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DOMATIC NUMBER AND DEGREES OF VERTICES OF A GRAPH

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A dominating set in an undirected graph G is a subset D of the vertex set $V(G)$ of G with the property that to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating sets in G. The maximal number of classes of a domatic partition of G is called the domatic number of G and denoted by $d(G)$. This concept was introduced by E. J. Cockayne and S. T. Hedetniemi [1].

The quoted authors have proved that $d(G) \leqq \delta(G) + 1$, where $\delta(G)$ is the minimal degree of a vertex of G. A natural question is, whether there exists also a lower bound for $d(G)$ in terms of $\delta(G)$. In particular, one may ask, whether the domatic number of a graph, all of whose vertices have infinite degree, must be infinite. The following theorem gives a negative answer to these questions.

Theorem 1. *For each non-zero cardinal number $\mathfrak{A}$ there exists a graph G in which each vertex has the degree at least $\mathfrak{A}$ and whose domatic number is 2. If $\mathfrak{A}$ is finite, then there exist both a finite graph with this property and an infinite one.*

Proof. Choose a cardinal number $\mathfrak{B} > 3\mathfrak{A}$. Let A be a set of the cardinality $\mathfrak{B}$, let B the set of all subsets of A which have the cardinality $\mathfrak{A}$. The vertex set of G will be $A \cup B$. A vertex $u \in A$ will be adjacent to a vertex $v \in B$ in G if and only if v is a set which contains the element u. No two vertices of A and no two vertices of B will be adjacent. Evidently the degree of any vertex of B is $\mathfrak{A}$ and the degree of any vertex of A cannot be less than $\mathfrak{A}$.

Suppose $d(G) \geqq 3$. Then there exist three pairwise disjoint dominating sets in G; let these sets be D_1, D_2, D_3. For $i = 1, 2, 3$ let $A_i = A \cap D_i$, $B_i = B \cap D_i$. If $B_1 = \emptyset$, then $D_1 = A_1 \subseteq A$. If it is a proper subset of A, then there exists at least one vertex $x \in A - D_1$; this vertex is adjacent to no vertex of D_1, because A is an independent set in G, and this implies that D_1 is not a dominating set in G. Hence $B_1 = \emptyset$ implies $D_1 = A_1 = A$. Then $A_2 = A_3 = \emptyset$ and analogously this implies $D_2 = D_3 = B$, which is a contradiction with the assumption that $D_2 \cap D_3 = \emptyset$. Therefore we must have $B_1 \neq \emptyset$ and analogously also B_2, B_3, A_1, A_2, A_3 are non-empty sets.

Let x be a vertex of B adjacent to no vertex of A_1. It is adjacent to no vertex of B_1, because $B_1 \subseteq B$ and B is an independent set in G. Therefore x is adjacent to no vertex of D_1 and this implies that $x \in D_1$. We have proved that B_1 contains all vertices of B which are adjacent to no vertex of A_1, i. e. all subsets of $A - A_1 = A_2 \cup A_3$ of the cardinality $\mathfrak{A}$. Analogously B_2 (or B_3) contains all subsets of $A_1 \cup A_3$ (or $A_1 \cup A_2$ respectively) of the cardinality $\mathfrak{A}$. If $|A_3| \geqq \mathfrak{A}$, then there exists a subset of A_3 which has the cardinality $\mathfrak{A}$; this set belongs to both B_1, B_2, which is a contradiction, because $B_1 \cap B_2 \subseteq D_1 \cap D_2 = \emptyset$. Hence $|A_3| < \mathfrak{A}$ and analogously $|A_1|\mathfrak{A}$, $|A_2| < \mathfrak{A}$. However, as $\{A_2, A_2, A_3\}$ is a partition of A, we have $|A| = |A_1| + |A_2| + |A_3| < 3\mathfrak{A}$, which is a contradiction with the assumption that $|A| = \mathfrak{B} > 3\mathfrak{A}$. Hence $d(G) \leqq 2$. As G does not contain isolated vertices, its domatic number is at least 2 and therefore $d(G) = 2$.

If $\mathfrak{A}$ is finite, then for $\mathfrak{B}$ an infinite cardinal number or a finite one may be chosen, therefore there exist both finite and infinite graphs with the required property.

We see that no lower bound for $d(G)$ in terms of $\delta(G)$ can be given. Nevertheless, there exists such a bound in terms of $\delta(G)$ and n, where n is the number of vertices of a graph.

Theorem 2. *Let G be a finite undirected graph with n vertices, let $\delta(G)$ be the minimum of degrees of vertices of G. Then*

$$d(G) \geqq [n/(n - \delta(G))].$$

Proof. Consider the complement $\bar{G}$ of G. A subset D of the vertex set $V(G)$ of G is a dominating set in G if for each $x \in V(G) - D$ there exists a vertex y which is not adjacent to x in $\bar{G}$. If some vertex has the degree r in G, it has the degree $n - r - 1$ in $\bar{G}$. Hence the maximum of degrees of vertices of $\bar{G}$ is $n - \delta(G) - 1$. Let D be a subset of $V(G)$ having at least $n - \delta(G)$ vertices. Then each vertex $x \in V(G) - D$ can be adjacent to at most $n - \delta(G) - 1$ vertices of D in $\bar{G}$ and there exists a vertex $y \in D$ which is not adjacent to x; this implies that each subset of $V(G)$ with at least $n - \delta(G)$ vertices is a dominating set in G. Consider a partition of $V(G)$ into classes having $n - \delta(G)$ vertices each, with the exception of at most one which would have more vertices. Evidently there exists such a partition having $[n/(n - \delta(G))]$ classes; this is a domatic partition. Hence $d(G) \geqq [n/(n - \delta(G))]$.

In the case of infinite graphs we cannot use the subtraction and the division. But by the same idea we may prove the following theorem.

Theorem 3. *Let G be an undirected graph whose vertex set thas the infinite cardinality $\mathfrak{A}$, let $\bar{G}$ be its complement. If the supremum of degrees of vertices of $\bar{G}$ is less than $\mathfrak{A}$, then $d(G) = a$.*

REFERENCE

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ДОМАТИЧЕСКОЕ ЧИСЛО И СТЕПЕНИ ВЕРШИН ГРАФА

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Резюме

Доматическое число $d(G)$ неориентированного графа G есть максимальное число классов разбиения множества вершин графа G в доминирующие множества. Доказано, что для произвольно большого значения минимальной степени графа G его доматическое число может быть 2. Далее дана нижняя оценка для $d(G)$ в зависимости от числа вершин и минимальной степени графа G и введено аналогичное утверждение для бесконечных графов.