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ON THE FUNCTIONAL INTEGRABILITY AND ASYMPTOTIC BEHAVIOURS OF A CERTAIN DIFFERENTIAL EQUATION WITH DELAY

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The present paper is a study of asymptotic properties of functionally integrable solutions of the differential equation with a deviating argument

$$(r(t)x'(t))' + f(t, x(g(t))) = h(t), \tag{1}$$

where the functions:

$$\begin{aligned}&r\colon [t_0, \infty) \to R\\ &h\colon [t_0, \infty) \to R\\ &f\colon [t_0, \infty) \times R \to R\\ &g\colon [t_0, \infty) \to R_+,\ g'(t) \geq 0 \quad \text{and}\\ &\lim_{t\to\infty} g(t) = \infty\end{aligned}$$

are continuous.

We restrict our attention to nontrivial solutions of (1), which exist on the interval $[t_0, \infty)$.

Definition 1. *A solution $x(t)$ of equation (1) is said to be oscillatory if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} t_n = \infty$. Otherwise, it is said to be nonoscillatory.*

Definition 2. (cf. [3]) *Let $x(t)$ be a solution of the differential equation (1). If*

$$0 < \int_{t_0}^{\infty} s^m W(|x(s)|)\, ds < \infty,$$

m-real number, where $W\colon [t_0, \infty) \to R$, $W(|u|) \geq 0$ is a given continuous nondecreasing function, then $x(t)$ belongs to class $L(m, W(\cdot))$.

If in the definition we put $m = 0$ and $W(|u|) = |u|^p$, $p > 0$, then we obtain the wellknown class $L(0, |\cdot|^p) = L_p[t_0, \infty)$, i.e.

$$0 < \int_{t_0}^{\infty} |u(s)|^p\, ds < \infty.$$

The results of this paper extend some results for the differential equation

$$(r(t)x'(t))' + a(t)x^{\alpha}(t) = b(t)$$

and for the class $L_p[t_0, \infty)$ obtained in paper [1]. An analogous problem was recently investigated in paper [2].

The proofs of theorems are based on the following lemma given in [3].

Lemma. *If the function $u(t)$ satisfies the following conditions*

$$|u^{(m)}(t)| \leq M \text{ for } t \geq t_0 > 0 \text{ and } m \geq 1, \; u \in L(m-1, W(\cdot)),$$

then $\lim_{t\to\infty} u(t) = 0$.

Let us start with the assumptions:

(I) $|f(t, u)| \leq a(t) W(|u|)$

(II) $|f(t, u)| \leq a(t)[W(|u|)]^{1/p}$, $p > 1$ where the functions a: $[t_0, \infty) \to R_+$, W: $[t_0, \infty) \to R$, $W(|u|) \geq 0$ are continuous and $W(|u|)$ is a nondecreasing function.

Theorem 1. *Let $h(t) \equiv 0$ and* (I) *be satisfied. If*

$$\int_{t_0}^{\infty} s^m |r(s)|^2 \, ds = \infty \quad \text{for} \quad m \in R \tag{2}$$

$$a(t) \leq Mg'(t)g^m(t), \tag{3}$$

then for arbitrary two solutions $x_1(t)$ and $x_2(t)$ of (1) *such that*

$$|W^{1/2}(|x_1(t)|)x_2'(t) - x_1'(t)W^{1/2}(|x_2(t)|)| \geq k > 0 \text{ for } t \geq t_0 > 0 \tag{4}$$

we have

$$x_1 \in L(m, W(\cdot)) \Rightarrow x_2 \notin L(m, W(\cdot)).$$

Proof. Assume that there exist two solutions $x_1(t)$ and $x_2(t)$ of equation (1) for which (4) holds and assume that $x_k \in L(m, W(\cdot))$ $(k=1, 2)$. Integrating (1) from t_0 to t we obtain $(k=1, 2)$

$$r(t)x_k'(t) = c - \int_{t_0}^{t} f(s, x_k(g(s))) \, ds,$$

where $c = r(t_0)\, x_k'(t_0)$. From (I) and (3) we get

$$|r(t)|\,|x_k'(t)| \leq |c| + \int_{t_0}^{t} |f(s, x_k(g(s)))| \, ds \leq |c| +$$

