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A NOTE ON HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS (II)

BOGDAN RZEPECKI

1. Introduction

In this paper we consider the Darboux boundary problem for the equation

$$(+)\qquad \frac{\partial^2 z}{\partial x\,\partial y}=f(x, y, z)$$

with continuous righ-hand side and conditions of the Krasnoselskii—Krein type. This part is closely related to Part I. In [18] there is a discussion of the existence and continuous dependence on initial functions and the right-hand side of the solution to the Darboux problem for the equation (+) with f satisfying the Kooi type conditions.

The questions of the unique solution (as a limit of successive approximations) and the continuous dependence of the solution on boundary data and right-hand side will be considered with use of the fixed point concept (given here as Proposition 1) due to Luxemburg [12]. For applications of the original Luxemburg theorem to hyperbolic partial differential equations with conditions of the Krasnoselskii—Krein type see: V. Ďurikovič [3]—[5] and J. S. W. Wong [19].

M. A. Krasnoselskii [9] has proved the following version of the well-known result of Schauder: If K is a non-empty bounded closed convex subset of a Banach space, A is a contraction and B is completely continuous on K, and $Ax+By\in K$ for x, y in K, then the equation $Ax+Bx=x$ has a solution in K. In Sec. 2 we give a modification of Krasnoselskii's theorem which enables us to get the global solutions of Equation (+) with $f=f_1+f_2$, where f_1, f_2 generate a contraction and a completely continuous transformation, respectively.

Next we give some remarks on the continuous dependence of solutions of our equation on the boundary data and on the function f.

The results of this paper are connected with the Bielecki method ([1], [2], [6]) of norm changing, and extend the facts of [18] and [19]. Let us remark that further results can be obtained if the concept of a metric space with the distance function taking its values in a normal cone in a Banach space and the Luxemburg concept will be used. See also [16] and [17].

2. Fixed point theorems

Let M be a non-empty set and let d be a function defined on $M \times M$ with $0 \leqslant d(x, y) \leqslant +\infty$. If d satisfies the usual axioms for metric space, then this function is called a generalized metric in M. Further, if every d-Cauchy sequence in M is d-convergent, then (M, d) is called [12] a *generalized complete metric space.* Moreover, we shall use the notations of $\mathscr{L}^*$-space, the $\mathscr{C}$-product of $\mathscr{L}^*$-spaces and a continuous mapping of $\mathscr{L}^*$-space into $\mathscr{L}^*$-space (see e.g. [11] pp. 83—90).

Proposition 1 (cf. [16]). *Let A be an arbitrary set, let B be an $\mathscr{L}^*$-space and let (M, d) be a generalized complete metric space. Suppose that $F\colon A \times B \to M$, $T\colon A \to M$ are one-to-one transformations and $F[A \times B] \subset T[A]$. Assume, moreover, that there exist $z_0 \in A$, $0 \leqslant k < 1$ such that for all y in B: $d(F(z_0, y), Tz_0) < \infty$, and $d(F(x_1, y), F(x_2, y)) \leqslant k \cdot d(Tx_1, Tx_2)$ for all $x_1, x_2 \in A$ with $d(Tx_1, Tx_2) < \infty$.*

Then there exists a unique function $\varphi\colon B \to A$ such that $F(\varphi(y), y) = T(\varphi(y))$ and $d(T(\varphi(y)), Tz_0) < \infty$ for each y in B. Further, if the function $F(x, \cdot)$ is continuous on B for all $x \in A$ with $d(Tx, Tz_0) < \infty$, then the function $T(\varphi(\cdot))$ is continuous on B.

Proposition 2. *Let E be a Banach space, let X be a non-empty subset of E, and let K be a non-empty convex closed subset of E. Suppose we are given: T — a one-to-one operator defined on X such that $T[X]$ is a closed subset of E and $T[X] \subset K$, S — a continuous mapping from K into a compact subset of E. Further, assume that F is a mapping from $X \times K$ to $T[X]$ satisfying the following conditions: (i) $\|F(x_1, y) - F(x_2, y)\| \leqslant k \cdot \|Tx_1 - Tx_2\|$ for every x_1, x_2 in X and $y \in K$, where k is a non-negative constant less than one, and (ii) $\|F(x, y_1) - F(x, y_2)\| \leqslant c \cdot \|Sy_1 - Sy_2\|$ for every $x \in X$ and y_1, y_2 in K, where c is a positive constant.*

Then there exists a point x_0 in X such that $F(x_0, Tx_0) = Tx_0$.

