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ON A PRODUCT OF METRIC SPACES

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Introduction

There is a natural way of introducing an algebraic structure on a product of algebraic structures of the same type. For example, if $(A, +)$ and $(B, \cdot)$ are groups, then $(A \times B, *)$, where $(a_1, b_1) * (a_2, b_2) = (a_1 + a_2, b_1 \cdot b_2)$ is a group as well. The application of this method to a collection $\{(A_t, d_t)\}_{t \in T}$ of metric spaces yields a mapping $(\varrho_d(x, y))(t) = d_t(x(t), y(t))$ which need not be a metric on $\prod_{t \in T} A_t$, since its values are in R^T. However, a metric can be obtained from that mapping by composing it with a suitable $f: R^T \to R$. In fact, the usual metrics on product spaces (as $\sqrt{(\varrho^2 + \sigma^2)}$, $\max(\varrho, \sigma)$, $\varrho + \sigma$, the Fréchet metric) can all be described in this way. Therefore it seems useful to investigate the set $\mathcal{M}(T)$ of all such mappings f: $R^T \to R$. A subset of $\mathcal{M}(T)$ (consisting of all nonnegative, monotone, subadditive mappings vanishing exactly at the constant zero function) is studied in [2]. In [1], the set $\mathcal{M}(T)$ is described in the special case when T has only one element. In the present paper we give a complete characterization of $\mathcal{M}(T)$ in the general case when T is an arbitrary set, and establish a necessary and sufficient condition for $f \circ \varrho_d$ to metrize the product topology. Theorem 2.11 was inspired by a suggestion of T. Šalát.

1. Preliminary considerations

1.1. Definition. *Let T be a set. Let $d = (d_t)_{t \in T}$ be a collection of mappings d_t: $A_t^2 \to B_t$, where $(A_t)_{t \in T}$, $(B_t)_{t \in T}$ are collections of sets. Define a mapping ϱ_d: $\left(\prod_{t \in T} A_t\right)^2 \to \prod_{t \in T} B_t$ by $(\varrho_d(x, y))(t) = d_t(x(t), y(t))$ for each x, $y \in \prod_{t \in T} A_t$, $t \in T$. Define a mapping σ_d: $\left(\prod_{t \in T} A_t\right)^3 \to \left(\prod_{t \in T} B_t\right)^3$ by $\sigma_d(x, y, z) = (\varrho_d(x, y), \varrho_d(x, z),$*

$\varrho_d(y, z))$ *for each* $x, y, z \in \prod_{t \in T} A_t$. *Denote* $E_d = \left\{\varrho_d(x, x): x \in \prod_{t \in T} A_t\right\}$, *and* $F_d =$
$= \left\{\varrho_d(x, y): x, y \in \prod_{t \in T} A_t, x \neq y\right\}$.

1.2. Theorem. *Let* $B \supset \operatorname{Im} \varrho_d$ *be a set (where* $\operatorname{Im} f = \{f(x): x \in X\}$ *for each mapping* $f: X \to Y$*). Let* $f: B \to R$. *Then* $f \circ \varrho_d$ *is a metric if and only if the following three conditions are satisfied:*

(1) $$E_d \cap F_d = \emptyset,$$

(2) $$\forall x \in \operatorname{Im} \varrho_d: f(x) = 0 \Leftrightarrow x \in E_d,$$

(3) $$\forall x, y, z \in \operatorname{Im} \varrho_d: (x, y, z) \in \operatorname{Im} \sigma_d \Rightarrow f(x) \leqq f(y) + f(z).$$

Proof. Necessity. Suppose that $a \in E_d \cap F_d$. Then $\exists x, y, z \in \prod_{t \in T} A_t, y \neq z$: $\varrho_d(x, x) = a = \varrho_d(y, z)$, therefore $0 = (f \circ \varrho_d)(x, x) = f(\varrho_d(x, x)) - f(\varrho_d(y, z)) = (f \circ \varrho_d)(y, z)$, a contradiction. This shows that $E_d \cap F_d - \emptyset$ Let $x \in \operatorname{Im} \varrho_d$. Then $\exists a, b \in \prod_{t \in T} A_t$: $x = \varrho_d(a, b)$, therefore $0 = f(x) = f(\varrho_d(a, b)) - (f\ \varrho_d)(a, b) \Leftrightarrow a = b \Leftrightarrow x = \varrho_d(a, b) \in E_d$.

Let $(x, y, z) \in \operatorname{Im} \sigma_d$. Then $\exists a, b, c \in \prod_{t \in T} A_t$: $x = \varrho_d(a, b)$, $\quad - \varrho_d(a, c)$, $z = \varrho_d(b, c)$, hence $f(x) = f(\varrho_d(a, b)) = (f \circ \varrho_d)(a, b) \leqq (f \circ \varrho_d)(a, c) + (f\ \varrho_d)(b, c) = f(\varrho_d(a, c)) + f(\varrho_d(b, c)) = f(y) + f(z)$.

Sufficiency. Let $x, y \in \prod_{t \in T} A_t$. Then $0 = (f \circ \varrho_d)(x, y) - f(\varrho_d(x, y)) \Leftrightarrow \varrho_d(x, y) \in E_d \Leftrightarrow x = y$. Let $x, y, z \in \prod_{t \in T} A_t$. Then $\sigma_d(x, y, z) \in \operatorname{Im} \sigma_d$, hence $(f\ \varrho_d)(x, y) = f(\varrho_d(x, y)) \leqq f(\varrho_d(x, z)) + f(\varrho_d(y, z)) = (f \circ \varrho_d)(x, z) + (f\ \varrho_d)(y, z)$.

1.3. Corollary. *Let* $h = (h_t)_{t \in T}$ *be a collection of mappings* $h_t\ C_t^2 \to D_t$, *where* $(C_t)_{t \in T}$, $(D_t)_{t \in T}$ *are collections of sets. Let* $E_h = E_d$, $E_h \cap F_h\ \ \emptyset$, $\operatorname{Im} \varrho_h \subset \operatorname{Im} \varrho_d$, $\operatorname{Im} \sigma_h \subset \operatorname{Im} \sigma_d$. *Let* $B \supset \operatorname{Im} \varrho_d$ *be a set. Let* $f: B \to R$ *be a mapping such that* $f\ \varrho_d$ *is a metric. Then* $f \circ \varrho_h$ *is a metric.*

1.4. Proposition. *Let* $f: A \to R$ *and* $g: B \to R$ *be mappings, where* $A, B \supset \operatorname{Im} \varrho_d$. *Define a mapping* $f + g: (A \cap B) \to R$ *by* $(f + g)(x) = f(x) + g(\ )$ *for each* $x \in A \cap B$. *Define a mapping* $\max(f, g): (A \cap B) \to R$ *b* $\max(f, g)(x) = \max(f(x), g(x))$ *for each* $x \in A \cap B$ *Let* $f \circ \varrho_d$ *and* $q \circ \varrho_d$ *b metrics Then* $(f + g) \circ \varrho_d$, $\max(f, g) \circ \varrho_d$ *are metrics.*

Proof. Let $x \in \operatorname{Im} \varrho_d$. Then $0 = (f + g)(x) = f(x) + g(x) \quad f(x) - 0 \& g(x) = 0 \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \operatorname{Im} \sigma_d$. Then $(f + g)(x) = f(x) + g(x) \leq f(y) + f(z) + g(y) + g(z) = (f + g)(y) + (f + g)(z)$. Then by 1.2, $(f + q) \circ \varrho_d$ is a metric.