$$+ \int_{t_0}^{t} a(s) W(|x_k(g(s))|) \, ds \leq |c| + M\int_{t_0}^{t} g'(s)g^m(s) W(|x_k(g(s))|) \, ds =$$

$$= |c| + M\int_{g(t_0)}^{g(t)} u^m W(|u|) \, du.$$

From this it follows that there exists a positive constant B such that

$$|r(t)|\,|x_k'(t)|\leqslant B \quad \text{if} \quad t\geqslant t_0>0.$$

Now we can write

$$\int_{t_0}^{t} |W^{1/2}(|x_1(s)|)x_2'(s)-x_1'(s)W^{1/2}(|x_2(s)|)|^2 s^m |r(s)|^2\,\mathrm{d}s = I(t).$$

From (4) we have

$$I(t)\geqslant k^2\int_{t_0}^{t} s^m |r(s)|^2\,\mathrm{d}s,$$

which from (2) implies that

$$\lim_{t\to\infty} I(t)=\infty. \tag{5}$$

On the other hand we have

$$I(t)\leqslant \sum_{k=0}^{2}\binom{2}{k}\int_{t_0}^{t} |r(s)x_2'(s)|^{2-k}|r(s)x_1'(s)|^k W^{(2-k)/2}(|x_1(s)|)\,W^{k/2}(|x_2(s)|)s^m\,\mathrm{d}s.$$

However, the integrals

$$\int_{t_0}^{t} |r(s)x_2'(s)|^{2-k}|r(s)x_1'(s)|^k W^{(2-k)/2}(|x_1(s)|)s^{m(2-k)/2}W^{k/2}(|x_2(s)|)s^{mk/2}\,\mathrm{d}s\leqslant$$

$$\leqslant B^2\int_{t_0}^{t} [s^m W(|x_1(s)|)]^{(2-k)/2}[s^m W(|x_2(s)|)]^{k/2}\,\mathrm{d}s\leqslant$$

$$\leqslant B^2\left(\int_{t_0}^{t} s^m W(|x_1(s)|)\,\mathrm{d}s\right)^{(2-k)/2}\left(\int_{t_0}^{t} s^m W(|x_2(s)|)\,\mathrm{d}s\right)^{k/2}$$

are finite as $t\to\infty$. Hence $I(t)$ is finite as $t\to\infty$, which contradicts (5). Hence the supposition that there exist two solutions of (1) satisfying (4) and both belonging to the class $L(m, W(\cdot))$ is not true.

Theorem 2. *Let $h(t)\equiv 0$ and (II) hold, and moreover assume that*

$$r(t)>0 \quad \textit{for} \quad t\in[t_0,\infty) \tag{6}$$

$$\int_{t_0}^{\infty}\frac{\mathrm{d}s}{r(s)}<\infty \tag{7}$$

$$\frac{a^p(t)}{g'(t)g^m(t)}\in L(0, |\cdot|^{1/(p-1)}) \quad \textit{for} \quad m\in R; \tag{8}$$

then every oscillatory solution of (1) does not belong to class $L(m, W(\cdot))$.

Proof. Let $x(t)$ be an oscillatory solution of (1). There exists a sequence $\{t_n\}_{n=1}^{\infty}$ of consecutive zeros of $x(t)$. Let t_{k-1}, t_k be two successive zeros of $x(t)$ and let $z_k \in [t_{k-1}, t_k]$ such that $|x(z_k)| = d_k$, where d_k is a true maximum of $|x(t)|$ in $[t_{k-1}, t_k]$ and let $d_k \geq \frac{3}{4} d$ for all $k \geq 1$ where $d = \text{const.} > 0$. Let a_k be the largest point before z_k and let b_k be the smallest point after z_k such that

$$|x(a_k)| = |x(b_k)| = \frac{d_k}{2} \quad \text{for} \quad k \geq 1. \tag{9}$$

The choice of a_k and b_k implies that

$$|x(t)| > \frac{d_k}{2} \quad \text{in} \quad (a_k, b_k).$$

Now

$$x(z_k) = x(a_k) + \int_{a_k}^{z_k} x'(s)\, ds$$

implies

$$|x(z_k)| \leq |x(a_k)| + \int_{a_k}^{z_k} |x'(s)|\, ds. \tag{10}$$

From (9) and (10)

