Mathematica Slovaca

Bohdan Zelinka

Elongation in a graph

Mathematica Slovaca, Vol. 32 (1982), No. 3, 291--296

Persistent URL: http://dml.cz/dmlcz/136300

Math. Slovaca 32, 1982, No. 3, 291—296

ELONGATION IN A GRAPH

BOHDAN ZELINKA

In [1] the concept of the elongation of two vertices in an undirected graph is defined.

Let u, v be two vertices of a finite undirected graph G. If $u \neq v$ and u, v belong to the same connected component of G, then the elongation $el_G(u, v)$ of the vertices u, v is the maximum of the lengths of all paths in G connecting u and v. If $u = v$, then $el_G(u, v) = 0$. If u, v belong to distinct connected components of G, then $el_G(u, v) = \infty$. Instead of $el_G(u, v)$ we shall write $el(u, v)$ if it does not cause a misunderstanding.

It is well known that the elongation el_G is a metric on the vertex set of a finite connected graph G.

Proposition 1. *The elongation in a finite graph G is equal to the distance in G for any two vertices of G if and only if G is a forest.*

The proof is left to the reader.

We shall define some concepts related to the elongation.

The elongation diameter $ed(G)$ of a finite connected graph G is the maximum of $el_G(u, v)$ taken over all the pairs u, v of vertices of G. The inner elongation diameter $ined(G)$ of G is the minimum of $el_G(u, v)$ taken ever all the pairs u, v of distinct vertices of G. An elongation centre of G is a vertex u of G for which $\max_{v \in V(G)} el_G(u, v)$ attains the minimum; this minimum is called the elongation radius of G and denoted by $er(G)$.

Proposition 2. *The elongation diameter of a connected graph G is equal to* 1 *if and only if* $G \cong K_2$.

Proposition 3. *The elongation diameter of a connected graph G is equal to* 2 *if and only if either* $G \cong K_3$ *or G is a star.*

Proofs are straightforward.

Theorem 1. *Let u, v be two adjacent vertices of a finite connected graph G with n vertices. Then the equality* $el_G(u, v) = ed(G)$ *implies* $ed(G) = n - 1$.

Proof. Suppose that $el_G(u, v) = ed(G)$ holds. Evidently always $ed(G) \leqq n - 1$, where n is the number of vertices of G. If $el_G(u, v) - 1$, then $ed(G) - 1$ and by Proposition 2 the graph G is isomorphic to K_2, hence $n - 2$ and $ed(G) - n \quad 1$. Suppose $2 \leqq el_G(u, v) < n \quad 2$. Let P be a path of the length $el_G(u, v)$ connecting u and v. The path P does not contain the edge uv; otherwise it would have the length 1. Therefore P together with the edge uv forms a circuit C of the length $ed(G) + 1$. We have $ed(G) + 1 < n \quad 1$ and therefore there exists at least one vertex w of G not belonging to C. As G is connected, there exists a vertex z of C such that there exists a path P_0 connecting w and z and having no common vertex with C except z; let its length be p_0. Let y be a vertex of C such that yz is an edge of C. The union of P_0 and the path obtained from C by deleting the edge yz is a path connecting y and z and having the length $p_0 + el_G(u \quad v)$ which is at least $el_G(u, v) + 1$. This is a contradiction.

Corollary 1. *For a finite connected graph G the equality $ined(G) - ed(G)$ implies that G is Hamiltonian connected (i e. any two distinct vertices of G are connected by a Hamiltonian path).*

Corollary 2. *In a finite connected graph G any two distinct vertices have the same elongation if and only if G is Hamiltonian-connected.*

Theorem 2. *Let a, b be positive integers, $a < b$. Then there exists a finite connected graph G such that $ined(G) \quad a$, $ed(G) \quad b$.*