Let $x \in \operatorname{Im} \varrho_d$. Then $0 = (\max(f, g))(x) = \max(f(x), g(x)) \Leftrightarrow f(x) = 0 \& g(x) = 0 \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \operatorname{Im} \sigma_d$. Then $f(x) \leqq f(y) + f(z) \leqq \max(f(y), g(y)) + \max(f(z), g(z))$, $g(x) \leqq g(y) + g(z) \leqq \max(f(y), g(y)) + \max(f(z), g(z))$, i.e. $(\max(f, g))(x) = \max(f(x), g(x)) \leqq \max(f(y), g(y)) + \max(f(z), g(z)) = (\max(f, g))(y) + (\max(f, g))(z)$. Then by 1.2, $\max(f, g)$ is a metric.

1.5. Proposition. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of mappings f_i: $C_i \to R$, where $C_i \supset \operatorname{Im} \varrho_d$. Let $\{f_i(x)\}_{i=1}^{\infty}$ converge for each $x \in \bigcap_{i=1}^{\infty} C_i$. Define a mapping $\lim_{i\to\infty} f_i$: $\bigcap_{i=1}^{\infty} C_i \to R$ by $(\lim_{i\to\infty} f_i)(x) = \lim_{i\to\infty} f_i(x)$ for each $x \in \bigcap_{i=1}^{\infty} C_i$. Let $\forall x \in F_d$: $(\lim_{i\to\infty} f_i)(x) \neq 0$. Let $f_i \circ \varrho_d$ be a metric for every $i \in N$. Then $(\lim_{i\to\infty} f_i) \circ \varrho_d$ is a metric.*

Proof. Let $(x, y, z) \in \operatorname{Im} \sigma_d$. Then $(\lim_{i\to\infty} f_i)(x) = \lim_{i\to\infty} f_i(x) \leqq \lim_{i\to\infty} (f_i(y) + f_i(z)) = \lim_{i\to\infty} f_i(y) + \lim_{i\to\infty} f_i(z) = (\lim_{i\to\infty} f_i)(y) + (\lim_{i\to\infty} f_i)(z)$. Then by 1.2, $(\lim_{i\to\infty} f_i) \circ \varrho_d$ is a metric.

1.6. Corollary. *Let $\sum_{i=1}^{\infty} f_i$ be a series of functions f_i: $C_i \to R$, where $C_i \supset \operatorname{Im} \varrho_d$. Let $\sum_{i=1}^{\infty} f_i(x)$ be convergent for each $x \in \bigcap_{i=1}^{\infty} C_i$. Let $f_i \circ \varrho_d$ be a metric for all $i \in N$. Then $\left(\lim_{n\to\infty} \sum_{i=1}^{n} f_i\right) \circ \varrho_d = \left(\sum_{i=1}^{\infty} f_i\right) \circ \varrho_d$ is a metric.*

Proof. By 1.4 $\left(\sum_{i=1}^{n} f_i\right) \circ \varrho_d$ is a metric for any $n \in N$. Let $x \in F_d$. Then $\forall i \in N$: $f_i(x) > 0$, therefore $\forall n \in N$: $\sum_{i=1}^{n} f_i(x) \geqq f_1(x)$, i.e. $\left(\lim_{n\to\infty} \sum_{i=1}^{n} f_i\right)(x) = \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x) \geqq f_1(x) > 0$. Then by 1.5 $\left(\lim_{n\to\infty} \sum_{i=1}^{n} f_i\right) \circ \varrho_d$ is a metric.

1.7. Proposition. *Let $f = (f_t)_{t \in I}$ be a collection of functions f_t: $C_t \to R$, where $C_t \supset \operatorname{Im} \varrho_d$ and $I \neq \emptyset$. Let the set $A_x = \{f_t(x): t \in I\}$ be bounded above for each $x \in \bigcap_{t \in I} C_t$. Define a function $\sup f$: $\bigcap_{t \in I} C_t \to R$ by $(\sup f)(x) = \sup A_x$ for each $x \in \bigcap_{t \in I} C_t$. Let $f_t \circ \varrho_d$ be a metric for every $t \in I$. Then $(\sup f) \circ \varrho_d$ is a metric.*

Proof. Let $x \in F_d$. Then $\forall t \in I$: $f_t(x) > 0$, thus $A_x \subset (0, \infty)$, i.e. $\sup A_x > 0$. Hence $\forall x \in \operatorname{Im} \varrho_d$: $(\sup f)(x) = 0 \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \operatorname{Im} \sigma_d$. Then $\forall t \in I$: $f_t(x) \leqq f_t(y) + f_t(z) \leqq \sup A_y + \sup A_z$. Then by 1.2 it follows that $(\sup f) \circ \varrho_d$ is a metric.

2. Characterization of $\mathcal{M}(T)$

2.1. Definition. *Let T be a nonempty set. Suppose R^T is ordered coordinate-wise, i.e. $x \leqq y$ $(x<y)$ if and only if $x(t) \leqq y(t)$ $(x(t)<y(t))$ for each x, $y \in R^T$, $t \in T$. Define a function Θ: $T \to R$ by $\Theta(t)=0$ for each $t \in T$. Denote $T^+ = \{x \in R^T : x \geqq \Theta\}$. Denote by $\mathcal{M}(T)$ the set of all functions f: $T^+ \to R$ such that $f \circ \varrho_d$ is a metric for every collection of metrics $d=(d_t)_{t \in T}$.*

2.2. Proposition. *Let f: $T^+ \to R$ be a function such that*
(*i*) $f(\Theta)=0$,
(*ii*) $\exists a>0\ \forall x \in T^+, x \neq \Theta$: $f(x) \in \langle a, 2a \rangle$.
Then $f \in \mathcal{M}(T)$.

