_b={}^0X_k\,{}^1X_y$ and according to Lemma 2.2 (iii)→(i), $Q({}^0X, {}^1X)$ is disjoint. (ii) The statement follows from the equivalency

$${}^0X_{a_0}\,{}^1X_{a_1}\ldots{}^nX_{a_n}=1 \quad \text{iff} \quad {}^tX_{a_t}\,{}^{t+1}X_{a_{t+1}}\ldots{}^{t+n}X_{a_{t+n}}=1.$$

(iii) Easily follows from (ii) and (i).

** If ${}^{i+1}X=R$, then ${}^{i+1}X^{-1}$ means R^{-1}, etc.

Lemma 2.4. *Let* $(Q, \cdot)$ *be a quasigroup and let* $X, Y \in \mathcal{T}$. *Then the following are equivalent*

(i) $Q(X, Y)$ *is disjoint*

(ii) $Q(Y^{-1}, X^{-1})$ *is disjoint;*

(iii) $Q(X^{-1}, X) = Q(Y, Y^{-1})$ *are disjoint:*

Proof. (i)$\rightarrow$(ii). By Lemma 2.2, $Q(X, Y, Y^{-1}, X^{-1})$ is id(3) and by Lemma 2.3(ii), $Q(Y^{-1}, X^{-1})$ is disjoint. (ii)$\rightarrow$(iii). Again by Lemma 2.2, $Q(Y^{-1}, X^{-1}, X, Y)$ is id(3), hence $Q(X^{-1}, X)$ and $Q(Y, Y^{-1})$ are disjoint. For any a, b and fixed $k \in Q$ there exists $c \in Q$ such that $Y_k^{-1}X_a^{-1}X_bY_c = 1$. Whence $X_a^{-1}X_b = Y_kY_c^{-1}$ and according to Lemma 2.2(i)$\rightarrow$(iii), $Q(X^{-1}, X) = Q(Y, Y^{-1})$. (iii)$\rightarrow$(i). In the equation $X_b^{-1}X_c = Y_aY_d^{-1}$ three of the indices can be arbitrary, therefore $Q(X^{-1}, X, Y, Y^{-1})$ is id(3), thus $Q(X, Y)$ is disjoint.

Corollary. *If* $Q(X, Y)$ *is disjoint, then* $Q(X^{-1}, X)$, $Q(X, X^{-1})$, $Q(Y, Y^{-1})$ *and* $Q(Y^{-1}, Y)$ *are all disjoint.*

Lemma 2.5. *Let* $(Q, \cdot)$ *be a quasigroup. Then the fact that* $Q(X, Y)$ *and* $Q(Y^{-1}, Z)$ *are disjoint implies that* $Q(X, Z)$ *is disjoint.*

Proof. By Lemma 2.4, $Q(X^{-1}, X) = Q(Y, Y^{-1})$ and $Q(Y, Y^{-1}) = Q(Z, Z^{-1})$ are disjoint, whence $Q(X^{-1}, X) = Q(Z, Z^{-1})$ are disjoint, therefore by Lemma 2.4(iii)$\rightarrow$(i), $Q(X, Z)$ is disjoint.

Corollary 1. *Let* $(Q, \cdot)$ *be a quasigroup. If any two of the conditions* $Q(X, Y)$ *is disjoint;* $Q(X, Z)$ *is disjoint;* $Q(Y^{-1}, Z)$ *is disjoint (or* $Q(X, Y)$ *is disjoint;* $Q(Z, Y)$ *is disjoint;* $Q(X, Z^{-1})$ *is disjoint) are satisfied, then they all are satisfied.*

Corollary 2. *Let* $(Q, \cdot)$ *be a quasigroup. If* $Q(X, X, Y^{-1}, X^{-1})$ *is* id(3), *then* $Q(Y, X)$ *is disjoint.*

Proof. By Lemmas 2.3 and 2.4, $Q(Y, X^{-1})$ and $Q(X, X)$ are disjoint, therefore by Lemma 2.5, $Q(Y, X)$ is disjoint.

Lemma 2.6. *Let* $(Q, \cdot)$ *be a quasigroup. If* $Q(X, Y, X^{-1})$ *is* id(2) *for* $X, Y \in \mathcal{T}$, *then* $Q(Y, Y)$ *is disjoint.*

Proof. By Lemma 2.3, $Q(Y, X^{-1})$ and $Q(X, Y)$ are disjoint, hence $Q(Y, Y)$ is also disjoint.

