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Sylvia Pulmannová
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# SEMIOBSERVABLES ON QUANTUM LOGICS

SYLVIA PULMANNOVÁ

A generalization of the notion of observables on a logic is given and its mathematical properties are described.

## 1. Introduction

Within the framework of the standard quantum theory "semiobservables" or "quantum measurements" can be described as (in general nonorthogonal) resolutions of the identity in a Hilbert space. Semiobservables were originally introduced by Davies and Lewis [2, 3] for the description of repeated measurements and conditional expectations in quantum systems. They were also used by Ingarden [13] to formulate the quantum information theory. In recent years there has been an increasing interest in quantum-mechanical variations of the problem of signal detection in background noise [9, 10, 11]. The basic concepts of the general theory of statistical decisions are stated by Holevo [10, 11]. Essentially new is the investigation of semiobservables as an analog of strategies in the classical theory of statistical decisions.

In the present paper, an analog of semiobservables in terms of the quantum logic approach to quantum mechanics is introduced and its mathematical description is given.

## 2. Basic concepts

In the quantum logic approach to quantum mechanics the basic concepts are the set $\mathscr{L}$ of all experimentally verifiable propositions of the physical system and the set $\mathscr{M}$ of physical states. $\mathscr{L}$ is usually supposed to be a partially ordered set with the greatest and least elements 1 and 0, respectively, with the orthocomplementation $a \mapsto a^{\perp}$, $a \in \mathscr{L}$, and with the orthomodularity property

$$a \leqslant b \Rightarrow b = a \vee (a^{\perp} \wedge b),$$

where we denote by $a \vee b$ and $a \wedge b$ the supremum and infimum of the elements $a$, $b \in \mathscr{L}$, respectively; and which is closed under the formations of the suprema $\vee a_i$

for any sequence $\{a_i\}$ $i=1, 2, \ldots,$ of elements from $\mathscr{L}$ such that $a_i \leqslant a_j^\perp$, $i \neq j$. The elements $a, b \in \mathscr{L}$ are *orthogonal* ($a \perp b$) if $a \leqslant b^\perp$, and they are *compatible* ($a \leftrightarrow b$) if they can be written in the form $a = a_1 \vee c$, $b = b_1 \vee c$, where $a_1, b_1, c$ are mutually orthogonal elements of $\mathscr{L}$. A set $\mathscr{L}$ with the properties just described is called a *logic*.

A *state* on $\mathscr{L}$ is a probability measure on $\mathscr{L}$, i.e. a map $m: \mathscr{L} \to [0, 1]$ such that $m(1) = 1$ and $m(\vee a_i) = \Sigma m(a_i)$ for any sequence $\{a_i\}$ of mutually orthogonal elements of $\mathscr{L}$. A logic $\mathscr{L}$ is *full* if there is a set $\mathscr{M}$ of states such that $m(a) \leqslant m(b)$ for all $m \in \mathscr{M}$ imply $a \leqslant b$, $a, b \in \mathscr{L}$. A logic $\mathscr{L}$ is *quite full* if the statement "$m(b) = 1$ *whenever* $m(a) = 1$" implies the statement "$a \leqslant b$". It can be easily seen that a quite full logic is full [6].

The set of states of a physical system is usually supposed to be closed under the formations of countable convex combinations, i.e. if $m_i \in \mathscr{M}$, $i = 1, 2, \ldots,$ then $m = \Sigma t_i m_i \in \mathscr{M}$, where $0 \leqslant t_i \leqslant 1$, $\Sigma t_i = 1$.

In the following we shall suppose that $\mathscr{L}$ is a quite full logic and $\mathscr{M}$ is closed under the formations of countable convex combinations. The pair $(\mathscr{L}, \mathscr{M})$ is called a *quantum logic*.

Let $\mathscr{L}$ and $\mathscr{K}$ be two logics. The mapping $h$ from $\mathscr{L}$ into $\mathscr{K}$ is called a *σ-homomorphism* if

(*i*) $h(1) = 1$,

(*ii*) $p \perp q$, $p, q \in \mathscr{L}$ implies $h(p) \perp h(q)$,

(*iii*) $h(\vee p_i) = \vee h(p_i)$ for any sequence $\{p_i\}$ of mutually orthogonal elements of $\mathscr{L}$.