Proof. Let $d=(d_t)_{t \in T}$ be a collection of metrics d_t: $M_t \times M_t \to R$. Then $E_d=\{\Theta\}$, $\Theta \notin F_d$, hence $E_d \cap F_d = \emptyset$. Let $x \in \operatorname{Im} \varrho_d$. Then $f(x)=0 \Leftrightarrow x=\Theta \Leftrightarrow x \in E_d$. Let $x, y, z \in \operatorname{Im} \varrho_d$, $(x, y, z) \in \operatorname{Im} \sigma_d$. If $x=\Theta$, then $f(x)=0 \leqq f(y)+f(z)$. If $y=\Theta$, then $x=z$, hence $f(x) = 0+f(z) = f(y)+f(z)$. If $z=\Theta$, then $x=y$, hence $f(x) = f(y)+0 = f(y)+f(z)$. If $\Theta \notin \{x, y, z\}$, then $f(x) \leqq 2a=a+a \leqq f(y)+f(z)$. Then by 1.2 we see that $f \circ \varrho_d$ is a metric.

2.3. Lemma. *Let $f \in \mathcal{M}(T)$. Then*

$$\forall x \in T^+ : f(x)=0 \Leftrightarrow x=\Theta.$$

Proof. Let ϱ: $S \times S \to R$ be a metric such that $\forall a \geqq 0 \exists x, y \in S$: $\varrho(x, y)=a$ (for example $S=R$, $\varrho(u, v) = |u-v|$ for every u, $v \in R$, $x=a$, $y=0$). Define a collection of metrics $d=(d_t)_{t \in T}$ by $d_t=\varrho$ for each $t \in T$. Then $\operatorname{Im} \varrho_d = T^+$, $E_d=\{\Theta\}$. Hence by 1.2 it follows that $\forall x \in T^+$: $f(x)=0 \Leftrightarrow x \in E_d \Leftrightarrow x=\Theta$.

2.4. Lemma. *Let $f \in \mathcal{M}(T)$. Then $\forall x, y, z \in T^+$:*

$$(x \leqq y+z \,\&\, y \leqq x+z \,\&\, z \leqq x+y) \Rightarrow f(x) \leqq f(y)+f(z).$$

Proof. Let ϱ: $S \times S \to R$ be a metric such that $\forall a, b, c \geqq 0$, $a \leqq b+c$, $b \leqq a+c$, $c \leqq a+b\ \exists x, y, z \in S$: $\varrho(x, y)=a$, $\varrho(y, z)=c$, $\varrho(x, z)=b$ (for example $S=R \times R$, $\varrho(u, v) = \|u-v\|$ for each u, $v \in R \times R$, $x=(a/2, 0)$, $y=(-a/2, 0)$, $z=((c^2-b^2)/(2a), (\sqrt{((a+b+c)\cdot(a+b-c)\cdot(a-b+c)\cdot(-a+b+c))})/(2a))$ for $a \neq 0$, $z=(b, 0)$ for $a=0$). Define a collection of metrics $d=(d_t)_{t \in T}$ by $d_t=\varrho$ for all $t \in T$. Let x, y, $z \in T^+$, $x \leqq y+z$, $y \leqq x+z$, $z \leqq x+y$. Since $\{(x, y, z) \in (T^+)^3 : x \leqq y+z, y \leqq x+z, z \leqq x+y\} \subset \operatorname{Im} \sigma_d$, by 1.2 we obtain $f(x) \leqq f(y)+f(z)$.

2.5. Lemma. *Let $f \in \mathcal{M}(T)$. Then*
(*i*) $\forall x, y \in T^+$: $f(x+y) \leqq f(x)+f(y)$,
(*ii*) $\forall x, y \in T^+$: $x \leqq 2y \Rightarrow f(x) \leqq 2f(y)$.

Proof. Let $x, y \in T^+$. Since $(x+y) \leqq x+y$, $x \leqq (x+y)+y$, $y \leqq (x+y)+x$, by

2.4 we have $f(x+y) \leqq f(x)+f(y)$. Let $x, y \in T^+$, $x \leqq 2y$. Since $x \leqq y+y$, $y \leqq x+y$, by 2.4 we get $f(x) \leqq f(y)+f(y) = 2f(y)$.

2.6. Theorem. *Let $f: T^+ \to R$. Then $f \in \mathcal{M}(T)$ if and only if*
(*i*) $\forall x \in T^+ : f(x)=0 \Leftrightarrow x=\Theta$,
(*ii*) $\forall x, y, z \in T^+ : (x \leqq y+z \,\&\, y \leqq x+z \,\&\, z \leqq x+y) \Rightarrow f(x) \leqq f(y)+f(z)$.

Proof. Sufficiency. Let $d=(d_t)_{t \in T}$ be a collection of metrics $d_t: M_t \times M_t \to R$. Then $E_d=\{\Theta\}$, $\Theta \notin F_d$, therefore $E_d \cap F_d = \emptyset$. Let $x \in \operatorname{Im} \varrho_d \subset T^+$. Then $f(x)=0 \Leftrightarrow x=\Theta \Leftrightarrow x \in E_d$. Let $(x, y, z) \in \operatorname{Im} \sigma_d$. Then $x \leqq y+z$, $y \leqq x+z$, $z \leqq x+y$, hence $f(x) \leqq f(y)+f(z)$.

Necessity. By 2.3 and 2.4.

2.7. Proposition. *Let $f, g \in \mathcal{M}(T)$. Then $f+g \in \mathcal{M}(T)$, $\max(f, g) \in \mathcal{M}(T)$. (See 1.4)*

2.8. Definition. *Let (M, d) be a metric space and let Ω denote the first uncountable ordinal number. The transfinite sequence*

$$\{a_\xi\}_{\xi<\Omega} \tag{1}$$

of elements of the space M is said to be convergent and to have a limit $a \in M$ if for each $\varepsilon>0$ there exists an ordinal number $\alpha<\Omega$ such that $d(a_\xi, a)<\varepsilon$ whenever $\alpha \leqq \xi < \Omega$. If (1) *has a limit a, we write $\lim_{\xi \to \Omega} a_\xi = a$ (or briefly $a_\xi \to a$). (See* [3], [4].)

2.9. Definition. *Let X be a set and let (Y, d) be a metric space. The transfinite sequence*

$$\{f_\xi\}_{\xi<\Omega} \tag{2}$$

of functions $f_\xi: X \to Y$ is said to be convergent and to have a limit function $f: X \to Y$ if for each $x \in X$ we have $\lim_{\xi \to \Omega} f_\xi(x)=f(x)$. If (2) *has a limit function f, we write $\lim_{\xi \to \Omega} f_\xi = f$ (or briefly $f_\xi \to f$). (See* [3], [4].)