Proof. Let us put $M = E$, $A = X$ and $B = K$. Then, all the assumptions of Proposition 1 are satisfied and therefore there exists a mapping $\varphi\colon K \to X$ such that $F(\varphi(y), y) = T(\varphi(y))$ for all y in K.

We define an operator Φ as $x \mapsto T(\varphi(x))$. Then Φ maps K into itself, and

$$\|\Phi x - \Phi y\| = \|F(\varphi(x), x) - F(\varphi(y), y)\| \leqslant c\|Sx - Sy\| + k\|\Phi x - \Phi y\|.$$

Hence $\|\Phi x - \Phi y\| \leqslant (1-k)^{-1} c\ \|Sx - Sy\|$ for $x, y \in K$, and therefore Φ is continuous on K. Now we prove that $\Phi[K]$ is conditionally compact in E.

Indeed, let (Φx_n) be a sequence with $x_n \in K$ for $n \geqslant 1$. From the above $\|\Phi x_i - \Phi x_j\| \leqslant (1-k)^{-1} c\ \|Sx_i - Sx_j\|$ for all $i, j \geqslant 1$. Since $S[K]$ is a conditionally compact set, (Sx_n) has a convergent subsequence (Sx_k) and therefore (Φx_k) is a Cauchy sequence. Consequently, (Φx_k) is a convergent subsequence of the sequence (Φx_n).

By the Schauder Fixed Point Theorem there exists at least one v_0 in K such that $\Phi v_0 = v_0$. Hence $T(\varphi(v_0)) = F(\varphi(v_0), v_0) = F(\varphi(v_0), \Phi v_0) = F(\varphi(v_0), T(\varphi(v_0)))$ and the proof is completed.

3. Assumptions and notations

Assumptions and notations given below are valid throughout this paper and will not be repeated in formulations of particular theorems.

Suppose that $G=(0, a]\times(0, b]$, $P=[0, a]\times[0, b]$, $Q=P\times(-\infty, \infty)$ and λ is a bounded function on P such that $\lambda(x, y)>0$ for all (x, y) in G.

Let us denote:

by X — the set of all continuous functions on P;

by $\mathscr{X}$ — the set of pairs (σ, τ) such that the functions σ and τ are, respectively, of the class $C^1[0, a]$ and $C^1[0, b]$ satisfying the condition $\sigma(0)=\tau(0)$;

by $\mathscr{F}_0$ — the set of all continuous functions on Q;

by $\mathscr{F}$ — the set of functions $f\in\mathscr{F}_0$ such that $|f(x, y, u) - f(x, y, v)| \leqslant L_f(x, y)\,|u-v|$ for $(x, y)\in G$ and $-\infty<u,\ v<+\infty$, where L_f is a function (depending on f) on P with $0\leqslant L_f(x, y)\leqslant+\infty$;

by $\mathscr{F}_1$ — the set of all $f\in\mathscr{F}$ with $L_f(x, y)\equiv A_f$ on P, where $A_f>0$ is a constant (depending on function f);

by $\mathscr{F}_2$ — the subset of $\mathscr{F}_1$ consisting of uniformly bounded functions;

by $\mathscr{X}_2$ — the subset of $\mathscr{X}$ consisting of all pairs (σ, τ) of equicontinuous functions on $[0, a]$ and $[0, b]$, respectively.

Moreover, we denote by $C(P)$ the Banach space of all continuous functions on P with the usual supremum norm $\|\cdot\|$.

Let us put

$$F(z, (f, \sigma, \tau))\,(x, y)=\sigma(x)+\tau(y)-\sigma(0)+\int_0^x\int_0^y f(u, v, z(u, v))\,du\,dv$$

for $f\in\mathscr{F}_0$, $(\sigma, \tau)\in\mathscr{X}$ and z in X.