Proof. If $2a \leq b$ then let G be a graph consisting of two blocks (with a common vertex) which are both complete graphs, one with $a + 1$ vertices, the other with $b \quad a + 1$ vertices. Any two distinct vertices belonging to the first block have the elongation a, any two distinct vertices of the second block have the elongation $b \quad a$, because they are connected by a Hamiltonian path of the corresponding block and each path connecting them must be contained in this block. The elongation of two vertices not belonging to the same block is b, because they are connected by a Hamiltonian path of G. We have $a \leq b \quad a < b$, therefore $ined(G) - a$, $ed(G) = b$. If $a < b < 2a$, take a complete graph G_0 with $a + 1$ vertices, choose two vertices u, v of it and connect them by a path P of the length $b - a + 1$ whose inner vertices do not belong to G_0; denote the resulting graph by G. Each path connecting u and v in G either is P, or is contained in G_0. A Hamiltonian path connecting u and v in G_0 has the length a; this path is the longest path connecting u and v in G_0 and is longer than P, hence $el_G(u, v) = a$. The supposed inequalities imply that the length of P is at least 2 and therefore the vertex w of P adjacent to u is distinct from v. There exists a Hamiltonian path of G connecting u and w; it is the union of a Hamiltonian path of G_0 connecting u and v and the path obtained from P by deleting the vertex u and the edge uw. Hence $el_G(u, w) = b$. Evidently the elongation of any two distinct vertices of G lies between a and b, therefore $ined(G) = a$, $ed(G) = b$. If $a \quad b$, then the required graph is a complete graph with $a + 1$ vertices.

Proposition 4. *Let a, b be two positive integers, $a \leqq b$. Then there exists a finite connected graph G with the diameter a and the elongation diameter b.*

Proof. If $a=1$, then a complete graph with $b+1$ vertices has the required property. If $a \geqq 2$, we take a complete graph with $b-a+2$ vertices and a path of the length $a-1$ disjoint with it and identify one terminal vertex of this path with an arbitrary vertex of this complete graph. The graph thus obtained has the required property.

Theorem 3. *For the elongation radius and the elongation diameter of a finite connected graph G the inequalities*

$$ed(G) \leqq ed(G) \leqq 2er(G)$$

hold. If a, b are two positive integers such that $a \leqq b \leqq 2a$, then there exists a finite connected graph G such that $er(G)=a$, $ed(G)=b$.

Proof. Let a finite connected graph G be given. The inequality $er(G) \leqq ed(G)$ follows immediately from the definition of $er(G)$ and $ed(G)$. Let c be an elongation centre of G. Let u, v be two vertices of G such that $el_G(u, v)=ed(G)$. Then $el_G(c, u) \leqq er(G)$, $el_G(c, v) \leqq (G)$. From the triangle inequality we have

$$ed(G)=el_G(u, v) \leqq el_G(c, u)+el_G(c, v) \leqq 2er(G).$$

Now let two pos tive integers a, b be given such that $a \leqq b \leqq 2a$. If $a=b$, then for a complete graph G with $a+1$ vertices $er(G)=ed(G)=a=b$. If $a<b$, take a graph G with two blocks (with a common vertex) which are both complete graphs, one with $a+1$ vertices, the other with $b-a+1$ vertices. This graph has a Hamiltonian path, therefore $ed(G)=b$. The cut vertex of G is evidently an elongation centre of G and a maximal elongation of a vertex of G from this vertex is a, hence $er(G)=a$.

Proposition 6. *Let a, b be two positive integers, $a \leqq b$. Then there exists a finite connected graph G such that $ined(G)=a$, $er(G)=b$.*

Proof. Let G be a graph with two blocks (with a common vertex) which are both complete graphs, one with $a+1$ vertices, the other with $b+1$ vertices. The elongation of any two vertices of the first (or second) block is a (or b respectively). The elongation of two vertices not belonging to the same block is $a+b$. Hence $ined(G)=a$. The cut vertex of G has the elongation a (or b) from each other vertex of the first (or second, respectively) block, while to each other vertex there exists a vertex having the elongation $a+b$ from it. Hence the cut vertex of G is an elongation centre of G and $er(G)=b$.

When we study some numerical invariants of a graph, it is usual to relate them to other numerical invariants. In the sequel we shall relate the invariants concerning the elongation with the vertex connectivity, the domatic number and the Hadwiger number of a graph. Obviously it would be possible to relate them also to other invariants However, for example for the chromatic number of a graph it seems that the results would not be interesting. By subdividing each edge of a graph by one vertex we obtain a bipartite graph, i.e. a graph with the chromatic number 2. Therefore we may have graphs with the chromatic number 2 and arbitrary large values of $ed(G)$, $er(G)$, $ined(G)$.

If G is not a complete graph, then the vertex connectivity of G is the minimal number of vertices by whose deleting from G a disconnected graph is obtained. If G is a complete graph with n vertices, then its vertex connectivity is by definition $n-1$.