$$\frac{d_k}{2} \leq \int_{a_k}^{z_k} |x'(s)|\, ds. \tag{11}$$

Proceeding similarly we obtain

$$\frac{d_k}{2} \leq \int_{z_k}^{b_k} |x'(s)|\, ds. \tag{12}$$

By summation of the inequalities (11) and (12) we have

$$d_k \leq \int_{a_k}^{b_k} |x'(s)|\, ds. \tag{13}$$

Squaring both sides of (13) we get by Schwarz's inequality

$$d_k^2 \leq \left\{\int_{a_k}^{b_k} |x'(s)|\, ds\right\}^2 = \left\{\int_{a_k}^{b_k} \frac{1}{\sqrt{r(s)}} \sqrt{r(s)} \sqrt{|x'(s)|} \sqrt{|x'(s)|}\, ds\right\}^2 \leq$$

$$\leq \int_{a_k}^{b_k} \frac{1}{r(s)}\, ds \int_{a_k}^{b_k} \{r(s)x'(s)\}x'(s)\, ds,$$

and by integrating by parts

$$\frac{d_k^2}{\int_{a_k}^{b_k} \frac{ds}{r(s)}} \leq \int_{a_k}^{b_k} \{r(s)x'(s)\}x'(s)\, ds = \tag{14}$$

$$= r(b_k)x'(b_k)x(b_k) - r(a_k)x'(a_k)x(a_k) - \int_{a_k}^{b_k} \{r(s)x'(s)\}'x(s)\,ds.$$

If $x(t) > 0$ in the interval $[t_{k-1}, t_k]$, then the choice of a_k and b_k in $[t_{k-1}, t_k]$ implies $x'(b_k) \leq 0$ and $x'(a_k) \geq 0$. Similarly if $x(t) < 0$ in the interval $[t_{k-1}, t_k]$, then the choice of a_k and b_k in $[t_{k-1}, t_k]$ implies $x'(b_k) \geq 0$ and $x'(a_k) \leq 0$. Thus in any case we have the following inequality for the first term on the right-hand side of (14), namely

$$r(b_k)x'(b_k)x(b_k) - r(a_k)x'(a_k)x(a_k) \leq 0. \tag{15}$$

From (14) and (15) we have

$$\frac{d_k^2}{\displaystyle\int_{a_k}^{b_k} \frac{ds}{r(s)}} \leq -\int_{a_k}^{b_k} \{r(s)x'(s)\}'x(s)\,ds. \tag{16}$$

Since $|x(t)| \leq d_k$ for $t \in (a_k, b_k)$

$$\frac{d_k}{\displaystyle\int_{a_k}^{b_k} \frac{ds}{r(s)}} \leq \int_{a_k}^{b_k} |f(s, x(g(s)))|\,ds$$

on the basis of equation (1). Since

$$\int_{t_0}^{\infty} |f(s, x(g(s)))|\,ds \leq \sum_{k=1}^{\infty} \int_{a_k}^{b_k} |f(s, x(g(s)))|\,ds,$$

we obtain the inequality

$$\frac{3}{4} d \sum_{k=1}^{\infty} \frac{1}{\displaystyle\int_{a_k}^{b_k} \frac{ds}{r(s)}} \leq \sum_{k=1}^{\infty} \frac{d_k}{\displaystyle\int_{a_k}^{b_k} \frac{ds}{r(s)}} \leq$$

$$\leq \sum_{k=1}^{\infty} \int_{a_k}^{b_k} |f(s, x(g(s)))|\,ds \leq \int_{t_0}^{\infty} |f(s, x(g(s)))|\,ds.$$

But then the left hand side of the last inequality tends to ∞ at $t \to \infty$ since $d > 0$, hence

$$\infty \leq \int_{t_0}^{\infty} |f(s, x(g(s)))|\,ds \leq \int_{t_0}^{\infty} a(s)[W(|x(g(s))|)]^{1/p}\,ds =$$

$$= \int_{t_0}^{\infty} \frac{a(s)}{[g'(s)g^m(s)]^{1/p}} [g'(s)g^m(s)\,W(|x(g(s))|)]^{1/p}\,ds \leq$$

$$\leq \left(\int_{t_0}^{\infty} \left[\frac{a^p(s)}{g'(s)g^m(s)}\right]^{1/(p-1)} ds\right)^{(p-1)/p} \left(\int_{g(t_0)}^{\infty} u^m W(|u|)\,du\right)^{1/p}.$$

On the basis of assumptions of the theorem it follows that $x \notin L(m, W(\cdot))$. With this the proof is achieved.