Theorem 2.1. *Let* $(Q, \cdot)$ *be a quasigroup. Then the following are equivalent*

(i) Q *is a transitive quasigroup;*

(ii) $a \cdot b = c \cdot d$, $a \cdot e = c \cdot f$ *and* $x \cdot b = y \cdot d$ *implies* $x \cdot e = y \cdot f$ *(i.e. the condition of* Reidmeister *holds);*

(iii) $Q(X^{-1}, X)$ *is disjoint for some* $X \in \mathcal{T}$;

(iv) $Q(X^{-1}, X)$ *is disjoint for all* $x \in \mathcal{T}$.

Proof. (i)$\leftrightarrow$(ii). See Theorem 11.3 in [1]. (ii)$\rightarrow$(iii). Let $R_d^{-1}R_b a = R_f^{-1}R_e a = c$, i.e. $ab = cd$, $ae = cf$. For every $x \in Q$ there exists $y \in Q$ such that $R_d^{-1}R_b x = y$, i.e. $xb = yd$. By the condition of Reidemeister $xe = yf$, i.e. $R_f^{-1}R_e x = y$, hence $R_f^{-1}R_e x = R_d^{-1}R_b x$ for all $x \in Q$. (iii)$\rightarrow$(ii) is obvious. (iii)$\rightarrow$(iv). It is known that all parastrophic quasigroups of a transitive quasigroup are also transitive. Now apply (i)$\rightarrow$(iii) of the theorem and Corollary of Lemma 2.4. (iv)$\rightarrow$(iii) is obvious.

Theorem 2.2. *Let a loop $(Q, \circ)$ be an isotope of a quasigroup $(Q, \cdot)$ by (1.2). Then the following are equivalent*

(i) $rb \cdot a = rd \cdot c$ *and* $x = x \rightarrow xb \cdot a = xd \cdot c$;

(ii) $Q(R, R)$ *is disjoint;*

(iii) $Q(T, L)$ *is disjoint;*

(iv) $(Q, \circ)$ *is a group and* α *is its quasiautomorphism.*

Proof. (i)$\rightarrow$(ii) is obvious. (ii)$\rightarrow$(iv). By Lemma 2.4, $(Q, \cdot)$ is transitive, hence $(Q, \circ)$ is a group. Let $b, r \in Q$ be fixed. For every $a, c \in Q$ there exists uniquely determined d such that $rb \cdot a = rd \cdot c$. Since $Q(R, R)$ is disjoint, $R_a R_b = R_c R_d$ and according to (1.1)

$$R^\circ_{\beta a} \alpha R^\circ_{\beta b} \alpha = R^\circ_{\beta c} \alpha R^\circ_{\beta d} \alpha,$$

whence

$$R^\circ_{I\beta c \circ \beta a} \alpha = \alpha R^\circ_{\beta d \circ I\beta b}.$$

If we put $I\beta c \circ \beta a = x$ and $\beta d \circ I\beta b = \eta x$, then η is a permutation of Q. Thus for every x, $R^\circ_x \alpha = \alpha R^\circ_{\eta x}$ and according to Theorem 1.1 (i), α is an quasiautomorphism of $(Q, \circ)$. (iii)$\rightarrow$(iv). The shortened proof: $Q(T, L, L^{-1}, T^{-1})$ is id(3), thus according to (1.1),

$$\beta^{-1} I L^\circ \alpha I^\circ \beta \beta^{-1} L^\circ \alpha^{-1} L^\circ I \beta = 1$$

(indices are omitted), whence

$$\beta^{-1} I L^\circ \alpha L^\circ L^\circ \alpha^{-1} L^\circ I \beta = 1, \quad I L^\circ \alpha L^\circ \alpha^{-1} L^\circ I = 1,$$
$$L^\circ \alpha I^\circ \alpha^{-1} L^\circ = 1, \quad L^\circ \alpha L^\circ \alpha^{-1} = 1, \quad L^\circ \alpha = \alpha I^\circ.$$

Now, apply Theorem 1.1 (i). (iv)$\rightarrow$(ii) and (iv)$\rightarrow$(iii) are obvious.

Corollary. *A quasigroup $(Q, \cdot)$ is left-hand linear iff the identity $(xy \cdot u)\backslash zy = (xs \cdot u)\backslash zs$ holds.*

Similarly we prove the following Theorems 2.3—2.7.