Theorem 4. *The elongation radius of a finite connected graph is greater than or equal to its vertex connectivity.*

Proof. Let G be a finite connected graph, let c be its elongation centre, let u be a vertex of G such that $el_G(c, u) = er(G)$. Let P be a path of the length $er(G)$ connecting c and u. If P is a Hamiltonian path of G, then G has $er(G)+1$ vertices and its vertex connectivity is at most $er(G)$. If P is not a Hamiltonian path of G, then there exists a vertex w of G not belonging to P. Let G_0 be the graph obtained from G by deleting all vertices of P except u; suppose that G_0 is connected. Then there exists a path P_0 in G_0 connecting u and w. The paths P, P_0 have no common vertex except u, therefore their union is a path in G connecting c and w and having the length at least $er(G)+1$, which is a contradiction with the assumption that c is an elongation centre of G. Hence G_0 is not connected and the vertex connectivity of G is at most $er(G)$. In the case of a complete graph the equality occurs.

The domatic number $d(G)$ of a graph G is the maximal number of classes of a partition of the vertex set of G, all of whose classes are dominating sets in G. (A dominating set in a graph G is a subset D of the vertex set $V(G)$ of G with the property that to each $x \in V(G)-D$ there exists $y \in D$ adjacent to x.)

Theorem 5. *For the elongation radius $er(G)$ and the domatic number $d(G)$ of a finite connected graph G we have*

$$er(G) \geqq d(G) - 1.$$

Proof. Let G be a finite connected graph, let its domatic number be d. Then there exists a partition $\{D_1, \ldots, D_d\}$ of the vertex set $V(G)$ of G such that D_i for $i=1, \ldots, d$ are dominating sets in G. Let u be a vertex of G; without loss of generality we may suppose that $u \in D_1$. Now we construct a sequence of vertices $v_1, \ldots, v_d$. We put $v_1 = u$, hence $v_1 \in D_1$. If v_i is constructed for some $i \leqq d-1$ and $v_i \in D_i$, then as D_{i+1} is a dominating set in G and $D_i \cap D_{i+1} = \emptyset$, there exists at least one vertex of D_{i+1} adjacent to v_i. Choose one of them and denote it by v_{i+1}. Then the vertices $v_1, \ldots, v_d$ are vertices of a path of the length $d-1$, one of whose terminal vertices is u. As u was chosen arbitrarily, we have $er(G) \geq d-1$. In the case of a complete graph the equality occurs.

The Hadwiger number (or contraction number) $\eta(G)$ of a connected graph G is the maximal number of vertices of a complete graph onto which G can be transformed by successive contractions of edges. The vertex set of G can be partitiones into $\eta(G)$ classes such that each class induces a connected subgraph of G and to any two of them there exists at least one edge joining a vertex of one of them with a vertex of the other.

Theorem 6. *The elongation radius $er(G)$ of a finite connected graph G is greater than or equal to $\eta(G)-1$, where $\eta(G)$ is the Hadwiger number of G.*

Proof. Instead of $\eta(G)$ we shall write only η. Then there exists a partition $\{V_1, \ldots, V_\eta\}$ of $V(G)$ with the above described properties. Let u be a vertex of G; without loss of generality we may suppose $u \in V_1$. We shall construct a finite sequence $v_1, w_1, v_2, w_2, \ldots, v_{\eta-1}, w_{\eta-1}, v_\eta$. Put $v_1 = u$. If we have constructed v_i for some $i \leqq \eta - 1$ and $v_i \in V_i$, then choose a vertex $w_i \in V_i$ which is adjacent to a vertex of V_{i+1}; this vertex of V_{i+1} will be denoted by v_{i+1}. By P_i denote the path connecting v_i and w_i in the subgraph of G induced by V_i for $i = 1, \ldots, \eta - 1$. Now take a path consisting of edges $w_i v_{i+1}$ for $i = 1, \ldots, \eta - 1$ and paths P_i. This is a path outgoing from u and having the length at least $\eta - 1$. As u was choosen arbitrarily, $er(G) \geqq \eta - 1$. For a complete graph the equality occurs.

Concluding the present paper we shall consider the direct product of graphs.