With the help of the concept of $\sigma$-homomorphism we introduce observables corresponding to physical quantities. Suppose $\mathscr{L}$ is a logic and $(\mathscr{U}, \mathscr{B})$ is a measureble space with the $\sigma$-algebra of subsets $\mathscr{B}$. An arbitrary $\sigma$-homomorphism $x: F \mapsto x(F)$ of the $\sigma$-algebra $\mathscr{B}$ into the logic $\mathscr{L}$ is called a $(\mathscr{U}, \mathscr{B})$-*observable on* $\mathscr{L}$. This definition can be interpreted as follows. We consider $\mathscr{U}$ as the space of possible states of some measuring device, and $\mathscr{B}$ as the class of events relating to this device, i.e. the results of measurements. The $\sigma$-homomorphism $x$ associates with each event some assertion (element of $\mathscr{L}$) about the physical system.

If $x$ is a $(\mathscr{U}, \mathscr{B})$-observable and $m$ is a state, then $m(x(\cdot))$ defines a probability measure on $(\mathscr{U}, \mathscr{B})$. The expectation of an observable $x$ in the state $m$ is $m(x) = \int tm[(x(dt)]$ if the integral exists.

If $\mathscr{R}$ is the real line and $\mathscr{B}(\mathscr{R})$ is the $\sigma$-algebra of Borel sets, then a $(\mathscr{R}, \mathscr{B}(\mathscr{R}))$-observable is called a real observable. The unique observable $I$: $E \mapsto I(E)$ defined by $I(\{1\}) = 1$, where $\{1\}$ is the one-point set consisting of unity, is called the *identity* on $\mathscr{L}$.

If $x$ is a real observable and $f: \mathscr{R} \to \mathscr{R}$ is a Borel function, then we can define the observable $f(x)$ by setting $f(x)(E) = x(f^{-1}(E))$. The *spectrum* $\sigma(x)$ of the real observable $x$ is the smallest closed set $C$ such that $x(C) = 1$. An observable $x$ is called a *simple observable* if $\sigma(x) \subset \{0, 1\}$, which is identical with $x^2 = x$. An observable $x$ is *bounded* if its spectrum $\sigma(x)$ is bounded. A real observable $x$ on a full logic is called the *sum* of the observables $y$, $z$ if $m(x) = m(y) + m(z)$ for any $m \in \mathcal{M}$. We shall suppose that for any two bounded real observables $y$, $z$ there is unique observable $x$ such that $x = y + z$. From the existence and uniqueness of the sums of observables it follows that $m(x) = m(y)$ for all $m \in \mathcal{M}$ implies $x = y$. If the logic $\mathcal{L}$ is quite full and the sums exist, then $\mathcal{L}$ is a lattice [6].

More details about the mentioned concepts can be found in [5, 6, 16, 17].

## 3. Semiobservables and their properties

Let $\mathcal{L}$ be a quite full logics with respect to the set of states $\mathcal{M}$. Let $X$ be the set of all bounded real observables on $\mathcal{L}$. We suppose that $X$ is closed under the formations of finite sums, so that $X$ can be considered as a real linear space. We define the norm on $X$ by setting

$$\|x\| = \sup \{|m(x)| : m \in \mathcal{M}\}.$$

It was shown in [5] that

$$\|x\| = \sup \{|t| : t \in \sigma(x)\}.$$

**Definition 1.** *Let $\{x_i\}$ be a sequence of bounded real observables. We shall say that an observable $x$ is the sum of $x_i$, i.e.*

$$x = \Sigma x_i,$$

*if*

$$m(x) = \Sigma m(x_i) \quad \text{for all} \quad m \in \mathcal{M}.$$

Clearly, if such an $x$ exists, it is uniquely defined. Now we shall consider the following generalization of the notion of an observable.

