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A NOTE ON EQUALITIES OF RADICALS IN A SEMIGROUP

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Let S be a semigroup with an ideal $J \subseteq S$. All ideals in the following are supposed to be two-sided. The principal twosided ideal of S generated by an element $a \in S$ is denoted by $J(a)$.

An element $x \in S$ is called nilpotent with respect to J if $x^n \in J$ for some positive integer n. An ideal, or a subsemigroup I of S is called nilpotent with respect to J if $I^n \subseteq J$ for some positive integer n. An ideal I of S is called a nilideal with respect to J if each element of I is nilpotent with respect to J. An ideal I, each finitely generated subsemigroup of which is nilpotent with respect to J, is called a locally nilpotent ideal with respect to J. An ideal P of S is called prime if for any two ideals A, B of S, $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$. An ideal P of S is called completely prime if for any $a, b \in S$ $ab \in P$ implies that either $a \in P$ or $b \in P$.

The set of all nilpotent elements of S with respect to J will be denoted by $N(J)$. The union $R(J)$ of all nilpotent ideals of S with respect to J is called the Schwarz radical of S with respect to J. The union $L(J)$ of all locally nilpotent ideals of S with respect to J is called the Ševrin radical of S with respect to J. The union $R^*(J)$ of all nilideals of S with respect to J is called the Clifford radical of S with respect to J. The intersection $M(J)$ of all prime ideals of S which contain J is called the McCoy radical of S with respect to J. The intersection $C(J)$ of all completely prime ideals of S which contain J is called the Luh radical of S with respect to J.

R. Šulka [4, Lemma 19] and J. Bosák [1, Theorem 2] proved that in an arbitrary semigroup S with an ideal J we have

$$R(J) \subseteq M(J) \subseteq L(J) \subseteq R^*(J) \subseteq N(J) \subseteq C(J). \tag{1}$$

In a commutative semigroup S as proved by R. Šulka [4, Theorem 7] and J. Bosák [1, Corollary 1] we have

$$R(J) = M(J) = L(J) = R^*(J) = N(J) = C(J).$$

A semigroup S is called a C_2-semigroup if $xyzyx = yxzxy$ for all x, y, z of S. J. E. Kuczkowski [2] proved that in a C_2-semigroup S we have $M(J) = L(J) = R^*(J) = N(J) = C(J)$ for every ideal J of S.

R. Šulka [5, p. 276] gave an example of a C_2-semigroup with $R(J) \neq M(J)$.

B. Pondělíček [3] studied in a semigroup S the necessary and sufficient condition in order that

$$M(J) = L(J) = R^*(J) = N(J) = C(J) \tag{2}$$

holds for every ideal J of S.

The purpose of this note is to give in an arbitrary semigroup S necessary and sufficient conditions which are equivalent to the condition of B. Pondělíček [3, Theorem] such that (2) holds for every ideal J of S.

Lemma 1 (B. Pondělíček [3, Theorem]). *Let S be a semigroup. Then* (2) *holds for every ideal J of S if and only if*

$$J(a) \cap J(b) \subseteq M(J(ab))$$

for all a, b of S.

Theorem 2. *Let S be a semigroup. Then the following statements are equivalent:*

(I) *The equalities $M(J) = L(J) = R^*(J) = N(J) = C(J)$ hold for every ideal J of S.*

(II) *$J(a) \cap J(b) \subseteq M(J(ab))$ holds for all a, b of S.*

(III) *$J(a)J(b) \subseteq M(J(ab))$ holds for all a, b of S.*

(IV) *Every prime ideal is a completely prime ideal of S.*

Proof. That (I) implies (II) is proved by B. Pondělíček (cf. Lemma 1). Evidently (II) implies (III).

We prove that (III) implies (IV). Let P be an arbitrary prime ideal of S and $ab \in P$. Then $J(ab) \subseteq P$. From this by Lemma 7 of [5] we have that $M(J(ab)) \subseteq M(P) = P$.

By the assumption then $J(a)J(b) \subseteq M(J(ab)) \subseteq P$. Since P is a prime ideal of S, this implies that either $J(a) \subseteq P$ or $J(b) \subseteq P$ and so either $a \in P$ or $b \in P$. It means that P is a completely prime ideal of S.

(IV) evidently implies (I).

Lemma 3 ([6, Corollary 3]). *In a finite semigroup S with an ideal J the equalities $R(J) = M(J) = L(J) = R^*(J) = N(J) = C(J)$ hold if and only if the set $N(J)$ is an ideal of S.*

Then from Theorem 2 and Lemma 3 there follows

Corollary 4. *In a finite semigroup S the following statements are equivalent:*

(I) *The set $N(J)$ is an ideal of S for every ideal J of S.*

(II) *$R(J) = M(J) = L(J) = R^*(J) = N(J) = C(J)$ holds for every ideal J of S.*

(III) *$J(a) \cap J(b) \subseteq M(J(ab))$ for all a, b of S.*

(IV) *$J(a)J(b) \subseteq M(J(ab))$ for all a, b of S.*

(V) *Every prime ideal is a completely prime ideal of S.*

REFERENCES

[1] BOSÁK, J.: On radicals of semigroups, Mat. časop. SAV 18, 1968, 204—212.

[2] KUCZKOWSKI, J. E.: On radicals in certain class of semigroups, Mat. časop. SAV 20, 1970, 278—280.

[3] PONDĚLÍČEK, B.: A note on radicals of semigroups, Mat. časop. SAV 23, 1973, 14—16.

[4] ŠULKA, R.: On nilpotent elements, ideals and radicals of a semigroup, Mat. fyz. časop. SAV 13, 1963, 209—222 (Russian).

[5] ŠULKA, R.: The maximal semilattice decomposition of a semigroup, Mat. časop. SAV 21, 1971, 269—276.

[6] KMEŤ, F.: On radicals in semigroups, Math. Slovaca 32, 1982, 183—188.

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ЗАМЕТКА К РАВЕНСТВАМ РАДИКАЛОВ В ПОЛУГРУППЕ

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Резюме

В статье изучаются необходимые и достаточные условия для равенства радикалов Маккойа, Шеврина, Клиффорда и Луга относительно произвольного идеала полугруппы.