Theorem 2.3. *Let a loop $(Q, \circ)$ be an isotope of a quasigroup $(Q, \cdot)$ by 1.2. Then the following are equivalent*

(i) $xy = rt$, $xz = ru$, $ta = ud \rightarrow ya = zd$;

(ii) $Q(R, T)$ *is disjoint;*

(iii) $Q(T, L^{-1})$ *is disjoint;*
(iv) $(Q, \circ)$ *is a group and* $\alpha\beta^{-1}$ *is its quasiautomorphism.*

Theorem 2.4. *Let a loop* $(Q, \circ)$ *be an isotope of a quasigroup* $(Q, \cdot)$ *by* 1.2. *Then the following are equivalent*
(i) $ar\cdot b = cr\cdot d$ *and* $x = x \rightarrow dx\cdot b = cx\cdot a$;
(ii) $Q(R, L)$ *is disjoint;*
(iii) $Q(T, R)$ *is disjoint;*
(iv) $(Q, \circ)$ *is a group and* α *is its antiquasiautomorphism.*

Theorem 2.5. *Let a loop* $(Q, \circ)$ *be an isotope of a quasigroup* $(Q, \cdot)$ *by* 1.2. *Then the following are equivalent*
(i) $a\cdot s = c\cdot t,\ s\cdot b = t\cdot d,\ y\cdot b = z\cdot d \rightarrow a\cdot y = c\cdot z$;
(ii) $Q(L, R^{-1})$ *is disjoint;*
(iii) $Q(T, T)$ *is disjoint;*
(iv) $(Q, \backslash)$ *is the left-hand linear quasigroup;*
(v) $(Q, \circ)$ *is a group and* $\alpha\beta^{-1}$ *is its antiquasiautomorphism.*

Theorem 2.6. *Let a loop* $(Q, \circ)$ *be an isotope of a quasigroup* $(Q, \cdot)$ *by* 1.2. *Then the following are equivalent*
(i) $r\cdot s = x\cdot y,\ r\cdot t = x\cdot z,\ a\cdot s = c\cdot t \rightarrow a\cdot y = d\cdot z$;
(ii) $Q(L, t)$ *is disjoint;*
(iii) $Q(T, R^{-1})$ *is disjoint;*
(iv) $Q(L^{-1}, R)$ *is disjoint;*
(v) $(Q, \circ)$ *is an abelian group i.e.* $\alpha\alpha^{-1} = \beta\beta^{-1} = 1$ *is the antiquasiautomorphism of* $(Q, \circ)$.

Summarizing results we get

Theorem 2.7. *A quasigroup* $(Q, \cdot)$ *is quasilinear iff* $Q(X, Y)$ *is disjoint for some* $X, Y \in \mathcal{T}$, $X \neq Y^{-1}$ *and* $(X, Y) \notin \{(R, T^{-1}), (T, R^{-1}), (L, T), (T^{-1}, L^{-1}), (R^{-1}, L), (L^{-1}, R)\}$.

Theorem 2.8. *Let* $(Q, \circ)$ *be a group isotopic to a linear quasigroup* $(Q, \cdot)$. *Then the following are equivalent*
(i) $Q(L^i, R^j)$ *is disjoint for some* $i, j \in \{1, -1\}$;
(ii) $(Q, \circ)$ *is an abelian group;*
(iii) $(Q, \cdot)$ *is a T-quasigroup.*

Proof. (i)$\rightarrow$(ii). Since $(Q, \cdot)$ is linear, according to Theorem 2.2 (iv)$\rightarrow$(ii) and its dual theorem, $Q(R, R)$ and $Q(L, L)$ are disjoint. Further, by Lemma 2.4 (i)$\rightarrow$(iii), $Q(R^{-1}, R) = Q(R, R^{-1})$ and $Q(L^{-1}, L) = Q(L, L^{-1})$. Since $Q(L^i, R^j)$ is disjoint, $Q(L^{-i}, L^i) = Q(R^j, R^{-j})$. Thus $Q(R^{-1}, R) = Q(L^{-1}, L)$. Now apply Theorem 4 of [4]. (ii)$\rightarrow$(i). By Theorem 2.6, $Q(L, T)$ and $Q(T, R^{-1})$ are disjoint, thus $Q(L, R^{-1})$ is disjoint.

Corollary. *Let $(Q, \cdot)$ be a quasigroup. Then $(Q, \cdot)$ is a T-quasigroup if and only if $Q(R, L)$, $Q(L, R)$ and $Q(L^{-1}, R)$ are all disjoint.*

Proof. Use Lemma 2.5 and Theorem 2.8.

Theorem 2.9. *A left-hand linear quasigroup is idempotent if and only if it is right-hand distributive.*

Proof. Let $(Q, \cdot)$ be a left-hand linear quasigroup. For arbitrary $x, y, r, d \in Q$ there exists c such that $ry \cdot x = rd \cdot c$ and so $zy \cdot x = zd \cdot c$ for all z in Q. In particular, $yy \cdot x = yd \cdot c$ and if $(Q, \cdot)$ is idempotent, then $yx \cdot yx = yd \cdot c$ and $zx \cdot yx = zd \cdot c$, therefore $zy \cdot x = zx \cdot yx$. The converse is obvious.

