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AN ENUMERATION THEOREM FOR ROOTED GRAPHS

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1. Introduction and notation

This paper presents a method for obtaining generating functions for some classes of rooted graphs, especially of those with a simple cyclic structure. Our method is based on the idea of an "$\mathcal{H}$-centre" of a graph. (A comprehensive survey of other methods of enumeration of rooted graphs can be found in [2].) In Section 2 we prove some results related to $\mathcal{H}$-centres. Section 3 contains an application of these results to the enumeration of rooted graphs.

All graphs discussed in this paper are finite, undirected, without loops and multiple edges (cf. [1]). The graph K_1 will be considered as a disconnected graph. When speaking about a set $\mathcal{S}$ of graphs, we always mean that graphs in $\mathcal{S}$ are pairwise non-isomorphic.

For any graph G and any vertex $v \in V(G)$, the set of vertices of G, the pair (G, v) is called a rooted graph. Two rooted graphs (G_1, v_1), (G_2, v_2) are isomorphic if there is a graph isomorphism f: $G_1 \to G_2$ such that $f(v_1) = v_2$.

Consider two graphs G, H and their vertices $u \in V(G)$, $v \in V(H)$. Suppose G and H have m and n vertices, respectively. A *1-amalgamation* of graphs G, H is the graph $K = (G, u, v, H)$ defined in the following way:

(i) K has $m + n - 1$ vertices, partitioned into two groups A, B such that $A \cap B = \{w\}$;

(ii) if K_A and K_B are the subgraphs of K induced by the sets A and B respectively, then the rooted graphs (K_A, w) and (G, u) are isomorphic, and the same holds for (K_B, w) and (H, v);

(iii) the number of edges of K is equal to the sum of numbers of edges of G and H.

In what follows we simply identify vertices of (G, u) and (H, v) with those of (K_A, w) and (K_B, w), respectively, i.e. we put $V(G) \cup V(H) = V(K)$, etc. The set of all pairwise non-isomorphic 1-amalgamations of graphs G and H will be denoted by $[G, H]$.

2. $\mathscr{H}$-decompositions and $\mathscr{H}$-centres of graphs

Let $\mathscr{H}$ be a set of graphs. A graph G is said to be *$\mathscr{H}$-decomposable* if there exist graphs G_1 and H_1 such that $H_1 \in \mathscr{H}$ and $G \in [G_1, H_1]$. *We say that* G is *$\mathscr{H}$-indecomposable if* G is not $\mathscr{H}$-decomposable.

Any maximal (in the sense of inclusion) subset $B \subseteq V(G)$ such that B induces in G a connected $\mathscr{H}$-indecomposable subgraph is said to be an *$\mathscr{H}$-centre* of G.

Example 1. Let P_n denote the path of length n. Put $\mathscr{H} = \{P_2\}$, $G = P_3$. Then G is $\mathscr{H}$-decomposable and has exactly 3 $\mathscr{H}$-centres, each of which induces the subgraph K_2 in G.

We see that, in general, a graph G can contain an arbitrary number of $\mathscr{H}$-centres. However, we are interested in cases when G has a unique $\mathscr{H}$-centre.

For any set $\mathscr{H}$ of graphs let $[\mathscr{H}]$ denote the smallest set $\mathscr{S}$ of graphs which satisfies the following conditions:

(a) $\mathscr{H} \subseteq \mathscr{S}$;

(b) $[H_1, H_2] \subseteq \mathscr{S}$ for any $H_1, H_2 \in \mathscr{S}$.

Furthermore, a set $\mathscr{H}$ of graphs will be called a *c-set* of graphs if $\mathscr{H}$ is a non-empty set of connected graphs such that for any $H \in \mathscr{H}$ and any connected induced subgraph H_1 of H we have $H_1 \in \mathscr{H}$.

Example 2. Put $\mathscr{H} = \{P_n;\ n \geqq 1\}$. It is easy to show that $\mathscr{H}$ is a c-set of graphs and that $[\mathscr{H}]$ consists of all trees of order at least 2.