**Definition 2.** *Let $(\mathcal{U}, \mathcal{B})$ be a measurable space with the $\sigma$-algebra of subsets $\mathcal{B}$. Let $\{x_B : B \in \mathcal{B}\}$ be an $X$-valued function on $\mathcal{B}$, such that*

*(i) $0 \leq m(x_B) \leq 1$ for all $m \in \mathcal{M}$ and all $B \in \mathcal{B}$,*

*(ii) $m(x_{\mathcal{U}}) = 1$ for all $m \in \mathcal{M}$,*

*(iii) if $\{B_i\}$ is any countable partition of $\mathcal{U}$, $B_i \in \mathcal{B}$, then $\Sigma x_{B_i} = x_{\mathcal{U}}$.*

*Then $\mathbf{X} = \{x_B : B \in \mathcal{B}\}$ will be called a $(\mathcal{U}, \mathcal{B})$-semiobservable on $\mathcal{L}$.*

From *(i)* it follows that $x_B$, $B \in \mathcal{B}$, are $([0, 1], \mathcal{B}[0, 1])$-observables on $\mathcal{L}$, where $\mathcal{B}[0, 1]$ is the $\sigma$-algebra of all Borel subsets of $[0, 1]$. Indeed, by [5] the set $V(x) = \{m(x) : m \in \mathcal{M}\}^-$ is the smallest closed interval containing $\sigma(x)$. By *(i)*, $V(x_B) = \{m(x_B) : m \in \mathcal{M}\}^- \subset [0, 1]$, so that $\sigma(x_B) \subset [0, 1]$. From *(ii)* it follows that $V(x_{\mathcal{U}}) = \{1\}$, so that $\sigma(x_{\mathcal{U}}) = \{1\}$, i.e. $x_{\mathcal{U}} = I$.

**Definition 3.** *A $(\mathcal{U}, \mathcal{B})$-semiobservable* $\boldsymbol{X}$ *on* $\mathcal{L}$ *is said to be simple, if all* $x_B$, $B \in \mathcal{B}$ *are simple observables.*

**Proposition 1.** *The set of all simple $(\mathcal{U}, \mathcal{B})$-semiobservables on $\mathcal{L}$ is in one-to-one correspondence with the set of all $(\mathcal{U}, \mathcal{B})$-observables on $\mathcal{L}$.*

Proof. Let $y\colon B \mapsto y(B)$ be a $(\mathcal{U}, \mathcal{B})$-observable on $\mathcal{L}$. For any $B \in \mathcal{B}$, let us define a real simple observable $y_B$ by setting $y_B(\{1\}) = y(B)$, i.e. $y_B = \chi_B(y)$, where $\chi_B$ is the characteristic function of the set $B \in \mathcal{B}$. Then $\{y_B\colon B \in \mathcal{B}\}$ is a simple $(\mathcal{U}, \mathcal{B})$-semiobservable. Indeed, *(i)* and *(ii)* are clearly fulfilled. Let $\{B_i\}$ be a partition of $\mathcal{U}$, then

$$\sum_i m(y_{B_i}) = \sum_i m[y_{B_i}(\{1\})] = \sum_i m[y(B_i)] = m\left[\bigvee_i y(B_i)\right] =$$

$$= m\left[y\left(\bigcup_i B_i\right)\right] = m[y(\mathcal{U})] = 1$$

for any $m \in \mathcal{M}$, which proves *(iii)* in Definition 2.

Now let $\{y_B\colon B \in \mathcal{B}\}$ be a simple $(\mathcal{U}, \mathcal{B})$-semiobservable on $\mathcal{L}$. Then by setting $y(B) = y_B(\{1\})$ we get a $(\mathcal{U}, \mathcal{B})$-observable on $\mathcal{L}$. Indeed, $y(\mathcal{U}) = y_{\mathcal{U}}(\{1\}) = 1$, $B \cap C = \emptyset$ implies $1 \geq m(y_{B\cup C}) = m(y_B) + m(y_C) = m(y(B)) + m(y(C))$, $m \in \mathcal{M}$, from which it follows that $y(B) \perp y(C)$. Now let $\{B_i\}$ be any sequence of disjoint sets of $\mathcal{B}$. Then $m\left[y\left(\bigcup_i B_i\right)\right] = m[y_{\cup_i B_i}(\{1\})] = m(y_{\cup_i B_i}) = \sum_i m(y_{B_i}) = \sum_i m[y(B_i)] = m\left[\bigvee_i y(B_i)\right]$ for all $m \in \mathcal{M}$, so that $y\left(\bigcup_i B_i\right) = \bigvee_i y(B_i)$.

Let $\mathcal{X}_{(\mathcal{U}, \mathcal{B})}$ be the set of all $(\mathcal{U}, \mathcal{B})$-semiobservables on $\mathcal{L}$. The following statement is straightforward.