2.10. Lemma. *Let (M, d) be a metric space, $a_\xi \in M (\xi<\Omega)$ and $a_\xi \to a$. Then there exists an ordinal number $\alpha<\Omega$ such that $a_\xi = a$ for each ξ with $\alpha \leqq \xi < \Omega$. (See* [4; *lemma* 1].)

2.11. Theorem. *Let $f_\xi \in \mathcal{M}(T)(\xi<\Omega)$ and let $f_\xi \to f$. Then $f \in \mathcal{M}(T)$.*

Proof. Let $a \in T^+$. Since $f_\xi \to f$, by 2.10 there exists an ordinal number $\alpha=\alpha(a)<\Omega$ such that $f_\xi(a)=f(a)$ whenever $\alpha \leqq \xi < \Omega$. Then $0=f(a)=f_\alpha(a) \Leftrightarrow a=\Theta$.

Let $a, b, c \in T^+$, $a \leqq b+c$, $b \leqq a+c$, $c \leqq a+b$. Since $f_\xi \to f$, by 2.10 there exists an ordinal number $\beta=\beta(a, b, c)<\Omega$ such that $f_\xi(a)=f(a)$, $f_\xi(b)=f(b)$, $f_\xi(c)=g(c)$ for each ξ with $\beta \leqq \xi < \Omega$. Then $f(a)=f_\beta(a) \leqq f_\beta(b)+f_\beta(c) = f(b)+f(c)$.

2.12. Proposition. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a sequence of functions* $f_i \in \mathcal{M}(T)$ *such that the sequence* $\{f_i(x)\}_{i=1}^{\infty}$ *converges for each* $x \in T^+$. *Let* $\forall x \in T^+$, $x \neq \Theta$: $(\lim_{i\to\infty} f_i)(x) \neq 0$. *Then* $\lim_{i\to\infty} f_i \in \mathcal{M}(T)$. (*See* 1.5)

2.13. Proposition. *Let* $\sum_{i=1}^{\infty} f_i$ *be a series of functions* $f_i \in \mathcal{M}(T)$ *such that the series* $\sum_{i=1}^{\infty} f_i(x)$ *converges for each* $x \in T^+$. *Then* $\left(\lim_{n\to\infty} \sum_{i=1}^{n} f_i\right) \in \mathcal{M}(T)$. (*See* 1.6)

2.14. Proposition. *Let* $f = (f_t)_{t\in I}$ *be a collection of functions* $f_t \in \mathcal{M}(T)$ *such that the set* $\{f_t(x)\colon t \in I\}$ *is bounded above. Then* $\sup f \in \mathcal{M}(T)$. (*See* 1.7)

2.15. Theorem. *Let* $f \in \mathcal{M}(T)$. *Then f is continuous if and only if f is continuous at the point* Θ.

Proof. Denote by $\mathcal{T}$ the usual topology on R. Denote by $\mathcal{S}_T$ the product topology on R^T. Let $\varepsilon > 0$. Then

$$\exists U \in \mathcal{S}_T, \quad \Theta \in U\ \forall x \in U \cap T^+ : f(x) < \varepsilon.$$

Therefore there exists a base element $V \subset U$, $\Theta \in V$, i.e. $\exists F \subset T$, F *is finite nonempty* $\forall t \in F\ \exists U_t \in \mathcal{T}$, $0 \in U_t$: $V = \bigcap_{t\in F} \pi_t^{-1}(U_t)$, where π_t is the projection from R^T into R, i.e. $\pi_t(x) = x(t)$ for each $x \in R^T$. Let $t \in F$. Then $\exists \gamma_t > 0$: $(-\gamma_t, \gamma_t) \subset U_t$. Denote $\gamma = \min_{t\in F} \gamma_t$. Then $\bigcap_{t\in F} \pi_t^{-1}((-\gamma, \gamma)) \subset V$, therefore $\forall x \in T^+$: $(\forall t \in F$: $x(t) < \gamma) \Rightarrow f(x) < \varepsilon$.

Let $x \in T^+$, $x \neq \Theta$. Denote $\delta = \gamma/2$. Let $y \in T^+$ be a function such that $\forall t \in F$: $|x(t) - y(t)| < \delta$.

Define a function $z\colon T \to R$ by

$$z(t) = \min(\delta, x(t) + y(t)) \quad \text{for} \quad t \in F,$$
$$z(t) = x(t) + y(t) \quad \text{for} \quad t \in T - F.$$

Then $z \in T^+$, $x \leqq y + z$, $y \leqq x + z$, $z \leqq x + y$, $\forall t \in F$: $z(t) < \gamma$, hence $|f(x) - f(y)| \leqq f(z) < \varepsilon$. Therefore $\forall x \in T^+$, $x \neq \Theta\ \forall \varepsilon > 0\ \exists W \in \mathcal{S}_T$, $x \in W\ \forall y \in W \cap T^+$: $|f(x) - f(y)| < \varepsilon$ $\left(W = \bigcap_{t\in F} \pi_t^{-1}\ (S(x(t), \delta))\right)$ and since, by the hypothesis, f is continuous at the point Θ, f is continuous.

2.16. Lemma. *Let* $f \in \mathcal{M}(T)$ *be continuous. Then*

$$\forall \varepsilon > 0\ \exists x \in T^+, \quad x > \Theta : f(x) < \varepsilon.$$

Proof. Let $\varepsilon>0$. Since f is continuous at the point Θ, we have $\exists U\in\mathscr{S}_T$, $\Theta\in U$ $\forall x\in U\cap T^+$: $f(x)<\varepsilon$. Since $U\in\mathscr{S}_T$ and $\Theta\in U$, $\exists\delta>0$ $\exists F\subset T$, *F is finite nonempty:* $\bigcap_{i\in F}\pi_i^{-1}(S(0,\delta))\subset U$. Define a function $x\colon T\to R$ by $x(t)=\delta/2$ for each $t\in T$. Then $x\in U\cap T^+$, therefore $f(x)<\varepsilon$.

2.17. Proposition. *Let T be a finite set. Let $f\in\mathcal{M}(T)$. Then f is continuous if and only if*

$$\forall\varepsilon>0\ \exists x\in T^+,\quad x>\Theta\colon f(x)<\varepsilon.$$

Proof. Sufficiency. Let $\varepsilon>0$. Then for $\varepsilon>0$ there is $a\in T^+$, $a>\Theta$: $f(a)<\varepsilon/2$. Since $\forall x\in T^+: x\leqq 2a \Rightarrow f(x)\leqq 2f(a)<\varepsilon$, hence for $U=\bigcap_{i\in T}\pi_i^{-1}(S(0,\min_{t\in T}a(t)))$ there holds $U\in\mathscr{S}_T$, $\Theta\in U$, $\forall x\in U\cap T^+: f(x)<\varepsilon$, therefore f is continuous at the point Θ and by 2.15 f is continuous. Necessity follows from 2.16.