Lemma 1. *Let $\mathscr{H}$ be a c-set of graphs. Suppose that $G \in [\mathscr{H}]$. Then G has no $\mathscr{H}$-centre.*

Proof. Let $B \subseteq V(G)$ such that the subgraph G_B of G induced by the set B is connected. One can easily see that if $\mathscr{H}$ is a c-set of graphs, then $[\mathscr{H}]$ has the same property. Thus, $G \in [\mathscr{H}]$ implies $G_B \in [\mathscr{H}]$. But clearly any graph in $[\mathscr{H}]$ is $\mathscr{H}$-decomposable. Lemma 1 follows.

Lemma 2. *Let $\mathscr{H}$ be a c-set of graphs. Suppose that G is a connected graph and $G \notin [\mathscr{H}]$. Then G has a unique $\mathscr{H}$-centre.*

Proof. Let n_0 be the minimum number such that there is a connected graph $G \notin [\mathscr{H}]$ with n_0 vertices. Consider such a graph G_0 with n_0 vertices. If G_0 is $\mathscr{H}$-indecomposable, then $V(G_0)$ is the unique $\mathscr{H}$-centre of G. Now, let G_0 be $\mathscr{H}$-decomposable, i.e. $G_0 \in [G_1, H_1]$ where $H_1 \in \mathscr{H}$. Clearly G_1 is connected, has less than n_0 vertices and $G_1 \notin [\mathscr{H}]$, which contradicts the choice of n_0.

We shall continue by induction. Assume that $n > n_0$ and that any connected graph $H \notin [\mathscr{H}]$ with less than n vertices has a unique $\mathscr{H}$-centre. Take a graph $G \notin [\mathscr{H}]$ which is connected and of order n (our assumptions guarantee that such a graph exists). Again, if G is $\mathscr{H}$-indecomposable, then $V(G)$ is the unique $\mathscr{H}$-centre of G. Let G be $\mathscr{H}$-decomposable. Then there are two connected graphs

G_1, H_1 such that $G_1 \notin [\mathscr{H}]$, $H_1 \in \mathscr{H}$ and $G \in [G_1, H_1]$. Clearly G_1 has less than n vertices. By the induction hypothesis, G_1 has aunique $\mathscr{H}$-centre B.

Suppose that a set $D \subseteq V(G)$ is an $\mathscr{H}$-centre of G. Consider the set $S = D \cap V(H_1)$. If S has at least two elements, then there exist connected graphs G_2, H_2 such that H_2 is an induced subgraph of H_1 and the subgraph G_D of G induced by the set D belongs to $[G_2, H_2]$. Since $\mathscr{H}$ is a c-set, we deduce that $H_2 \in \mathscr{H}$, i.e. D cannot be an $\mathscr{H}$-centre of G. Therefore $|S| \leqq 1$ and $D \subseteq V(G_1)$. But then $D \subseteq B$ since B is a maximal $\mathscr{H}$-indecomposable subgraph inducing set in $V(G_1)$. We see that B is the unique $\mathscr{H}$-centre of G, q.e.d.

Lemma 3. *Let $\mathscr{H}$ be a c-set of graphs. Suppose that a connected graph G has an $\mathscr{H}$-centre $B \subseteq V(G)$. Then any connected induced subgraph $H \subseteq G$ such that $|B \cap V(H)| \leqq 1$ belongs to $[\mathscr{H}]$.*

Proof. Let n_0 be the minimum number such that there is a connected graph $G \notin [\mathscr{H}]$ with n_0 vertices. Any such graph must be $\mathscr{H}$-indecomposable, i.e. $B = V(G)$. Now assume that $n > n_0$ and the claim of lemma 3 holds for any connected graph $H \notin [\mathscr{H}]$ of order less than n. Take a connected graph $G \notin [\mathscr{H}]$ of order n. We may suppose that G is $\mathscr{H}$-decomposable and $G \in [G_1, H_1]$, where $G_1 \notin [\mathscr{H}]$ and $H_1 \in \mathscr{H}$. Lemma 2 implies that G and G_1 have the same unique centre B. Let H be a connected subgraph of G such that $|B \cap V(H)| \leqq 1$. Then either $H \subseteq G_1$ or $H \in [G_2, H_2]$, where $G_2 \subseteq G_1$, $H_2 \subseteq H_1$ and H_2 is connected. In the first case $H \in [\mathscr{H}]$ by the induction hypothesis. In the second case $H_2 \in [\mathscr{H}]$. If $G_2 = K_1$, then $H = H_2$, whence $H \in [\mathscr{H}]$. Finally, if $G_2 \neq K_1$, then G_2 is connected and $G_2 \in [\mathscr{H}]$ by the induction hypothesis, whence again $H \in [\mathscr{H}]$. The proof is finished.