**Proposition 2.** *The set $\mathcal{X}_{(\mathcal{U}, \mathcal{B})}$ is convex.*

Let $\boldsymbol{X} \in \mathcal{X}_{(\mathcal{U}, \mathcal{B})}$ and $m \in \mathcal{M}$. Then $B \mapsto m(x_B)$, $B \in \mathcal{B}$ defines a probability measure on $\mathcal{B}$. Let $\mathcal{P}(\mathcal{B})$ be the set of all probability measures on $\mathcal{B}$. Then the map $v\colon m \mapsto m(x_B)$ is a convex homomorphism from the set $\mathcal{M}$ into the set $\mathcal{P}(\mathcal{B})$, i.e.

$$v[\alpha m_1 + (1-\alpha)m_2] = \alpha v(m_1) + (1-\alpha)v(m_2), \quad 0 \leq \alpha \leq 1.$$

In the following we shall need some definitions. As before, $x$ is the set of all bounded real observables on $\mathcal{L}$.

**Definition 4.** *We shall say that the logic $\mathcal{L}$ has the property $(A)$ if for any sequence $\{x_n\} \subset X$ such that*

*(i)* $\|x_n\| \leq K < \infty$, $n = 1, 2, \ldots$,

*(ii)* $\lim_{n\to\infty} m(x_n) = \alpha_m$, $\alpha_m \in \mathcal{R}$ *for any* $m \in \mathcal{M}$, *there is an observable* $x \in X$ *such that* $\alpha_m = m(x)$, $m \in \mathcal{M}$.

**Definition 5.** *We shall say that the logic $\mathscr{L}$ has the property $(B)$ if to any measurable space $(\mathscr{U}, \mathscr{B})$ and any convex homomorphism $v$: $\mathscr{M} \to \mathscr{P}(\mathscr{B})$ there is a unique semiobservable $\boldsymbol{X} = \{x_B: B \in \mathscr{B}\}$ such that $v(m)(B) = m(x_B)$ for any $m \in \mathscr{M}$ and any $B \in \mathscr{B}$.*

**Definition 6.** *We shall say that the logic $\mathscr{L}$ has the property $(C)$ if to any affine functional $\varphi$: $\mathscr{M} \to \mathscr{R}$ such that $\sup_{m \in \mathscr{M}} |\varphi(m)| \leq K$, $K \in \mathscr{R}$, there is an observable $x \in X$ such that $\varphi(m) = m(x)$, $m \in \mathscr{M}$.*

**Proposition 3.** *If the logic $\mathscr{L}$ has the property $(C)$, then it has also the properties $(A)$ and $(B)$.*

Proof. Let $\{x_n\} \subset X$ be such that $\|x_n\| \leq K$, $n = 1, 2, \ldots$ and $\lim_{n \to \infty} m(x_n) = \alpha_m$, $\alpha_m \in \mathscr{R}$, $m \in \mathscr{M}$. Then $m \mapsto \alpha_m$ is an affine mapping from the set $\mathscr{M}$ into $\mathscr{R}$. From $\|x_n\| \leq K$, $k = 1, 2, \ldots$ it follows that $|m(x_n)| \leq K$, $n = 1, 2, \ldots$, so that $|\alpha_m| \leq K$ for any $m \in \mathscr{M}$. Then by $(C)$ there is an $x \in X$ such that $\alpha_m = m(x)$. Hence $(C) \Rightarrow (A)$. Now let $v$: $\mathscr{M} \to \mathscr{P}(\mathscr{B})$ be a convex homomorphism. Then for any fixed $B \in \mathscr{B}$, $f_B(m) = v(m)(B)$ is an affine mapping from $\mathscr{M}$ into $\mathscr{R}$ such that $\sup_{m \in \mathscr{M}} |f_B(m)| \leq 1$. Then by $(C)$ there is an observable $x_B \in X$ such that $f_B(m) = m(x_B)$, $m \in \mathscr{M}$. Clearly, $m(x_B) \in [0, 1]$ and $m(x_{\mathscr{U}}) = 1$ for any $m \in \mathscr{M}$. Let $\{B_i\}$ be a partition of $\mathscr{U}$, then $\Sigma m(x_{B_i}) = \Sigma v(m)(B_i) = v(m)(\cup B_i) = v(m)(\mathscr{U}) = 1$ for any $m \in \mathscr{M}$, so that $\Sigma x_{B_i} = I$. Hence $\{x_B: B \in \mathscr{B}\}$ is a semiobservable. Thus we have proved that $(C) \Rightarrow (B)$.