We ask for a function z in X satisfying the equation $(+)$ on P, and such that $z(x, 0) = \sigma(x)$ for $0\leqslant x\leqslant a$ and $z(0, y)=\tau(y)$ for $0\leqslant y\leqq b$. If $f\in\mathscr{F}_0$ and $(\sigma, \tau)\in\mathscr{X}$, then the above Darboux problem for $(+)$ is equivalent to the solution of the following equation

$$(*)\qquad z(x, y)=F(z, (f, \sigma, \tau))\,(x, y)$$

in the set X.

4. Class $\mathscr{F}$ of functions

Let $f \in \mathscr{F}$. We say that a function f satisfies:

(i) *Lipschitz—Bielecki conditions* ([1], [2]), if $f \in \mathscr{F}_1$ and $L_f(x, y) \equiv A_f$, $\lambda(x, y) = \exp(p(x+y))$ on P, where $p \geq 0$ is a constant;

(ii) *Rosenblatt—Kooi—Luxemburg conditions* ([15], [7], [13], [19]), if $|f(x, y, z)| \leq M(x \cdot y)^r$ for $(x, y, z) \in Q$ and $L_f(x, y) = B_f(x \cdot y)^{-1}$, $\lambda(x, y) = (x \cdot y)^{r+1}$ on P, where $M > 0$, $r > -1$ are constants and $B_f > 0$ is a constant (depending on f) such that $B_f < (r+1)^2$;

(iii) *Krasnoselskii—Krein—Luxemburg conditions* ([10], [8], [12], [14], [19]), if f is a bounded function on Q, $(x \cdot y)^{\beta} \cdot |f(x, y, u) - f(x, y, v)| \leq D_f |u - v|^{\alpha}$ on Q and $L_f(x, y) = C_f \cdot (x \cdot y)^{-1}$, $\lambda(x, y) = (x \cdot y)^{p\sqrt{C_f}}$ on P, where $C_f > 0$, $D_f > 0$, $\alpha > 0$, β and $p > 1$ are constants such that $\alpha < 1$, $\beta < \alpha$, $(1-\alpha) \cdot \sqrt{C_f} < 1 - \beta$ and $p \cdot C_f \cdot (1-\alpha)^2 < (1-\beta)^2$.

In X we define the distance function d as follows: for each z_1, z_2 in X we put

$$d(z_1, z_2) = \sup\left\{\frac{|z_1(x, y) - z_2(x, y)|}{\lambda(x, y)} : (x, y) \in G\right\}.$$

Obviously, d is a generalized metric in X such that $\sup_G \lambda(x, y))^{-1} \|z_1 - z_2\| \leq d(z_1, z_2)$ for all z_1, z_2, and therefore (cf. [19]) (X, d) is a generalized complete metric space.

Let F be the transformation defined in Sec. 3. We introduce the following Assupmtion (0):

(0). There exists a function z_0 in X such that for $f \in \mathscr{F}$ and $(\sigma, \tau) \in \mathscr{X}$ we have

$$z_0(x, y) - F(z_0, (f, \sigma, \tau))(x, y) = O(\lambda(x, y))$$

for each (x, y) in G.

Let Assumption (0) be satisfied. The above defined F is said to satisfy the $\mathscr{L}^*$-condition, if the sets $\mathscr{F}$, $\mathscr{X}$ are considered as $\mathscr{L}^*$-spaces, $\mathscr{F} \times \mathscr{X}$ as their $\mathscr{L}^*$-product, and for every fixed z in X with $d(z, z_0) < \infty$ the transformation $F(z, \cdot)$ maps $\mathscr{F} \times \mathscr{X}$ continuously into (x, d).