Corollary 1. *A quasigroup is an idempotent left-hand linear quasigroup if and only if it is a transitive right-hand distributive quasigroup.*

Proof. Let $(Q, \cdot)$ be a transitive right-hand distributive quasigroup. Then for every x, y, z, $xy \cdot z = xz \cdot yz$, i.e. $R_z^{-1}R_{yz} = R_yR_z^{-1}$, whence $Q(R^{-1}, R) = Q(R, R^{-1})$. According to Theorem 2.1 (i)$\to$(iv), $Q(R^{-1}, R)$ and $Q(R, R^{-1})$ are disjoint. Thus by Lemma 2.4 (iii)$\to$(i), $Q(R, R)$ is disjoint.

Recall that a quasigroup is called medial if it satisfies the identity $xy \cdot zt = xz \cdot yt$.

Corollary 2. *An idempotent quasigroup is linear iff it is medial.*

Theorem 2.10. *Let $(Q, \cdot)$ be a quasigroup. Then the following are equivalent*

(i) *$(Q, \cdot)$ is medial;*

(ii) *$ar \cdot b = c \cdot rd$ and $x = x \to ax \cdot b = c \cdot xd$;*

(iii) *there exist an abelian group $(Q, +)$, its commuting automorphisms φ, ψ and $f \in Q$ such that for all $x, y \in Q$*

$$x \cdot y = \varphi x + \psi y + f. \tag{2.1}$$

Proof. (i)$\to$(ii). Let $ar \cdot b = c \cdot rd$. There exists m such that $b = md$. Then $ar \cdot md = c \cdot rd$ i.e. $am \cdot rd = c \cdot rd$ hence $am = c$. Thus $ar \cdot md = am \cdot rd$ and also $ax \cdot md = am \cdot xd$ for all $x \in Q$ i.e. $ax \cdot b = c \cdot xd$. (ii)$\to$(iii). Obviously, (ii) is equivalent to the condition $Q(R^{-1}, L^{-1}, R, L)$ is id(3). By Lemma 2.3 (iii), $Q(L, R)$, $Q(R, L)$ and $Q(L^{-1}, R)$ are all disjoint. By Corollary of Theorem 2.8, $(Q, \cdot)$ is a T-quasigroup, therefore there exist an abelian group $(Q, +)$, its automorphisms φ, ψ and $f \in Q$ such that (2.1) holds. If the equation $ax \cdot b = c \cdot xd$, is rewritten with $+$ operation, then

$$\varphi^2 a + \varphi f + \psi b + \varphi\psi x = \varphi c + \psi^2 d + \psi f + \psi\varphi x.$$

If we put $x = e$ (e is the identity of $+$), then $\varphi^2 a + \varphi f + \psi b = \varphi c + \psi^2 d + \psi f$, therefore $\varphi\psi x = \psi\varphi x$. (iii)$\to$(i) is easy. The proof is finished.

It is well known that (i)$\leftrightarrow$(iii) is Toyoda's theorem. There are many other proofs of the theorem, for example in [1], [3], [4].

Theorem 2.11. *Let* $(Q, \cdot)$ *be a quasigroup. Then the following are equivalent*

(i) $ra \cdot b = cr \cdot d$ *and* $x = x \rightarrow xa \cdot b = cx \cdot d$;

(ii) $e \cdot fr = g \cdot rh$ *and* $x = x \rightarrow e \cdot fx = g \cdot xh$;

(iii) $ra \cdot b = g \cdot rh$ *and* $x = x \rightarrow xa \cdot b = g \cdot xh$;

(iv) $e \cdot fr = cr \cdot d$ *and* $x = x \rightarrow e \cdot fx = cx \cdot d$;

(v) $(Q, \cdot)$ *is a commutative medial quasigroup.*

Proof. (i)$\rightarrow$(v). Obviously (i) is equivalent to the statement $Q(R, R, L^{-1}, R^{-1})$ is id(3). By Corollary 2 of Lemma 2.5, $Q(L, R)$ is disjoint. By Lemma 2.3, $Q(R, R)$, $Q(R, L^{-1})$ and $Q(R, L)$ are all disjoint. If we now apply Corollary of Theorem 2.8, we get, $(Q, \cdot)$ is a T-quasigroup. Thus (2.1) holds. Obviously $ra \cdot b = cr \cdot b$ implies $xa = cx$ for all $x \in Q$. If the last equality is rewriten with the $+$ operation, then $\varphi x + \psi a = \varphi c + \psi x$. Put $x = e$ (the identity of $+$); then $\psi a = \varphi c$, hence $\varphi = \psi$. (v)$\rightarrow$(i). Use the commutativity of $(\cdot)$ and Theorem 2.10 (i)$\rightarrow$(ii). (ii)$\leftrightarrow$(iii). The proof is dual to (i)$\leftrightarrow$(iii). By analogy we do the rest of the proof.