2.18. Example. Let $f\colon\{0,1\}^+\to R$ be defined as follows:

$$f(\{(0,x),(1,y)\})=1 \quad\text{for}\quad x\neq 0,$$
$$f(\{(0,x),(1,y)\})=\min(1,y) \quad\text{for}\quad x=0.$$

Then $f\in\mathcal{M}(\{0,1\})$, f is not continuous and we have

$$\forall\varepsilon>0\ \exists x\in\{0,1\}^+,\quad x\neq\Theta\colon f(x)<\varepsilon$$

(for example $x=\{(0,0),(1,\min(1/2,\varepsilon/2))\}$).

2.19. Corollary. *Let T be a finite set. Let $f\in\mathcal{M}(T)$. Then f is not continuous if and only if*

$$\exists\eta>0\ \forall x\in T^+,\quad x>\Theta\colon f(x)\geqq\eta.$$

2.20. Lemma. *Let $f\colon S\to T$ be a bijective mapping. Define a mapping $f^*\colon T^+\to S^+$ by $f^*(a)=a\circ f$ for all $a\in T^+$. Let $g\colon S^+\to R$. Then $g\in\mathcal{M}(S)$ if and only if $(g\circ f^*)\in\mathcal{M}(T)$.*

Proof. Necessity. Let $a\in T^+$. Then $0=(g\circ f^*)(a) = g(f^*(a)) \Leftrightarrow f^*(a)=\Theta \Leftrightarrow \forall t\in S\colon a(f(t))=0 \Leftrightarrow \forall t\in T\colon a(t)=0 \Leftrightarrow a=\Theta$.

Let a, b, $c\in T^+$, $a\leqq b+c$, $b\leqq a+c$, $c\leqq a+b$. Then $f^*(a)\leqq f^*(b)+f^*(c)$, $f^*(b)\leqq f^*(a)+f^*(c)$, $f^*(c)\leqq f^*(a)+f^*(b)$, hence $(g\circ f^*)(a) = g(f^*(a))\leqq g(f^*(b)) + g(f^*(c)) = (g\circ f^*)(b) + (g\circ f^*)(c)$. Then by 2.6. we obtain $(g\circ f^*)\in\mathcal{M}(T)$.

Sufficiency. Since $f^{-1}\colon T\to S$ is a bijective mapping, we have $g=(g\circ f^*)\circ(f^{-1})^*\in\mathcal{M}(S)$.

2.21. Lemma. *Let $S\subset T$ be a nonempty set, $i\colon S\to T$, $i(x)=x$.* Define a mapping $i_*\colon S^+\to T^+$ for each $a\in S^+$ by $(i_*(a))(t)=a(t)$ for $t\in S$ and $(i_*(a))(t)=0$ for $t\in T-S$. Let $f\in\mathcal{M}(T)$. Then $(f\circ i_*)\in\mathcal{M}(S)$.

Proof. Let $a \in S^+$. Then $0 = (f \circ i*)(a) = f(i*(a)) \Leftrightarrow i*(a) = \Theta \Leftrightarrow a = \Theta$.

Let $a, b, c \in S^+$, $a \leqq b + c$, $b \leqq a + c$, $c \leqq a + b$. Then $i*(a) \leqq i*(b) + i*(c)$, $i*(b) \leqq i*(a) + i*(c)$, $i*(c) \leqq i*(b) + i*(a)$, therefore $(f \circ i*)(a) = f(i*(a)) \leqq f(i*(b)) + f(i*(c)) = (f \circ i*)(b) + (f \circ i*)(c)$. Then $(f \circ i*) \in \mathcal{M}(S)$ (by 2.6).

2.22. Proposition. *Let S be a nonempty set. Let f: $S \to T$ be an injective mapping. Define a mapping i: $\operatorname{Im} f \to T$ by $i(x) = x$. Let g: $\operatorname{Im} f \to S$ be a bijective mapping such that $f = i \circ g^{-1}$. Define a mapping $f*$: $S^+ \to T^+$ by $f* = i* \circ g^*$. Let $h \in \mathcal{M}(T)$. Then $(h \circ f*) \in \mathcal{M}(S)$.*

Proof. Since by 2.21 $(h \circ i*) \in \mathcal{M}(\operatorname{Im} f)$, it follows from 2.20 that $h \circ f* = (h \circ i*) \circ g^* \in \mathcal{M}(S)$.

2.23. Remark. Let $\mathcal{K}$ and $\mathcal{S}$ be categories whose objects are nonempty sets and morphisms are injective mappings and mappings, respectively. Assign the set $\mathcal{M}(T)$ to each object T of $\mathcal{K}$. For every morphism f: $S \to T$ of the category $\mathcal{K}$ define a mapping $\mathcal{M}(f)$: $\mathcal{M}(T) \to \mathcal{M}(S)$ by $(\mathcal{M}(f))(g) = g \circ f*$ whenever $g \in \mathcal{M}(T)$. Thus we have described a countervariant functor $\mathcal{M}$: $\mathcal{K} \to \mathcal{S}$.

3. Metrization of the product topology

3.1. Lemma. *Let $d = (d_t)_{t \in T}$ be a collection of metrics d_t: $M_t^2 \to R$. Let $f \in \mathcal{M}(T)$. Denote by $\mathcal{T}_S$ the product topology on $\prod_{t \in T} M_t$ and denote by $\mathcal{T}_f$ the topology generated by the metric $f \circ \varrho_d$. Then $T_S \subset T_f$.*

Proof. Let $t \in T$. Let U_t be an open set in M_t. Let $x \in \pi_t^{-1}(U_t)$, where π_t is the projection from $\prod_{t \in T} M_t$ into M_t, i.e. $\pi_t(x) = x(t)$ for each $x \in \prod_{t \in T} M_t$. Then $x(t) \in U_t$, therefore $\exists \varepsilon > 0$: $S(x(t), \varepsilon) \subset U_t$. Define a function a: $T \to R$ by $a(t) = 2\varepsilon$ and $a(i) = 0$ for each $i \in T - \{t\}$.

Put $\delta = f(a)/2$. Let $y \in S(x, \delta) \in \mathcal{T}_f$. Then $(f \circ \varrho_d)$ $(x, y) < \delta$, therefore $f(\varrho_d(x, y)) < \delta = f(a)/2$. By 2.6 we have $\forall b \in T^+$: $a \leqq 2b \Rightarrow f(a) \leqq 2f(b)$, or equivalently $\forall b \in T^+$: $f(b) < f(a)/2 \Rightarrow \neg(b \geqq a/2)$.