Let $\mathscr{L}(\mathscr{H})$ be the logic of all closed subspaces of a complex separable Hilbert space $\mathscr{H} \cdot \mathscr{L}(\mathscr{H})$ is a quite full logic with respect to the set of all its states. By the Gleason theorem [17] there is a one-to-one correspondence between the set of all states of $\mathscr{L}(\mathscr{H})$ and the set of all nonnegative trace-class operators with the trace 1. For any state $\varrho$ and any observable (i.e. Hermitean operator) $x$ we have $\varrho(x) = tr(\varrho x)$, where $\varrho$ is the trace-class operator corresponding to the state $\varrho$.

**Proposition 4.** *The logic $\mathscr{L}(\mathscr{H})$ has the property $(C)$.*

Proof. Let $f$ be an affine real functional on the set of all states such that $|f(\varrho)| \leq K$ for all $\varrho$. Then $f$ can be considered as the affine functional on the set of all positive trace-class operators with $tr\varrho = 1$. It can be uniquely extended to the linear functional $\tilde{f}$ on the set of all trace-class operators such that

$$|\tilde{f}(\sigma)| \leq 2K tr|\sigma|, \quad |\sigma| = \sqrt{\sigma\sigma^*}.$$

By [15, Theorem 2, p. 47] there is a unique bounded operator $x$ such that $\tilde{f}(\sigma) = tr(\sigma x)$. As $\tilde{f}$ is real on Hermitean trace-class operators, we get that $x = x^*$, so that $x$ is an observable.

From Propositions 3 and 4 it follows that the logic $\mathscr{L}(\mathscr{H})$ has also the properties $(A)$ and $(B)$.

Let $X_1=\{x\in X: \|x\|\leq 1\}$, where $X$ is the set of all bounded real observables on the logic $\mathscr{L}$. Let $x_0\in X_1$, $m_1, m_2, \ldots, m_k\in\mathscr{M}$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ be positive numbers. Let us define the *weak topology* on $X_1$ by the neighbourhoods

$$U(x_0; m_1, \ldots, m_k; \varepsilon_1, \ldots, \varepsilon_k)= \\ =\{x\in X_1: |m_i(x_0)-m_i(x)|<\varepsilon_i,\ i=1, 2, \ldots, k\}.$$

This topology is Hausdorffian, because the set $\mathscr{M}$ separates points of $X_1$.

To any $x\in X_1$ we define the function $\bar{x}$: $\mathscr{M}\to[-1, 1]$ by setting $\bar{x}(m)=m(x)$, $m\in\mathscr{M}$. The mapping $x\to\bar{x}$ is one-to-one. Let $\bar{X}_1=\{\bar{x}: x\in X_1\}\subset[-1, 1]^{\mathscr{M}}$. Then $X_1$ with the weak topology and $\bar{X}_1$ with the relative product topology are topologically isomorphic.

Let the logic $\mathscr{L}$ have the property $(A)$. Then to each sequence $\{x_n\}\subset X_1$ which is Cauchy in the weak topology there is an $x\in X_1$ such that $m(x_n)\to m(x)$ for any $m\in\mathscr{M}$.

Let $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$ be the set of all $(\mathscr{U}, \mathscr{B})$-semiobservables on $\mathscr{L}$. Clearly, $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}\subset X_1^{\mathscr{B}}$.

**Definition 7.** *Let $\{\mathbf{X}^n\}$, $\mathbf{X}^n=\{x_B^n: B\in\mathscr{B}\}$ be a sequence of semiobservables. We shall say that $\{\mathbf{X}^n\}$ converges to the semiobservable $\mathbf{X}$, in symbols $\mathbf{X}^n\to\mathbf{X}$, if $x_B^n\to x_B$ weakly, i.e. if $m(x_B^n)\to m(x_B)$ for any $m\in\mathscr{M}$ and any $B\in\mathscr{B}$.*

**Proposition 5.** *The set of all semiobservables $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$ is sequentially closed in $X_1^{\mathscr{B}}$.*