Theorem 3. *Suppose that* (II) *holds. If*

$$|r(t)| \geq \varrho > 0 \quad \text{for} \quad t \geq t_0 > 0 \tag{17}$$

$$h \in L(0, |\cdot|) \tag{18}$$

$$\frac{a^p(t)}{g'(t)} \in L(0, |\cdot|^{1/(p-1)}), \tag{19}$$

then for every solution $x \in L(0, W(\cdot))$ *of* (1)

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Let us show first that the derivative of the solution is bounded for $t \geq t_0$. Integrating both sides of (1) from t_0 to t it follows that

$$r(t)x'(t) = B + \int_{t_0}^{t} h(s)\, \mathrm{d}s - \int_{t_0}^{t} f(s, x(g(s)))\, \mathrm{d}s,$$

where $B = r(t_0)x'(t_0)$. By the Hölder inequality

$$|r(t)|\,|x'(t)| \leq \int_{t_0}^{t} |h(s)|\, \mathrm{d}s + \int_{t_0}^{t} |f(s, x(g(s)))|\, \mathrm{d}s + |B| \leq \tag{20}$$

$$\leq \int_{t_0}^{t} |h(s)|\, \mathrm{d}s + \int_{t_0}^{t} a(s)[W(|x(g(s))|)]^{1/p}\, \mathrm{d}s + |B| =$$

$$= \int_{t_0}^{t} |h(s)|\, \mathrm{d}s + \int_{t_0}^{t} \frac{a(s)}{[g'(s)]^{1/p}} [g'(s) W(|x(g(s))|)]^{1/p}\, \mathrm{d}s + |B| \leq$$

$$\leq \left(\int_{t_0}^{t} \left[\frac{a^p(s)}{g'(s)}\right]^{1/(p-1)} \mathrm{d}s\right)^{(p-1)/p} \left(\int_{g(t_0)}^{g(t)} W(|u|)\, \mathrm{d}u\right)^{1/p} + |B| + \int_{t_0}^{t} |h(s)|\, \mathrm{d}s$$

and by the assumption of the theorem we have the estimation $|x'(t)| \leq M$ for $t \geq t_0$. On the basis of the Lemma it follows that $\lim_{t\to\infty} x(t) = 0$. With this the proof is achieved.

Theorem 4. *Assume* (I) *and let*

$$h \in L(0, |\cdot|) \tag{21}$$

$$|r(t)| \geq \varrho > 0 \quad \text{for} \quad t \geq t_0 > 0 \tag{22}$$

$$a(t) \leq Mg'(t); \tag{23}$$

then for every solution $x \in L(0, W(\cdot))$ *of* (1) $\lim_{t\to\infty} x(t) = 0$ *holds.*

Proof. It follows from (1) and (I) that

$$|r(t)|\,|x'(t)| \leq \int_{t_0}^{t} |h(s)|\, ds + \int_{t_0}^{t} |f(s, x(g(s)))|\, ds + D \leq$$

$$\leq \int_{t_0}^{t} |h(s)|\, ds + \int_{t_0}^{t} a(s)\, W(|x(g(s))|)\, ds + D,$$

where $D = |r(t_0)|\,|x'(t_0)|$. On the basis of assumptions of the theorem

$$\varrho |x'(t)| \leq \int_{t_0}^{t} |h(s)|\, ds + M \int_{g(t_0)}^{g(t)} W(|u|)\, du + D.$$

Since $h \in L(0, |\cdot|)$ and $x \in L(0, W(\cdot))$, we have $|x'(t)| \leq N$. Applying Lemma we obtain $\lim_{t \to \infty} x(t) = 0$.

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О ФУНКЦИОНАЛЬНОЙ ИНТЕГРИРУЕМОСТИ И АСИМПТОТИЧЕСКОМ ПОВЕДЕНИИ НЕКОТОРОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ЗАПАЗДЫВАНИЕМ

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Резюме

В статье даются достаточные условия, при которых нелинейное дифференциальное уравнение с запаздыванием (1) имеет колеблющиеся решения, принадлежащие или непринадлежащие классу $L(m, W(\cdot))$. Даются также условия стремления к нулю при $t \to \infty$ решений (1), принадлежащих классу $L(m, W(\cdot))$.