3. Varietes of t_M-quasigroups

Definition 3.1. *Let* M *be a non-empty subset of the set* $\{Q(X, Y, Z, U)$: $X, Y, Z, U \in \mathcal{T}\}$. *A quasigroup* $(Q, \cdot)$ *is called* t_M*-quasigroup if every* $s \in M$ *has the property* id(3).

Lemma 3.1. *Let a quasigroup* $(P, \circ)$ *be a homomorphic image of a quasigroup* $(Q, \cdot)$. *If* $(Q, \cdot)$ *is a* t_M*-quasigroup, then* $(P, \circ)$ *is also* t_M*-quasigroup.*

Proof. Let φ be a homomorphism of Q onto P. From $\varphi(x \cdot y) = \varphi x \circ \varphi y$ we have $X^\circ_{\varphi a} = \varphi X_a \varphi^{-1}$ for every $X \in \mathcal{T}$. Let $Q(X, Y, Z, U)$ be id(3). Then $1 = X_a Y_b Z_c U_d$, where three of the indices can be arbitrary. Obviously also

$$1 = \varphi X_a \varphi^{-1} \varphi Y_b \varphi^{-1} \varphi Z_c \varphi^{-1} \varphi U_d \varphi^{-1} = X^\circ_{\varphi a} Y^\circ_{\varphi b} Z^\circ_{\varphi c} U^\circ_{\varphi d},$$

where every three of the indices φ_a, φ_b, φ_c, φ_d can be arbitrary. Thus $P(X^\circ, Y^\circ, Z^\circ, U^\circ)$ is id(3). The proof is finished.

A condition $Q(X, Y, Z, U)$ is id(3) can be expressed as the quasiidentity in a quasigroup Q. For example $Q(R, R, R^{-1}, R^{-1})$ is id(3) is equivalent (see Lemma 2.2) to the following quasiidentity: $ra \cdot b = rc \cdot d$ and $x = x$ implies $xa \cdot b = xc \cdot d$. Thus a class of all t_M-quasigroups is a quasivariety. It is known (see [9]) that a class of an algebraic system R, that is quasivariety, is a variety if and only if every homomorphic image of an R-system is an R-system. As an immediate consequence of this statement and Lemma 3.1 we have

Theorem 3.1. *The class of all* t_M*-quasigroups is a variety.*

Theorem 3.2. *The variety of all* t_M*-quasigroups possesses a basic which consists of a finite number of identities.*

Proof. It is known (see [9]) that every variety possessing a basis which consists of a finite number of quasiidentities has a finite generating set of identities.

4. Some classes of quasilinear quasigroups

A *quasigroup word* is a formal expression consisting of variables and some of the three binary operation symbols $\cdot$, \, /.

Throughout this section we shall use the following notations

$(Q, \cdot)$ — quasigroup;

$V = \{x_i\colon i \in A\}$ is a set of variables x_i, A is an index set;

$w_1, w_2, \ldots$ are quasigroup words;

$V(w_i)$ is the set of all variables occurring in w_i, $V(w_i) \subset V$;

$w_i(x)$ is a word w_i in which exactly one variable symbol x_j is replaced by the variable x, for example, if $w_i = x_1/(x_2 \cdot x_3 x_1)$, then $w_i(x) \in \{x/(x_2 \cdot x_3 x_1),\ x_1/(x \cdot x_2 x_1), \ldots\}$;

$w_i(x) \sim w_j(y)$ means that if x in $w_i(x)$ is replaced by y, then we get $w_j(y)$.

Theorem 4.1. *Let $(Q, \cdot)$ be a quasigroup that satisfies the following identity*

$$w_1 \cdot (w_2 \cdot x) = w_3 \cdot (w_4 \cdot x).$$

Let $\{i_1, i_2, i_3\} \subset \{1, 2, 3, 4\}$. If there exists a solution of simultaneous equations $w_{i_r} = a_r$, $r \in \{1, 2, 3\}$ for arbitrary $a_1, a_2, a_3 \in Q$, then $(Q, \cdot)$ is a right-hand linear quasigroup.

Proof. Obviously $Q(L, L)$ is disjoint.