Proof. Let $\{\mathbf{X}^n\}\subset\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$ be such that $\mathbf{X}^n\to\mathbf{Y}$, where $\mathbf{Y}=\{y_B: B\in\mathscr{B}\}\subset X_1^{\mathscr{B}}$. For any $m\in\mathscr{M}$ we define the probability measure $\mu_m^n$ on $\mathscr{B}$ by setting

$$\mu_m^n(B)=m(x_B^n),\quad B\in\mathscr{B},\quad n=1, 2, \ldots.$$

Then

$$m(x_B^n)\to m(y_B)\quad\text{implies}\quad \mu_m^n(B)\to m(y_B),\quad n\to\infty.$$

By [8, § 40, p. 170], the map $B\mapsto m(y_B)$ is a probability measure on $\mathscr{B}$. Let $\{B_i\}$ be a countable partition of $\mathscr{U}$. Then

$$1=\mu_m(\mathscr{U})=\mu_m(\cup B_i)=\Sigma\mu_m(B_i)=\Sigma m(y_{B_i}),$$

where

$$\mu_m(B)=m(y_B),\quad B\in\mathscr{B},\quad m\in\mathscr{M}.$$

From this it follows that $\{y_B: B\in\mathscr{B}\}\in\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$.

**Proposition 6.** *Let $X_1$ be sequentially compact, i.e. to each sequence $\{x_n\}\subset X_1$ there is a subsequence $\{x_{n_k}\}\subset\{x_n\}$ and an element $x\in X_1$ such that $x_{n_k}\to x$ weakly. Let $\mathscr{B}$ be countable. Then the set of all $(\mathscr{U}, \mathscr{B})$-semiobservables $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$ is sequentially compact.*

Proof. By [14, 7D, p. 238], the set $X_1^{\mathscr{B}}$, as a product of a countable number of sequentially compact spaces is sequentially compact. By Proposition 5, $\mathscr{X}_{(\mathscr{U},\mathscr{B})}$ is sequentially closed, from which it follows that it is sequentially compact.

We note that the set $X_1$ of the Hilbert space logic $\mathscr{L}(\mathscr{H})$ is sequentially compact. Indeed, from [1, p. 105] and the fact that the limit of the sequence of Hermitean operators in the weak operator topology is a Hermitean operator, it follows that to any sequence $\{x_n\}\subset X_1$ there is a subsequence $\{x_{n_k}\}$ and an $x\in X_1$ such that $(x_{n_k}\varphi, \psi)\to(x\varphi, \psi)$ for any $\varphi, \psi\in\mathscr{H}$. Let $\varrho=\Sigma w_i P[\varphi_i]$ be a state on $\mathscr{L}(\mathscr{H})$. Then

$$|tr\varrho(x_{n_k}-x)|=\left|\sum_{i=1}^{\infty} w_i((x_{n_k}-x)\varphi_i, \varphi_i)\right|\leqslant$$

$$\leqslant\sum_{i=1}^{\infty} w_i|((x_{n_k}-x)\varphi_i, \varphi_i)|\leqslant\sum_{i=1}^{\infty} w_i\cdot 2,$$

because

$$|((x_{n_k}-x)\varphi_i, \varphi_i)|\leqslant\|x_{n_k}-x\|\cdot\|\varphi_i\|^2\leqslant 2.$$

Let $s$ be such that

$$\sum_{i=s+1}^{\infty} w_i|((x_{n_k}-x)\varphi_i, \varphi_i)|<\frac{\varepsilon}{2}\quad\text{for any } n_k.$$

Then

$$|tr\varrho(x_{n_k}-x)|\leqslant\sum_{i=1}^{\infty} w_i|((x_{n_k}-x)\varphi_i, \varphi_i)|=$$

$$=\sum_{i=1}^{s} w_i|((x_{n_k}-x)\varphi_i, \varphi_i)|+\sum_{i=s+1}^{\infty}|((x_{n_k}-x)\varphi_i, \varphi_i)|<\varepsilon$$

for $n_k>n_0$ if we choose $n_0$ such that

$$\sum_{i=1}^{s} w_i|((x_{n_k}-x)\varphi_i, \varphi_i)|<\frac{\varepsilon}{2}\quad\text{for}\quad n_k>n_0.$$

**Definition 8.** *We shall say that a functional $f$ on $\mathscr{X}_{(\mathscr{U},\mathscr{B})}$ is continuous if $f(\mathbf{X}^n)\to f(\mathbf{X})$ provided $\mathbf{X}^n\to\mathbf{X}$, $n\to\infty$.*

The following statement is a modification of the Weierstrass theorem. The proof of it is standard and we omit it.