Notice that if $\sup\left\{(\lambda(x, y))^{-1} \int_0^x \int_0^y \lambda(u, v)\, du\, dv : (x, y) \in G\right\} < \infty$ and the sets $\mathscr{X}$, $\mathscr{F}$ are endowed with the convergence, respectively:

$$\lim_{n\to\infty} (\sigma_n, \tau_n) = (\sigma_0, \tau_0) \text{ meaning}$$

$$\lim_{n\to\infty} \sup \{|(\sigma_n(x) + \tau_n(y) - \sigma_n(0)) -$$

$$-(\sigma_0(x)+\tau_0(y)-\sigma_0(0))|\,(\lambda(x, y))^{-1}:(x, y)\in G\}=0$$

and

$$\lim_{n\to\infty} f_n = f_0 \text{ meaning}$$

$$\lim_{n\to\infty} \sup\,\{(\lambda(x, y))^{-1}|f_n(x, y, z)-f_0(x, y, z)|:(x, y)\in G,\, z\in\Omega\}=0$$

$$\text{for every compact } \Omega \text{ in } (-\infty, \infty),$$

then our transformation F satisfies the $\mathscr{L}^*$-condition. The proof of this fact is similar to the proof of Remark given in [18]. Therefore it will be omitted.

The following theorem holds:

Theorem 1. *Let Assumption* (0) *be satisfied, let the functions* $\lambda\cdot L_f$ $(f\in\mathscr{F})$ *be integrable on P, and let*

$$k_f=\sup\left\{\frac{1}{\lambda(x, y)}\int_0^x\int_0^y \lambda(u, v)L_f(u, v)\,du\,dv:\,(x, y)\in G\right\}<1.$$

Then, for an arbitrary $f\in\mathscr{F}$ *and* $(\sigma, \tau)\in\mathscr{X}$ *there exists a unique function* $z_{(f,\sigma,\tau)}$ *in X satisfying the equation* $(*)$ *on P and such that* $d(z_0, z_{(f,\sigma,\tau)})<\infty$.

Moreover, if F satisfies the $\mathscr{L}^*$*-condition and* $\sup\{k_f: f\in\mathscr{F}\}<1$ *then* $(f, \sigma, \tau)\mapsto z_{(f,\sigma,\tau)}$ *maps* $\mathscr{F}\times\mathscr{X}$ *continuously into* (X, d).

Proof. Let $B=\mathscr{F}\times\mathscr{X}$. Evidently, F mapr $X\times B$ into X and $d(z_0, F(z_0, \xi))<\infty$ for each ξ in B. We prove that $d(F(z_1, \xi), F(z_2, \xi))\leqslant k\cdot d(z_1, z_2)$ for $d(z_1, z_2)<\infty$, where $k=\sup\{k_f: f\in\mathscr{F}\}$.

Indeed, for $(x, y)\in G$, $\xi\in B$ and $z_1, z_2\in X$, we have

$$|F(z_1, \xi)\,(x, y)-F(z_2, \xi)\,(x, y)|\leqslant$$

$$\leqslant d(z_1, z_2)\cdot\int_0^x\int_0^y \lambda(u, v)L_f(u, v)\,du\,dv$$

hence $d(F(z_1, \xi), F(z_2, \xi))\leqslant k\cdot d(z_1, z_2)$ when $d(z_1, z_2)<\infty$. The application of Proposition 1 completes the proof.

Remark. Each of the conditions given below implies the assumptions of Theorem 1 for function f:

1° Lipschitz—Bielecki conditions;

2° Rosenblatt—Kooi—Luxemburg conditions;

3° Krasnoselskii—Krein—Luxemburg conditions.

Now we prove this. The case 1° is obvious. If 2° is satisfied and $\eta(x, y) = \sigma(x) + \tau(y) - \sigma(0)$ with $(\sigma, \tau)\in\mathscr{X}$, then

$$k_f = B_f \cdot \sup \left\{ (x \cdot y)^{-(r+1)} \cdot \int_0^x \int_0^y (u \cdot v)^r \, du \, dv \colon (x, y) \in G \right\} = (r+1)^{-2} B_f < 1$$

and

$$z_0(x, y) - F(z_0, (f, \sigma, \tau))\,(x, y) = O((x \cdot y)^{r+1}) \text{ on } G$$

for each $z_0 \in X$ such that $z_0(x, y) - \eta(x, y) = O((x \cdot y)^{r+1})$ on G (in particular, for $z_0(x, y) = \eta(x, y) + M(r+1)^{-1}(x \cdot y)^{r+1}$ on P). Finally, from 3° we obtain $k_f < 1$ and if $(\sigma, \tau) \in \mathscr{X}$, $w_0 \in X$, $w_{n+1}(x, y) = F(w_n, (f, \sigma, \tau))\,(x, y)$ for $n = 0, 1, \ldots$, then (cf. [12], [19]) there exists an index N such that $d(w_N, w_{N+l}) < \infty$ for $l \geq 1$ and, in particular,

$$w_N(x, y) - F(w_N, (f, \sigma, \tau))\,(x, y) = O((x \cdot y)^{p \vee c_f})$$

on G.