A quasigroup $(Q, \cdot)$ is called B_1-quasigroup (see [8]) if it satisfies the identity $x \cdot yz = y \cdot xz$. The following theorem shows that the converse of Theorem 4.1 is false.

Theorem 4.2. *Let $(Q, \cdot)$ be a quasigroup. Then the following are equivalent*

(i) *$(Q, \cdot)$ is a B_1-quasigroup;*

(ii) *there exist an abelian group $(Q, +)$ and a permutation α of Q such that for all $x, y \in Q$, $x \cdot y = \alpha x + y$.*

Proof. See Theorem 13 of [7].

A quasigroup $(Q, \cdot)$ is called left-hand transitive if it satisfies the identity $xy \cdot xz = yz$ (see [2]). The following theorem shows that there exists a right-hand linear quasigroup satisfying an identity different from the one in Theorem 4.1.

Theorem 4.3. *Let $(Q, \cdot)$ be a quasigroup. Then the following are equivalent*

(i) *$(Q, \cdot)$ is left transitive;*

(ii) *there exist a group $(Q, \circ)$, its automorphism ψ and $k \in Q$ such that for all $x, y \in Q$, $x \cdot y = I\psi x \circ \psi y \circ k$, where $Ix = x^{-1}$ in the group.*

Proof. (i)→(ii). From the identity we have $R_{xz}L_xR_z^{-1}=1$ for all $x, z\in Q$. Hence $Q(R, L, R^{-1})$ is id(2). If we apply Lemmas 2.3, 2.6, the dual theorem of Theorem 2.2, Theorem 2.4, we have $x\cdot y=\varphi x\circ\psi y\circ k$ for an antiautomorphism φ and an automorphism ψ of a group $(Q, \circ)$. If the identity $xy\cdot xz=yz$ is rewritten with $\circ$ operation, then

$$\varphi(\varphi x\circ\psi y\circ k)\circ\psi(\varphi x\circ\psi z\circ k)\circ k=\varphi y\circ\psi z\circ k,$$
$$\varphi k\circ\varphi\psi y\circ\varphi^2x\circ\psi\varphi x\circ\psi^2z\circ\psi k=\varphi y\circ\psi z;$$

if $x=y=z=e$ (e is the identity of $(Q, \circ)$), then $\varphi k\circ\psi k=e$; if $y=z=e$, then $\varphi k\circ\varphi^2x\circ\psi\varphi x\circ\psi k=e$, whence $\varphi^2x\circ\psi\varphi x=e$, $\varphi x\circ\psi x=e$ i.e. $\varphi=I\psi$. The converse is easy.

Let us note that Theorem 4.2 can also be proved in a similar way.

Theorem 4.5. *Let $(Q, \cdot)$ be an elastic quasigroup (such quasigroups satisfy the identity $x\cdot yx=xy\cdot x$) in which the map $x\mapsto x\cdot x$ is Q onto Q. Then the following are equivalent*

(i) *Q is a B_1-quasigroup;*

(ii) *Q is left-hand transitive;*

Proof. Use Theorems 4.2 and 4.3.

Other properties of left-hand transitive quasigroups are in [5].

Definition 4.1. *Let w_i, $i=1, 2, 3, 4$ be quasigroup words and let $A, B\in\{R, L\}$. A quasigroup $(Q, \cdot)$ is called a t_1-quasigroup if it satisfies the identity*

$$(t_1)\qquad A_xw_1(z)\cdot B_yw_2(t)=A_xw_3(t)\cdot B_yw_4(z).$$

If $w_1(z)\sim w_3(t)$, $w_2(t)\sim w_4(z)$, then Q is called a t_2-quasigroup. If, moreover, $A=L$, $B=R$ ($A=R$, $B=L$, respectively) and

$$w_1(z)=A_{x_1}A_{x_2}\dots A_{x_{n-1}}(z),\quad w_2(t)=B_{x_n}B_{x_{n+1}}\dots B_{x_{2n-2}}(t),$$

then Q is called an α_n (a β_n, respectively)-quasigroup.

Thus t_1, t_2-quasigroups are generalizations of α_n, β_n-quasigroups that were studied by P. Nemec and T. Kepka in [8], [10].

Theorem 4.6. *Let $A, B, C, D\in\{R, L\}$ and let a quasigroup $(Q, \cdot)$ satisfy the identity*

$$\text{(i)}\qquad A_xw_1(z)\cdot B_yw_2(t)=C_xw_3(t)\cdot D_yw_4(z).$$

Then Q is a T-quasigroup. If $A\neq C$ or $B\neq D$, then Q is a commutative t_1-quasigroup.