**Proposition 8.** *A continuous functional on a sequentially compact set $\mathscr{X}_{(\mathscr{U},\mathscr{B})}$ of semiobservables achieves a maximum.*

## 4. Extremal points of the set $\mathscr{X}_{(\mathscr{U},\mathscr{B})}$

Now we shall study the extremal points of the convex set $\mathscr{X}_{(u,\mathscr{B})}$ of all $(\mathscr{U},\mathscr{B})$-semiobservables. The methods of proofs are similar to those used in [12].

First we shall define a pseudoproduct on the set of all bounded observables $X$ of the logic $\mathscr{L}$ by setting

$$x \circ y = \frac{1}{2}[(x+y)^2 - x^2 - y^2].$$

**Lemma 1.** *$x \leftrightarrow y$ implies $x \circ y = xy$.*

Proof. $x \leftrightarrow y$ implies that there is a $z \in X$ and Borel functions $u_1$, $u_2$ such that $x = u_1(z)$, $y = u_2(z)$. But then $x \circ y = u_1(z) \circ u_2(z) = u_1(z)u_2(z) = xy$.

**Lemma 2.** *An observable $x$ is simple, i.e. $x = x^2$ if and only if $x \circ (I - x) = 0$.*

Proof. It follows from the equation $x \circ (I - x) = x - x^2$.

**Lemma 3.** *A semiobservable $\{x_B \colon B \in \mathscr{B}\}$ is simple if and only if $x_B \circ x_C = 0$ for any $B$, $C \in \mathscr{B}$ such that $B \cap C = \emptyset$.*

Proof. Let $\{x_B \colon B \in \mathscr{B}\}$ be simple. Then $x_B = x_B^2$ for any $B \in \mathscr{B}$ implies that

$$x_B \circ x_C = \frac{1}{2}[(x_B + x_C)^2 - x_B^2 - x_C^2] = \frac{1}{2}[x_{B \cup C} - x_B - x_C] = 0,$$

provided $B \cap C = \emptyset$.

Now let $B \cap C = \emptyset$ imply $x_B \circ x_C = 0$. Then $x_B \circ (I - x_B) = x_B \circ x_{U-B} = 0$, so that $x_B = x_B^2$, $B \in \mathscr{B}$.

We shall write $x \leqslant y$ if $m(x) \leqslant m(y)$ for all $m \in \mathscr{M}$. Then $(X, \leqslant)$ is a partially ordered linear space.

**Proposition 9.** *If the pseudoproduct is distributive (relative to addition of observables), then simple observables are extremal points in the set $X_1^+ = \{x \in X_1 \colon m(x) \geqslant 0$ for any $m \in \mathscr{M}\}$.*

Proof. Let $x \in X_1^+$, $x = x^2$ and let $x = \frac{1}{2}(y + z)$ for some $y$, $z \in X_1^+$. Then

$$m(y^2) = \int_0^1 \lambda^2 m_y(d\lambda) \leqslant \int_0^1 \lambda m_y(d\lambda) = m(y)$$

for any $m \in \mathscr{M}$ implies that $y^2 \leqslant y$ and similarly $z^2 \leqslant z$. Then

$$\frac{1}{2}(y^2 + z^2) \leqslant \frac{1}{2}(y + z) = x = x^2 = \frac{1}{4}(y + z)^2.$$

From this we get $2(y^2 + z^2) \leqslant (y + z)^2$, i.e. $y^2 + z^2 \leqslant 2(y \circ z)$, from which it follows that $(y - z)^2 \leqslant 0$, i.e. $y = z$.

From Proposition 9 it follows that the simple semiobservables are extremal points in $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$.

We recall that a $\sigma$-algebra is *discrete* if it is generated by an at most countable set of atoms.