For example, we apply Lipschitz—Bielecki conditions. Let us denote by $\mathscr{X}_1$ the set $\mathscr{X}$ with the product metric generated by the usual supremum metrics. The set $\mathscr{X}_2$ shall be considered with the pointwise convergence. We endow the sets $\mathscr{F}_1, \mathscr{F}_2$ with the almost uniform convergence and pointwise convergence on Q, respectively.

Using the Lebesgue Bounded Convergence Theorem and proceeding similarly as in the proof of Corollary 2 from [18], we obtain the following result as a consequence of Theorem 1:

Let $i = 1,2$. For an arbitrary $f \in \mathscr{F}_i$ and $(\sigma, \tau) \in \mathscr{X}$ there exists a unique function $z_{(f, \sigma, \tau)}$ in X satisfying the equation $(*)$ on P. Moreover, if $\sup \{A_f \colon f \in \mathscr{F}_i\} < \infty$ *then $(f, \sigma, \tau) \mapsto z_{(f, \sigma, \tau)}$ maps continuously the $\mathscr{L}^*$-product $\mathscr{F}_i \times \mathscr{X}_i$ into $C(P)$.*

5. Class $\mathscr{F}_0$ of functions

Assume that $g \in \mathscr{F}$ and $h \in \mathscr{F}_0$ are bounded functions on Q. We prove that if $(\sigma, \tau) \in \mathscr{X}$ and L_g is an integrable function on P, then there exists a function z in X satisfying Equation $(*)$ with $f = g + h$.

Without loss of generality we may suppose that $\sigma(x) = \tau(y) \equiv 0$ for (x, y) in P. Let us put:

$$X = \{ z \in C(P) \colon \|z\| \leq ab(M_1 + M_2) \},$$

$$K = \left\{ z \in C(P) \colon |z(x, y)| \leq ab(M_1 + M_2) \cdot \exp\left(-p \int_0^x \int_0^y L_g(u, v)\, du\, dv \right.\right. \text{ for } (x, y) \in P \Big\},$$

$$(Tz)\, x, y) = \exp\left(-p \cdot \int_0^x \int_0^y L_g(u, v)\, du\, dv \right) \in z(x, y) \text{ for } z \in X,$$

$$(Sz)(x,y)=\int_0^x\int_0^y h\left(u,v,\exp\left(p\cdot\int_0^u\int_0^v L_g(t,s)\,\mathrm{d}t\,\mathrm{d}s\right)\cdot z(u,v)\right)\mathrm{d}u\,\mathrm{d}v \text{ for } z\in K,$$

$$G(w,z)(x,y)=\exp\left(-p\cdot\int_0^x\int_0^y L_g(u,v)\,\mathrm{d}u\,\mathrm{d}u\right)\cdot\Big[(Sz)(x,y)+$$

$$+\int_0^x\int_0^y g(u,v,w(u,v))\,\mathrm{d}u\,\mathrm{d}v\Big] \text{ for } (w,z)\in X\times K,$$

where $p>1$ is a constant and M_1, M_2 are numbers thet bound the functions g and h, respectively.

It can be easily seen that $G[X\times K]\subset T[X]\subset K$, $T[X]$ is closed and K is a closed convex subset of $C(P)$. Obviously, S is continuous on K and by Ascoli—Arzela Theorem the set $S[K]$ is conditionally compact. For w_1, $w_2\in X$, $z\in K$ and $(x,y)\in P$, we have

$$\left|\int_0^x\int_0^y (g(u,v,w_1(u,v))-g(u,v,w_2(u,v)))\,\mathrm{d}u\,\mathrm{d}v\right|\leqslant$$

$$\leqslant\int_0^x\int_0^y L_g(u,v)\,|w_1(u,v)-w_2(u,v)|\,\mathrm{d}u\,\mathrm{d}v=$$