We shall summarize the above results in the following:

Theorem 1. *Let $\mathscr{H}$ be a c-set of graphs and G be a connected graph. Then G has no $\mathscr{H}$-centre iff $G \in [\mathscr{H}]$ and G has a unique $\mathscr{H}$-centre iff $G \notin [\mathscr{H}]$. If B is the $\mathscr{H}$-centre of G, then any connected induced subgraph $H \subseteq G$ with $|B \cap V(H)| \leqq 1$ belongs to $[\mathscr{H}]$.*

3. $\mathscr{H}$-centres and enumeration

There is a natural connection between $\mathscr{H}$-centres and enumeration of certain classes of rooted graphs. Several enumeration theorems can be derived. We consider only one example.

Let $\mathscr{G}$ be the set of all connected graphs and $\mathscr{H}$ be a c-set of graphs such that $\mathscr{H} = [\mathscr{H}] \neq \mathscr{G}$. Let $\mathscr{K} \subseteq \mathscr{G} - \mathscr{H}$ be a non-empty set of connected graphs. It follows from Theorem 1 that any graph in $\mathscr{K}$ has a unique $\mathscr{H}$-centre.

Denote by $\mathscr{H}^*$, or $\mathscr{K}^*$ the set of all rooted graphs (H, v) such that $H \in \mathscr{H}$, or $H \in \mathscr{K}$, respectively. Consider a rooted graph $(G, u) \in \mathscr{K}^*$. The root u will be called *simple* if there is at most one path P in G joining u with a vertex of the $\mathscr{H}$-centre of G such that P contains exactly one vertex of the $\mathscr{H}$-centre of G. Let

$$H(x)=x+\sum_{n=2}^{\infty}h_n x^n; \tag{1}$$

$$B(x)=\sum_{n-2}^{\infty}b_n x^n; \tag{2}$$

$$K(x)=\sum_{n=2}^{\infty}k_n x^n \tag{3}$$

be generating functions such that

(1) h_n is the number of graphs of order n in $\mathscr{H}^*$;

(2) b_n is the number of graphs $G\in\mathscr{H}^*$ of order n such that their root belongs to the $\mathscr{H}$-centre of G;

(3) k_n is the number of graphs of order n of $\mathscr{H}^*$ such that their root is simple.

Theorem 2. $K(x)=(1-H(x))^{-1}B(x)$.

Proof. We prove by induction that the number of graphs $G\in\mathscr{H}^*$ with a simple root such that the distance between the root and the $\mathscr{H}$-centre of G is equal to n is determined by the generating function $K_n(x)=B(x)\cdot H^n(x)$. The case $n=0$ is trivial because $K_0(x)=B(x)$.

Denote by $\mathscr{H}_n^*$ the set of graphs $G\in\mathscr{H}^*$ such that G has a simple root and the distance between this root and the $\mathscr{H}$-centre of G is equal to n. Let K_2 be a fixed graph with $V(K_2)=\{u_1, v_1\}$. Put $\mathscr{H}^\dagger=\mathscr{H}^*\cup\{(K_1, w)\}$, where (K_1, w) is the trivial rooted graph. Consider a mapping f: $\mathscr{H}_n^*\times\mathscr{H}^\dagger\rightarrow\mathscr{H}_{n+1}^*$ defined as follows:

$$f((G, u), (H, v))=(((G, u, u_1, K_2), v_1, v, H), v).$$

We shall show that f is a bijection.