Proof. Let in (i) all variables, besides x, y, z, t, be replaced by fixed elements of Q. Then

$$\text{(ii)}\qquad A_x\alpha(z)\cdot B_y\beta(t)=C_x\gamma(t)\cdot D_y\delta(z)$$

for all $x, y, z, t \in Q$ and certain permutations $\alpha, \beta, \gamma, \delta$ of Q. If (ii) is rewritten with translations of Q, then

$$R_{B_y\beta(t)}A'_{\alpha(z)} = R_{D_y\delta(z)}C'_{\gamma(t)},$$
$$L_{A_x\alpha(z)}B'_{\beta(t)} = L_{C_x\gamma(t)} \cdot D'_{\delta(z)},$$
$$R_{B_y\beta(t)}A_x\alpha = L_{C_x\gamma(t)}D_y\delta,$$
$$L_{A_x\alpha(z)}B_y\beta = R_{D_y\delta(z)}C_x\gamma,$$

for all $x, y, z, t \in Q$ where A' is the dual symbol of A, etc. These equations are equivalent, respectively, with the following ones

(iii) $$(C'_{\gamma(t)})^{-1}R^{-1}_{D_y\delta(z)}R_{B_y\beta(t)}A'_{\alpha(z)} = 1,$$

(iv) $$(D'_{\delta(z)})^{-1}L^{-1}_{C_x\gamma(t)}L_{A_x\alpha(z)}B'_{\beta(t)} = 1,$$

(v) $$D_y^{-1}L^{-1}_{C_x\gamma(t)}R_{B_y\beta(t)}A_x = \delta\alpha^{-1},$$

(vi) $$C_x^{-1}R^{-1}_{D_y\delta(z)}L_{A_x\alpha(z)}B_y = \gamma\beta^{-1}.$$

Obviously, every three of the four indices in each of the above equations can be arbitrary, therefore we can use Lemma 2.3. Then from (iii) and (vi) it follows, respectively, that $Q((C')^{-1}, R^{-1})$, $Q(C^{-1}, R^{-1})$ are disjoint. Since $R \in \{C', C\}$, both $Q(R^{-1}, R^{-1})$ and $Q(R, R)$ are disjoint. Similarly, from (iv) and (v) we get that $Q(L, L)$ is disjoint, hence Q is a linear quasigroup. From (v) it follows that $Q(L^{-1}, R)$ is disjoint, therefore by Theorem 2.8, $(Q, \cdot)$ is a T-quasigroup. Thus there exist an abelian group $(Q, +)$, its automorphism φ, ψ such that for all x, y and some $k \in Q$

(4.1) $$x \cdot y = \varphi x + \psi y + k.$$

Now let $A = L$, $C = R$. If in (i) all variables, besides x, are replaced by e — the identity of $(Q, +)$ and then (i) rewritten with $+$ operation we get

$$\varphi(\varphi x + \psi a + k) + \psi b + k = \varphi(\varphi c + \psi x + k) + \psi d + k$$

for some a, b, c, d. Put $x = e$, then

$$\varphi\psi a + \varphi k + \psi b + k = \varphi^2 c + \varphi k + \psi d + k,$$

therefore $\varphi^2 x = \varphi\psi x$ for all x, hence $\varphi = \psi$.

Corollary. *Every t_1-quasigroup is a T-quasigroup.*

Theorem 4.7. *Let $(Q, \cdot)$ be a quasigroup. Then the following identities are equivalent*

(i) $A_xw_1(z) \cdot B_yw_2(t) = A_xw_1(t) \cdot B_yw_2(z)$, *where* $w_1(z) \sim w_1(t)$, $w_2(t) \sim w_2(z)$ *and* $V(w_1) = V(w_2) = \{x_1\}$;

(ii) $A_xw_3(z) \cdot B_yw_4(t) = A_xw_3(t) \cdot B_yw_4(z)$, *where* $w_3(z)$, $w_4(t)$, $w_3(t)$ *and* $w_4(z)$ *are* *such words that if all their variables, besides t, z, are replaced by x_1, then we get $w_1(z)$, $w_2(t)$, $w_1(t)$ and $w_2(z)$, respectively, and $w_3(z)\sim w_3(t)$, $w_4(t)\sim w_4(z)$.*

Proof. A quasigroup $(Q, \cdot)$ that satisfies (i) or (ii) is a T_1-quasigroup, hence Q is a T-quasigroup. (i)$\to$(ii). If (i), rewritten with $+$ operation (see 4.1), then all variables, besides t, z, will be absent. Thus these variables can be replaced in the same position in words $w_1(z)$ and $w_1(t)$, etc. by arbitrary variables from V. The converse is obvious.