$$=\int_0^x\int_0^y L_g(u,v)\cdot\exp\left(p\cdot\int_0^u\int_0^v L_g(t,s)\,\mathrm{d}t\,\mathrm{d}s\right)\cdot$$

$$\cdot\exp\left(-p\cdot\int_0^u\int_0^v L_g(t,s)\,\mathrm{d}t\,\mathrm{d}s\right)|w_1(u,v)-w_2(u,v)|\,\mathrm{d}u\,\mathrm{d}v\leqslant$$

$$\leqslant\|Tw_1-Tw_2\|\cdot\int_0^x\int_0^y L_g(u,v)\cdot\exp\left(p\cdot\int_0^u\int_0^v L_g(t,s)\,\mathrm{d}t\,\mathrm{d}_s\right)\mathrm{d}u\,\mathrm{d}v\leqslant$$

$$\leqslant p^{-1}\cdot\exp\left(p\cdot\int_0^x\int_0^y L_g(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)\cdot\|Tw_1-Tw_2\|$$

and it follows $\|G(w_1,z)-G(w_2,z)\|\leqslant p^{-1}\cdot\|Tw_1-Tw_2\|$. Since

$$\|G(w,z_1)-G(w,z_2)\|=$$

$$=\sup\left\{\exp\left(-p\int_0^x\int_0^y L_g(u,v)\,\mathrm{d}u\,\mathrm{d}v\right)|(Sz_1)(x,y)-(Sz_2)(x,y)|:(x,y)\in P\right\}\leqslant$$

$$\leqslant\|Sz_1-Sz_2\|$$

for $w\in X$ and z_1, z_2 in K, so all the conditions of Proposition 2 are satisfied. Therefore, there exists a function $z_0\in X$ such that $G(z_0, Tz_0\,(x,y) = (Tz_0)(x,y)$ for each (x,y) in P, and the proof is finished.

So we have proved the following:

Theorem 2. *Denote by $(++)$ the equation $(+)$ with $f=g+h$. Suppose that $g\in\mathcal{F}$ is a bounded function with L_g integrable on P, $h\in\mathcal{F}_0$ is a bounded function on Q and $(\sigma,\tau)\in\mathcal{X}$. Then there exists at least one function z in $C(P)$ satisfying Equation $(++)$ on P, and such that $z(x,0)=\sigma(x)$ for $0\leqslant x\leqslant a$ and $z(0,y)=\tau(y)$ for $0\leqslant y\leqslant b$.*

6. Remarks about continuous dependence

The solution of $(*)$ depends on the functions f, σ and τ. This solution is an operator (multivalued, in general) defined on the space of points (f,σ,τ). In this section we give some sufficient conditions for this operator to be continuous. We leave the details to the reader.

Let us denote:

by $S(f,\sigma,\tau)$ — the set of all continuous solutions of Equation $(*)$ with f in $\mathcal{F}_0$ and (σ,τ) in $\mathcal{X}$;

by $\mathcal{V}$ — the class of all operators $F(\cdot,(f,\sigma,\tau))$ that f, (σ,τ) ranges over $\mathcal{F}_0$ and $\mathcal{X}$, respectively.

We shall deal with the set $\mathcal{V}$ as the $\mathcal{L}^*$-space endowed with the continuous convergence [11, p. 93], i. e.,

$$\lim_{n\to\infty} F(\cdot,(f_n,\sigma_n,\tau_n))=F(\cdot,(f_0,\sigma_0,\tau_0))$$

meaning

$$\lim_{n\to\infty} \|F(z_n,(f_n,\sigma_n,\tau_n))-F(z_0,(f_0,\sigma_0,\tau_0))\|=0$$

for any sequence (z_n) in $C(P)$ that $\|z_n-z_0\|\to 0$ as $n\to\infty$. Moreover, to be precise, we define the function D in the family of non-empty bounded subsets of $C(P)$ by:

$$D(U,V)=\sup\{\varrho(u,V)\colon u\in U\},$$

where $\varrho(u,V)=\inf\{\|u-v\|\colon v\in V\}$.