If G and H are undirected graphs with the vertex sets $V(G)$ and $V(H)$ respectively, then their direct product $G \times H$ is the graph whose vertex set is $V(G) \times V(H)$ and in which the vertices $[u_1, u_2]$, $[v_1, v_2]$ (for $u_1 \in V(G)$, $u_2 \in V(H)$, $v_1 \in V(G)$, $v_2 \in V(H)$) are adjacent if and only if either $u_1 = v_1$ and the vertices u_2, v_2 are adjacent in H, or $u_2 = v_2$ and the vertices u_1, v_1 are adjacent in G.

Theorem 7. *Let G, H be two finite connected graphs, let u_1, v_1 be two vertices of G and u_2, v_2 be two vertices of H. Then*

$$el_{G \times H}([u_1, u_2], [v_1, v_2]) \geqq$$
$$\geqq el_G(u_1, v_1) \cdot el_H(u_2, v_2) + \max(el_G(u_1, v_1), el_H(u_2, v_2)).$$

Proof. For each vertex x of G let $H(x)$ be the subgraph of $G \times H$ induced by the set of vertices whose first coordinate is x. For each vertex y of H let $G(y)$ be the subgraph of $G \times H$ induced by the set of vertices whose second coordinate is y. Evidently $H(x) \cong H$, $G(y) \cong G$ for each $x \in V(G)$ and $y \in V(H)$. Let P be a path of the length $el_G(u_1, v_1)$ connecting u_1 and v_1 in G, let Q be a path of the length $el_H(u_2, v_2)$ connecting u_2 and v_2 in H. Let the vertices of P be $u_1 = x_0, x_1, \ldots, x_r = v_1$ and let the vertices of Q be $u_2 = y_0, y_1, \ldots, y_s = v_2$, where $r = el_G(u_1, v_1)$, $s = el_H(u_2, v_2)$. Suppose $r \leqq s$. For $i = 0, 1, \ldots, s$ let P_i be the path in $G(y_i)$ whose vertices are $[x_0, y_i], [x_1, y_i], \ldots, [x_r, y_i]$. If s is even, then the vertices and edges of all paths P_i for $i = 0, 1, \ldots, s$ together with the edges connecting $[x_r, y_j]$ with $[x_r, y_{j+1}]$ for j even and $[x_0, y_j]$ with $[x_0, y_{j+1}]$ for j odd form a path connecting $[u_1, u_2]$ with

$[v_1, v_2]$ in $G \times H$ of the length $rs + r + s$, which is greater than or equal to $rs + s = el_G(u_1, v_1) \cdot el_H(u_2, v_2) + \max(el_G(u_1, v_1), el_H(u_2, v_2))$. If s is odd, then the vertices and edges of all paths P_i for $i = 0, 1, \ldots, s - 1$ together with the above described edges form a path connecting $[u_1, u_2]$ with $[v_1, v_2]$ in $G \times H$ of the length $rs + s = el_G(u_1, v_1) \cdot el_H(u_2, v_2) + \max(el_G(u_1, v_1), el_H(u_2, v_2))$. This implies the inequality. If $r > s$, we proceed analogously, interchanging G and H.

Corollary 3. *For any two finite connected graphs G, H the following inequalities hold:*

$$ed(G \times H) \geq ed(G) \cdot ed(H) + \max(ed(G), ed(H)),$$
$$ined(G \times H) \geq ined(G) \cdot ined(H) + \max(ined(G), ined(H)),$$
$$er(G \times H) \geq er(G) \cdot er(H) + \max(er(G), er(H)).$$

In the further investigation of the elongation it would be interesting to relate it to other numerical invariants of graphs (e.g. clique number, thickness) and to apply considerations analogous to those for the distance in a graph (e.g. to characterize metric spaces which are isometric to the metric space formed by the vertex set of a graph and the elongation on it).

REFERENCE

[1] ORE, O.: Theory of Graphs. AMS, Providence 1962.

Received September 2, 1980

Katedra matematiky
Vysokej školy strojnej a textilnej
Komenskeho 2
460 01 Liberec

ПРОТЯЖЕННОСТЬ В ГРАФЕ

Богдан Зелинка

Резюме

Пусть u, v — две вершины конечного связаного неориентированного графа G. Если $u \neq v$, то протяженность $el_G(u, v)$ вершин u, v есть максимум длин всех цепей в G, соединняющих u и v. Если $u = v$, то $el_G(u, v) = 0$. Введены понятия диаметра протяженности, внутреннего диаметра протяженности и радиуса протяженности и исследованы их свойства. Эти понятия тоже изучены в связи с другими численными инвариантами графов