Hence $\neg(\varrho_d(x, y) \geqq a/2)$ and therefore, by definition a, we have $d_t(x(t), y(t)) = (\varrho_d(x, y))(t) < a(t)/2 = \varepsilon$. Therefore $y \in \pi_t^{-1}(S(x(t), \varepsilon)) \subset \pi_t^{-1}(U_t)$. Then

$$\forall x \in \pi_t^{-1}(U_t)\ \exists V \in \mathcal{T}_f, x \in V: V \subset \pi_t^{-1}(U_t) \quad (V = S(x, \delta)),$$

which implies $\mathcal{T}_S \subset \mathcal{T}_f$.

3.2. Proposition. *Let $d = (d_t)_{t \in T}$ be a collection of metrics d_t: $M_t^2 \to R$. Put $H_d = \{t \in T: M'_t \neq \emptyset\}$ (where M' is the set of all accumulation points of the metric space (M, ϱ)). Let F, $H_d \subset F \subset T$, be such a set that $T - F$ is a finite set. Let i: $F \to T$ be a mapping defined by $i(x) = x$. Let $f \in \mathcal{M}(T)$. Let $f \circ i*$ be a continuous mapping. Then $\mathcal{T}_S = \mathcal{T}_f$.*

Proof. Since by 3.1 we have $\mathcal{T}_S \subset \mathcal{T}_f$, it is sufficient to prove that $\mathcal{T}_f \subset \mathcal{T}_S$.

Let $x \in \prod_{t \in T} M_t$ and $\varepsilon > 0$. The function $f \circ i*$ is continuous at the point Θ, i.e. $\exists K \subset F$, $K \neq \emptyset$ *finite* $\exists \gamma > 0\ \forall y \in F^+$: $(\forall t \in K: y(t) < \gamma) \Rightarrow (f \circ i*)(y) < \varepsilon$.

The set $T - F$ is finite, this implies that there exists $\beta > 0$ such that $\forall t \in T - F$ $\forall y \in M_t$, $y \neq x(t)$: $d_t(x(t), y) \geqq \beta$. Denote $\delta = \min(\beta, \gamma)$ and $L = K \cup (T - F)$.

Put $V = \bigcap_{t \in L} \pi_t^{-1}(S(x(t), \delta))$. Then $V \in \mathcal{T}_S$ and $x \in V$. Let $y \in V$.

Then $\forall t \in T - F$: $(\varrho_d(x, y))(t) = 0$, this implies $i*(\varrho_d(x, y)|_F) = \varrho_d(x, y)$. Since $\varrho_d(x, y)|_F \in F^+$ and $\forall t \in K$: $(\varrho_d(x, y)|_F)(t) = d_t(x(t), y(t)) < \gamma$, we have $(f \circ \varrho_d)(x, y) = f(\varrho_d(x, y)) = (f \circ i*)(\varrho_d(x, y)|_F) < \varepsilon$, i.e. $y \in S(x, \varepsilon)$. Therefore $\forall x \in \prod_{t \in T} M_t\ \forall \varepsilon > 0\ \exists V \in \mathcal{T}_S$: $x \in V \subset S(x, \varepsilon)$, i.e. $\mathcal{T}_f = \mathcal{T}_S$.

3.3. Corollary. *Let $d = (d_t)_{t \in T}$ be a collection of metrics d_t: $M_t^2 \to R$. Let $f \in \mathcal{M}(T)$ be a continuous mapping. Then $\mathcal{T}_S = \mathcal{T}_f$.*

3.4. Proposition. *Let $d = (d_t)_{t \in T}$ be a collection of metrics d_t: $M_t^2 \to R$. Denote $I_d = \{t \in T$: $\sup \operatorname{Im} d_t \in R\}$. Let M_t be a nonempty set for each $t \in T$. Let $H_d \cap I_d$ be a finite set. Let i: $H_d \to T$ be a mapping defined by $i(x) = x$. Let $f \in \mathcal{M}(T)$ be a mapping such that $\mathcal{T}_S = \mathcal{T}_f$. Then $f \circ i*$ is a continuous mapping.*

Proof. If $H_d = \emptyset$, then the statement is true. Suppose $H_d \neq \emptyset$. Then by 2.21 $f \circ i* \in \mathcal{M}(H_d)$, therefore it is sufficient to prove that $f \circ i*$ is continuous at the point Θ. Since M_t is nonempty for all t in T, there exists x in $\prod_{t \in T} M_t$ such that $\forall t \in H_d$: $x(t) \in M'_t$. Let $\varepsilon > 0$. Then $S(x, \varepsilon/2) \in \mathcal{T}_f \subset \mathcal{T}_S$, hence

$$\exists K \subset T,\quad K \neq \emptyset \quad \textit{finite} \quad \exists \gamma > 0: \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \gamma)) \subset S(x, \varepsilon/2).$$

Let F be a nonempty finite set such that $H_d \cap (K \cup I_d) \subset F \subset H_d$. Let $t \in F$. Since $x(t) \in M'_t$, there exists $y_t \in M_t$ with $0 < d_t(x(t), y_t) < \gamma$. Put $\delta = \min_{t \in F} d_t(x(t), y_t)$. Let $z \in H_d^+$, $z \in \bigcap_{t \in F} \pi_t^{-1}(S(0, \delta))$. Then $\forall t \in H_d - F\ \exists y_t \in M_t$: $z(t) \leqq d_t(x(t), y_t)$. Define a mapping y: $T \to \bigcup_{t \in T} M_t$ by $y(t) = y_t$ for $t \in H_d$, $y(t) = x(t)$ for $t \in T - H_d$. Then $y \in \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \gamma))$ and $i*(z) \leqq 2\varrho_d(x, y)$, hence

$$(f \circ i*)(z) = f(i*(z)) \leqq 2f(\varrho_d(x, y)) = 2(f \circ \varrho_d)(x, y) < \varepsilon.$$

Therefore $\forall \varepsilon>0\ \exists F\subset H_d,\ F\neq\emptyset$ *finite* $\exists\delta>0$

$$\forall z\in\bigcap_{t\in F}\pi_t^{-1}(S(0,\delta)):(f\circ i*)(z)<\varepsilon,$$

i.e. $f\circ i*$ is continuous at the point Θ.

3.5. Corollary. *Let d_t be the usual metric on R for all $t\in T$. Let $f\in\mathcal{M}(T)$. Then $\mathcal{T}_S=\mathcal{T}_f$ if and only if f is continuous.*

3.6. Theorem. *Let T be a finite set. Let $d=(d_t)_{t\in T}$ be a collection of metrics d_t: $M_t^2\to R$. Let M_t be a nonempty set for all $t\in T$. Let i: $H_d\to T$ be a mapping defined by $i(x)=x$. Let $f\in\mathcal{M}(T)$. Then $\mathcal{T}_S=\mathcal{T}_f$ if and only if $f\circ i*$ is continuous.*

Proof. Necessity follows by 3.4. Sufficiency follows by 3.2.