Let $\mathfrak{A}$ be a closed subset of the space $\mathcal{V}$ such that each $S(f,\sigma,\tau)$ is non-empty for (f,σ,τ) with $F(\cdot,(f,\sigma,\tau))\in\mathfrak{A}$. The following theorem holds:

Suppose that $F(\cdot,(f_n,\sigma_n,\tau_n))\in\mathfrak{A}$ for $n\geqslant 1$, $\lim_{n\to\infty} F(\cdot,(f_n,\sigma_n,\tau_n))=F(\cdot,(f_0,\sigma_0,\tau_0))$ and, moreover, that $\bigcup_{n=1}^{\infty} S(f_n,\sigma_n,\tau_n)$ *is conditionally compact set. Then*

$$D(S(f_n,\sigma_n,\tau_n),S(f_0,\sigma_0,\tau_0))\to 0$$

as $n\to\infty$. *(Hence, for any $\varepsilon>0$ there exists a natural number N such that*

$$S(f_n,\sigma_n,\tau_n)\subset\{w\in C(P)\colon \inf_{z\in S(f_0,\sigma_0,\tau_0)}\|w-z\|<\varepsilon\}$$

for every $n>N$.)

Proof. Let us put $\xi_m = (f_m, \sigma_m, \tau_m)$ for $m = 0, 1, \ldots$. Assume the existence of $\varepsilon > 0$ and a subsequence (ξ_i) of sequence (ξ_0) with $D(S(\xi_i), S(\xi_0)) \geq \varepsilon$ for $i \geq 1$.

Fix an index i. Then there exists a sequence $(z_k^{(i)})$ of functions in $S(\xi_i)$ with $\varrho(z_k^{(i)}, S(\xi_0)) + k^{-1} > D(S(\xi_i), S(\xi_0)) \geq \varepsilon$ for $k = 1, 2, \ldots$. Since the set $S(\xi_i)$ is compact, $(z_k^{(i)})$ has a convergent subsequence $(z_l^{(i)})$. We have: $\varrho(z_l^{(i)}, S(\xi_0)) + l^{-1} > \varepsilon$ for $l \geq 1$, and $\|z_l^{(i)} - z_i\| \to 0$ as $l \to \infty$. From this it follows that there exists z_i in $S(\xi_i)$ such that $\varrho(z_i, S(\xi_0)) \geq \varepsilon$.

Proceeding similarly we conclude that the sequence (z_i) contains a subsequence (z_j) such that $\|z_j - z_0\| \to 0$ as $j \to \infty$, and therefore $\varrho(z_0, S(\xi_0)) \geq \varepsilon$. Obviously

$$\|z_0 - F(z_0, \xi_0)\| \leq \|z_0 - z_j\| + \|F(z_j, \xi_j) - F(z_0, \xi_0)\|$$

for $j \geq 1$, and $\lim_{j \to \infty} F(z_j, \xi_j) = F(z_0, \xi_0)$. Hence $z_0 \in S(\xi_0)$, and $\varrho(z_0, S(\xi_0) \geqq \varepsilon$ with $\varepsilon > 0$. This contradiction completes the proof.

From the above theorem we obtain as a corrolary:

Let the assumptions of the above result be satisfied, let $z_n \in S(f_n, \sigma_n, \tau_n)$ for $n \geq 1$ and let Equation (*) have exactly one solution z_0 for $f = f_0$, $\sigma = \sigma_0$ and $\tau = \tau_0$. Then $\|z_n - z_0\| \to 0$ as $n \to \infty$.

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ЗАМЕТКА ОБ ГИПЕРБОЛИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЯХ ВТОРОГО ПОРЯДКА (II)

Бодан Жепецки

Резюме

В работе даны условия существования и еднинственности решения задачи Дарбу для гиперболических уравнений второго порядка и установеные свойства непрерывности этого решения. Наша задача поставлена корректно в некоторых $\mathscr{L}^*$-пространствах правых частей и граничных условий. Полученные результаты связаны с методом Белецкого о изменении нормы в теории дифференциальных уравнений и являются итогом применения концепциообобщенного метрического пространства (расстояние не обязательно должно быть конечньем) и теорем о неподвижной точке.