Let $(K, z)\in\mathscr{H}_{n+1}^*$ and B denote the $\mathscr{H}$-centre of K. Since z is a simple root, there is exactly one path $zz_1\ldots z_{n+1}$ in K such that $z_i\in V(K)-B$ for $1\leqq i\leqq n$ and $z_{n+1}\in B$. It follows that the graph K_0 obtained from K by removing the edge zz_1 has exactly two connected components, namely $K(z)$ and $K(z_1)$, containing z or z_1, respectively. Theorem 1 implies that $K(z)\in\mathscr{H}$ except the case when $K(z)=K_1$. Hence $(K(z), z)\in\mathscr{H}^\dagger$ and $(K(z_1), z_1)\in\mathscr{H}_n^*$. Moreover $f((K(z_1), z_1), (K(z), z))=(K, z)$. It follows from the uniqueness of the path $zz_1\ldots z_{n+1}$ that there is exactly one pair $((G, u), (H, v))\in\mathscr{H}_n^*\times\mathscr{H}^\dagger$ such that $f((G, u), (H, v))=(K, z)$, i.e. f is a bijection.

The last result immediately implies the relation $K_{n+1}(x)=B(x)\cdot H^{n+1}(x)$. Consequently,

$$K(x)=\sum_{n=0}^{\infty}K_n(x)=B(x)\sum_{n=0}^{\infty}H^n(x)=(1-H(x))^{-1}B(x).$$

The proof of Theorem 2 is finished.

According to Theorem 2, to compute the generating function for simple-rooted graphs in $\mathcal{K}^*$ it suffices to know the generating function for graphs in $\mathcal{H}^*$ and the symmetry groups of graphs in $\mathcal{K}^*$ having the root in their $\mathcal{H}$-centre. To show how Theorem 2 applies, consider the simplest case — enumeration of rooted unicyclic graphs.

Let $\mathcal{H}$ be the set of all trees with at least two vertices. Obviously $\mathcal{H}$ is a c-set of graphs and $\mathcal{H}=[\mathcal{H}]$. Let $\mathcal{K}$ denote the set of all connected unicyclic graphs. It is easily seen that the $\mathcal{H}$-centre of any unicyclic graph is exactly the set of all vertices of the unique cycle of G. Further, the root of any graph in $\mathcal{K}^*$ is simple. Thus, the generating function $K(x)$ enumerates all rooted unicyclic graphs, $B(x)$ enumerates rooted unicyclic graphs such that their root is contained in the cycle, and $H(x)$ enumerates rooted trees (cf. [2]):

$$H(x)=x+x^2+2x^3+4x^4+9x^5+\dots$$

The generating function $B(x)$ is easily computable using Pólya's theorem. The symmetry group of a rooted cycle C_n^* of order $n\geqq 3$ is isomorphic to the group Z_2 and for its cycle index $Z(C_n^*)$ we obtain:

$$2Z(C_n^*)=\begin{cases} s_1^n+s_1^2 s_2^{(n-2)/2} & \text{for } n \text{ even,}\\ s_1^n+s_1 s_2^{(n-1)/2} & \text{for } n \text{ odd.}\end{cases}$$

The Pólya's enumeration theorem implies

$$B(x)=\sum_{n=3}^{\infty} Z(C_n^*, H(x)).$$

The right-hand side of the last equation can be easily modified as follows:

$$B(x)=\left(\frac{1}{2}\, s_1\, \frac{s_1^2(1-2s_2)+s_2}{(1-s_1)(1-s_2)},\, H(x)\right).$$

Using Theorem 2 we immediately obtain the desired generating function $K(x)$.

Theorem 3. *The number of rooted connected unicyclic graphs is given by the generating function* $K(x)$, *where*

$$K(x)=\left(\frac{1}{2}\, s_1\, \frac{s_1^2(1-2s_2)+s_2}{(1-s_1)^2(1-s_2)},\, H(x)\right),$$

$H(x)$ *is the generating function for rooted trees and* $s_i=H(x^i)$ *for* $i=1, 2$.

After a short computation we obtain

$$K(x)=x^3+4x^4+15x^5+50x^6+164x^7+520x^8+1632x^9+\dots$$

REFERENCES

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ОДНА ТЕОРЕМА ПЕРЕЧИСЛЕНИЯ КОРНЕВЫХ ГРАФОВ

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Резюме

В статье описаны разбирения графов при помощи амалгамации. Полученные результаты применены для нахождения перечисляющих рядов для многих классов корневых графов.