3.7. Example. Let $d=(d_n)_{n\in N}$ be a collection of metrics d_n: $\langle 0,1/n\rangle^2\to R$, $d_n(u,v)=|u-v|$ for each $u,v\in\langle 0,1/n\rangle$, where N is the set of all positive integer numbers. Let i: $H_d\to N$, $i(x)=x$ (therefore i is the identity, since $H_d=N$). Let f: $N^+\to R$ be a function defined by $f(x)=\sup_{n\in N}(\min(1,x(n)))$ for all $x\in N^+$. Then we can verify that $f\in\mathcal{M}(N)$, $\mathcal{T}_S=\mathcal{T}_f$ but $f\circ i*$ is not continuous.

3.8. Example. Let $d=(d_n)_{n\in N}$ be a collection of metrics d_n: $\{0,1\}^2\to R$, $d_n(0,1)=1$ for all $n\in N$. Let i: $H_d\to N$, $i(x)=x$ (since $H_d=\emptyset$, i is the empty mapping). Let f: $N^+\to R$ be a function defined by $f(\Theta)=0$ and $f(x)=1\ \forall x\in N^+$, $x\neq\Theta$. Then we can show that $f\in\mathcal{M}(N)$, $f\circ i*$ is continuous but $\mathcal{T}_S\neq\mathcal{T}_f$.

3.9. Proposition. *Let $d=(d_t)_{t\in T}$ be a collection of metrics d_t: $M_t^2\to R$. Let E, $H_d\subset E\subset I_d$, be such a set that $T-E$ is a finite set. Let $f\in\mathcal{M}(T)$ be a mapping such that $\forall\varepsilon>0\ \exists c\in T^+$:*

(i) $\exists F\subset E$ finite $\forall t\in E-F$: $c(t)\geqq\sup\operatorname{Im} d_t$,

(ii) $\forall t\in E$: $c(t)>0$,

(iii) $f(c)<\varepsilon$.

Then $\mathcal{T}_S=\mathcal{T}_f$.

Proof. Let $x\in\prod_{t\in T}M_t$ and $\varepsilon>0$. Since $T-E$ is a finite set, there exists $\delta>0$ such that $\forall t\in T-E\ \forall y\in M_t,\ y\neq x(t)$: $d_t(x(t),y)\geqq\delta$. Further, since $\varepsilon/2>0$, there exists $c\in T^+$ such that $(\forall t\in E$: $c(t)>0)$ & $(\exists F\subset E$ *finite* $\forall t\in E-F$: $c(t)\geqq\sup\operatorname{Im} d_t)$ & $(f(c)<\varepsilon/2)$. Since $F\subset E$, we have $\forall t\in F$: $c(t)>0$. Since F is a finite set, there exists $\gamma>0$ such that $\forall t\in F$: $c(t)\geqq\gamma$. Let K be a nonempty finite set such that $((T-E)\cup F)\subset K\subset T$. Put $V=\bigcap_{t\in K}\pi_t^{-1}(S(x(t),\min(\gamma,\delta)))$. Let $y\in V$. Then $\forall t\in E-F$: $d_t(x(t),y(t))\leqq\sup\operatorname{Im} d_t\leqq c(t)$, $\forall t\in T-E$: $d_t(x(t),y(t))=0\leqq c(t)$, $\forall t\in F$: $d_t(x(t),y(t))\leqq\gamma\leqq c(t)$, i.e. $\varrho_d(x,y)\leqq c$. Then $\varrho_d(x,y)\leqq 2c$, hence $(f\circ\varrho_d)(x,y)=f(\varrho_d(x,y))\leqq 2f(c)<2\varepsilon/2=\varepsilon$, i.e. $y\in S(x,\varepsilon)$. Therefore $x\in V\subset S(x,\varepsilon)$, $V\in\mathcal{T}_S$. Then $\mathcal{T}_f\subset\mathcal{T}_S$.

3.10. Corollary. *Let $d=(d_t)_{t\in T}$ be a collection of metrics $d_t\colon M_t^2\to R$. Let $\forall\varepsilon>0$ $\exists H\subset T$ finite $\forall t\in T-H$: $\sup \operatorname{Im} d_t<\varepsilon$. Let E, $H_d\subset E\subset T$, be such a set that $T-E$ is a finite set. Let $f\in\mathcal{M}(T)$ be a mapping such that*

$$\forall\varepsilon>0\ \exists\gamma>0\ \exists c\in T^+ : (f(c)<\varepsilon)\,\&\,(\forall t\in E\colon c(t)\geqq\gamma).$$

Then $\mathcal{T}_S=\mathcal{T}_f$.

Proof. Let $\varepsilon>0$. Then $\exists\gamma>0$ $\exists c\in T^+$: $(f(c)<\varepsilon)\,\&\,(\forall t\in E\colon c(t)\geqq\gamma)$. Then $\exists H\subset T$ *finite* $\forall t\in T-H$: $\sup\operatorname{Im} d_t<\gamma$. Put $F=H\cap E$. Then $F\subset E$, F is a finite set and $\forall t\in E-F$: $c(t)\geqq\gamma>\sup\operatorname{Im} d_t$. Therefore $\mathcal{T}_S=\mathcal{T}_f$ by 3.9.

3.11. Example. Let $d=(d_n)_{n\in N}$ be a collection of metrics $d_n\colon\langle 0, 1/n^2\rangle^2\to R$, $d_n(u,v)=|u-v|\ \forall u,v\in\langle 0,1/n^2\rangle$. Let $f\colon N^+\to R$ be a function defined by $f(x)=\sup_{n\in N}(\min(1, n\cdot x(n)))$ for each $x\in N^+$. Then by 2.6 $f\in\mathcal{M}(N)$, by 3.9 $\mathcal{T}_S=\mathcal{T}_f$, but d and f do not satisfy the hypothesis of 3.10.

3.12. Theorem. *Let $d=(d_t)_{t\in T}$ be a collection of metrics $d_t\colon M_t^2\to R$. Let M_t be a nonempty set for all t in T. Let $f\in\mathcal{M}(T)$. Then $\mathcal{T}_S=\mathcal{T}_f$ if and only if*

$$\forall\varepsilon>0\ \exists F\subset T\ \ \textit{finite}\ \ \exists\delta>0\ \forall\alpha\in N^{(T-(I_d\cup F))}\ \exists a\in T^+ :$$

(*i*) $\forall t\in(T-(I_d\cup F))\colon a(t)\geqq\alpha(t)$,
(*ii*) $\forall t\in(I_d-F)\colon a(t)\geqq\sup\operatorname{Im} d_t$,
(*iii*) $\forall t\in(F\cap H_d)\colon a(t)\geqq\delta$,
(*iv*) $f(a)<\varepsilon$.