Theorem 4.8. *Let $(Q, +)$ be a loop isotopic to a quasigroup $(Q, \cdot)$. Then the following are equivalent*

(i) *$(Q, \cdot)$ is t_2-quasigroup;*

(ii) *$(Q, +)$ is an abelian group and there exist its automorphisms φ, ψ and α, β in the group generated by the set $\{I, \varphi, \psi\}$ such that $\varphi\alpha=\psi\beta$ and (4.1) holds.*

Proof. (i)$\to$(ii). Obviously, $(Q, \cdot)$ is a T-quasigroup, hence (4.1) holds. If in the identity (t_1) of Definition 4.1 all variables, besides t, z, are replaced by elements of Q, then we get

(iii)
$$A_{a_0}\,{}^1X_{a_1}\,{}^2X_{a_2}\ldots{}^rX_{a_r}(z)\cdot B_{b_0}\,{}^1Y_{b_1}\,{}^2Y_{b_2}\ldots{}^sY_{b_s}(t)=$$
$$=A_{a_0}\,{}^1X_{a_1}\,{}^2X_{a_2}\ldots{}^rX_{a_r}(t)\cdot B_{b_0}\,{}^1Y_{b_1}\,{}^2Y_{b_2}\ldots{}^sY_{b_s}(z)$$

for appropriate $a_0, b_0, \ldots, a_r, b_s\in Q$, ${}^1X, {}^1Y, \ldots, {}^rX, {}^sY\in\mathcal{T}$ and for all $t, z\in Q$. It is an easy task to show that $L^+_{\psi a}\psi=\psi L^+_a$. According to (1.1)

$$A_{a_0}\,{}^1X_{a_1}\,{}^2X_{a_2}\ldots{}^rX_{a_r}=L^+_a\alpha,\quad B_{b_0}\,{}^1Y_{b_1}\,{}^2Y_{b_2}\ldots{}^sY_{b_s}=L^+_b\beta$$

where α, β are products of φ, ψ, I. Thus from (iii) we have

$$L^+_a\alpha(z)\cdot L^+_b\beta(t)=L^+_a\alpha(t)\cdot L^+_b\beta(z).$$

If we rewrite this equation with $+$ operation, then

$$\varphi a+\varphi\alpha(z)+\psi b+\psi\beta(t)+k=\varphi a+\varphi\alpha(t)+\psi b+\psi\beta(z)+k,$$
$$\varphi\alpha(z)+\psi\beta(t)=\varphi\alpha(t)+\psi\beta(z),$$

and if $z=e$ (the identity of $+$), then $\varphi\alpha=\psi\beta$. (ii)$\to$(i). If

$$\alpha=\varphi^{i_1}\psi^{j_1}\ldots\varphi^{i_m}\psi^{j_m}I=\varphi^{i_1}\psi^{j_1}\ldots\varphi^{i_m}\psi^{j_m-1}\varphi(\varphi^{-1}\psi I),$$

then we put

$$S_\alpha=R_xR_u^{i_1-1}L_u^{j_1}\ldots R_u^{i_m}L_u^{j_m-1}R_uT_u.$$

Similarly we get S_β. It is not difficult to show that

$$S_\alpha(z)\cdot S_\beta(t)=S_\alpha(t)\cdot S_\beta(z),$$

for all z, t, hence $(Q, \cdot)$ is a t_2-quasigroup.

Corollary 1. *Let $(Q, +)$ be an abelian group. If φ, ψ are its automorphisms of finite orders, then a quasigroup $(Q, \cdot)$, defined by (4.1), is a t_2-quasigroup.*

Proof. If m, n are the orders of φ, ψ respectively, then $\varphi\varphi^{m-1} = \psi\psi^{n-1} = 1$.

Corollary 2. *Every finite T-quasigroup is a t_2-quasigroup.*

Example 4.1. Let $(C, +)$ be the group of complex numbers and let $\varphi x = -x$, $\psi x = ix$. Then $\varphi^2 = \psi^4 = 1$. The quasigroup $(C, \cdot)$, defined by (4.1), satisfies the identity

$$zx \cdot y(u \cdot ut) = tx \cdot y(u \cdot uz)$$

that is equivalent to the identity

$$zx \cdot y(u \cdot vt) = tx \cdot y(u \cdot vz).$$

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Received December 14, 1981

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О КВАЗИТОЖДЕСТВАХ ТРАНЗИТИВНЫХ КВАЗИГРУПП

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Резюме

В работе дана характеристика некоторых многообразий транзитивных квазигрупп с помощью квазитождеств. Полученные результаты использованы для изучения некоторых свойств линейных квазигрупп.