Proof. Necessity. Let $t\in H_d$. Then $\exists x_t\in M_t$ $\forall\varepsilon>0$ $\exists y\in M_t$: $0<d_t(x_t,y)<\varepsilon$. Since $\forall t\in T$: $M_t\neq\emptyset$, we have $\forall t\in(T-H_d)$ $\exists x_t\in M_t$. Define a mapping $x\colon T\to\bigcup_{t\in T}M_t$ by $x(t)=x_t$ for all $t\in T$. Let $\varepsilon>0$. Since $\mathcal{T}_S=\mathcal{T}_f$, $S(x,\varepsilon/4)\in\mathcal{T}_S$. Therefore

$$\exists F\subset T,\ \ F\neq\emptyset\ \ \textit{finite}\ \ \exists\gamma>0\colon\bigcap_{t\in F}\pi_t^{-1}(S(x(t),\gamma))\subset S(x,\varepsilon/4).$$

Let $t\in F\cap H_d$. Then $\exists y_t\in M_t$: $0<d_t(x(t),y_t)<\gamma$. If $F\cap H_d\neq\emptyset$ put $\delta=\min_{t\in F\cap H_d}d_t(x(t),y_t)>0$. If $F\cap H_d=\emptyset$, put $\delta=1$. Let $\alpha\in N^{(T-(I_d\cup F))}$. Let $t\in T-(I_d\cup F)$. Then $\exists y_t\in M_t$: $d_t(x(t),y_t)\geqq\alpha(t)$. Let $t\in I_d-F$. If $\sup\operatorname{Im} d_t>0$, there exists $y_t\in M_t$:

$$d_t(x(t),y_t)>(1/4)\cdot\sup\operatorname{Im} d_t.$$

If $\sup\operatorname{Im} d_t=0$, put $y_t=x(t)$. Put $y_t=x(t)$ for each $t\in F-H_d$.

Define a mapping $y\colon T\to\bigcup_{t\in T}M_t$ by $y(t)=y_t$ for all $t\in T$. Put $a=4\varrho_d(x,y)$. Then

$$f(a)\leqq 4\cdot f(\varrho_d(x,y))=4\cdot(f\circ\varrho_d)(x,y)<4\cdot\varepsilon/4=\varepsilon.$$

Sufficiency. Let $x \in \prod_{t \in T} M_t$ and $\varepsilon > 0$. Since $\varepsilon/2 > 0$, there exists a finite set $F \subset T$ such that

$$\exists \delta > 0 \; \forall \alpha \in N^{(T-(I_d \cup F))} \; \exists a \in T^+ : (\forall t \in (T-(I_d \cup F)) : a(t) \geqq \alpha(t)) \,\&\, (\forall t \in I_d - F : a(t) \geqq \sup \operatorname{Im} d_t) \,\& \,\&\, (\forall t \in F \cap H_d : a(t) \geqq \delta) \,\&\, (f(a) < \varepsilon/2).$$

Since $F - H_d$ is a finite set

$$\exists \gamma > 0 \; \forall t \in (F - H_d) \; \forall y \in M_t, \quad y \neq x(t) : d_t(x(t), y) \geqq \gamma.$$

Let K, $F \subset K \subset T$, be a nonempty finite set.

Put $V = \bigcap_{t \in K} \pi_t^{-1}(S(x(t), \min(\gamma, \delta)))$. Let $y \in V$. Let $t \in (T-(I_d \cup F))$. Then there exists a positive integer n_t such that $d_t(x(t), y(t)) \leqq n_t$. Define a mapping α: $(T-(I_d \cup F)) \to N$ by $\alpha(t) = n_t$ for each $t \in T-(I_d \cup F)$. Then $\exists a \in T^+$: $(\forall t \in (T-(I_d \cup F)) : a(t) \geqq \alpha(t))$ & $(\forall t \in I_d - F : a(t) \geqq \sup \operatorname{Im} d_t)$ & $(\forall t \in F$: $a(t) \geqq \delta)$ & $(f(a) < \varepsilon/2)$.

Then $\forall t \in I_d - F$: $d_t(x(t), y(t)) \leqq \sup \operatorname{Im} d_t \leqq a(t)$, $\forall t \in F \cap H_d$: $d_t(x(t), y(t)) < \delta \leqq a(t)$, $\forall t \in (T-(I_d \cup F))$: $d_t(x(t), y(t)) \leqq \alpha(t) \leqq a(t)$, $\forall t \in F - H_d$: $d_t(x(t), y(t)) = 0 \leqq a(t)$, i.e. $\varrho_d(x, y) \leqq a$. Then $\varrho_d(x, y) \leqq 2a$, hence $(f \circ \varrho_d)(x, y) = f(\varrho_d(x, y)) \leqq 2f(a) < 2 \cdot \varepsilon/2 = \varepsilon$, i.e. $y \in S(x, \varepsilon)$. Therefore $x \in V \subset S(x, \varepsilon)$, $V \in \mathcal{T}_S$. Then $\mathcal{T}_f \subset \mathcal{T}_S$.

3.13. Example. Let d be a collection of metrics from example 3.8. Define a function k: $N^+ - \{\Theta\} \to N$ by $k(x) = \min\{n \in N : x(n) \neq 0\}$. Define a function f: $N^+ \to R$ by $f(x) = e^{-k(x)}$ for $x \in N^+$, $x \neq \Theta$ and $f(\Theta) = 0$. Then by 2.6 $f \in \mathcal{M}(N)$, by 3.12 $\mathcal{T}_S = \mathcal{T}_f$, but f and d do not satisfy the hypotheses of either 3.2 or 3.9.

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О ПРОИЗВЕДЕНИИ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

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Резюме

Пусть T является непустым множеством. Обозначим $\mathcal{M}(T)$ множество всех отображений f: $\{x \in R^T;\ \forall t \in T\colon x(t) \geqq 0\} \to R$, для которых

(1) $$d(x, y) = f(\{d_t(x_t, y_t)\}_{t \in T})$$

является метрикой на множестве

$$\prod_{t \in T} M_t$$

для каждого семейства метрических пространств $\{(M_t, d_t)\}_{t \in T}$. В этой работе мы предлагаем характеризацию множества $\mathcal{M}(T)$, а также необходимое и достаточное условие метризации топологии произведения на

$$\prod_{t \in T} M_t$$

при помощи метрики (1).