**Proposition 10.** *Let the logic $\mathscr{L}$ be a Boolean $\sigma$-algebra and let $(\mathscr{U}, \mathscr{B})$ be a measurable space with the discrete $\sigma$-algebra $\mathscr{B}$. Then the simple semiobservables are the only extremal points in the set $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$.*

Proof. Let $\{A_i: i \in D\}$ be the set of atoms in $\mathscr{B}$, where $D$ is at most countable indexed set, and let $\boldsymbol{X}$ be an extremal point in $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$. If $\boldsymbol{X}$ is not simple, then there are sets $B$, $C \in \mathscr{B}$, $B \cap C = \emptyset$ such that $x_B \circ x_C \neq 0$. As $\mathscr{L}$ is a Boolean algebra, $x_B \leftrightarrow x_C$ and $x_B \circ x_C = x_B x_C$. Let $E \in \mathscr{B}$, then $E = \cup\{A_i: A_i \subset E\}$, and $x_E = \sum_{A_i \subset E} x_{A_i}$. As all $x_{A_i}$ are pairwise compatible, they can be considered as functions of an observable $x$. Let us set $x_{A_i} = f_i(x)$, $i \in D$. Then

$$x_B x_C = \sum_{\{i:\, A_i \subset B\}} f_i(x) \sum_{\{i:\, A_j \subset C\}} f_j(x) = \sum_{\{i:\, A_i \subset B\}} \sum_{\{j:\, A_j \subset C\}} f_i(x) f_j(x) > 0$$

implies that $f_i(x) f_j(x) > 0$ for some $i$, $j$. Let $f_i(x) = x_i$, $f_j(x) = x_j$. Then $(x_i + x_j)^2 > \zeta(x_i - x_j)^2$, so that $(x_i + x_j) > |x_i - x_j|$. Let us set $z = \frac{1}{2}(x_i + x_j - |x_i - x_j|)$. Then $z > 0$, $z \leq x_i$, $z \leq x_j$ [7]. We can define semiobservables $\boldsymbol{Y} = \{y_k: k \in D\}$ by setting $y_k = x_k$ if $k \neq j, i$ and $y_i = x_i + z$, $y_j = x_j - z$, and $\boldsymbol{Z} = \{z_k: k \in D\}$ by setting $z_k = x_k$ if $k \neq i, j$ and $z_i = x_i - z$, $z_j = x_j + z$. Then $\boldsymbol{X} = \frac{1}{2}(\boldsymbol{Y} + \boldsymbol{Z})$, which is impossible as $\boldsymbol{X}$ was supposed to be an extremal point.

## 5. Integrals of functions with respect to a semiobservable

Let $(\mathscr{U}, \mathscr{B})$ be a measurable space and $\mathscr{X}_{(\mathscr{U}, \mathscr{B})}$ be the set of all $(\mathscr{U}, \mathscr{B})$-semiobservables on $\mathscr{L}$. Let $\mathscr{L}$ have the property $(A)$. Let $f: \mathscr{U} \to \mathscr{R}$ be a simple function of the form

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \quad A_i \in \mathscr{B}.$$

Let us set for any $\boldsymbol{X} \in \mathscr{X}_{(\mathscr{U}, \mathscr{B})}$

$$f(\boldsymbol{X}) = \sum_{i=1}^{n} \alpha_i x_{A_i}.$$

Clearly, $f(\boldsymbol{X})$ is a bounded observable on $\mathscr{L}$. For any $m \in \mathscr{M}$ we get that

$$m(f(\boldsymbol{X})) = m\left(\sum_{i=1}^{n} \alpha_i x_{A_i}\right) = \sum_{i=1}^{n} \alpha_i m(x_{A_i}) = \int f(t) m(x_{dt}).$$

Now let $f: \mathscr{U} \to \mathscr{R}$ be a nonnegative Borel function, $f \leq K < \infty$ and let $\{f_n\}$ be an increasing sequence of nonnegative functions converging to $f$ [8, §20, Th. B]. Then for every $m \in \mathcal{M}$

$$m(f_n(\boldsymbol{X})) = \int_{\mathcal{U}} f_n(t) m(x_{dt}) \leq K,$$

so that $\{m(f_n(\boldsymbol{X}))\}$ is an increasing bounded sequence. By the property $(A)$ then there is a bounded observable, let us denote it by $f(\boldsymbol{X})$, such that $\lim_{n\to\infty} m(f_n(\boldsymbol{X})) = m(f(\boldsymbol{X}))$ for every $m \in \mathcal{M}$. By [8, § 27, Th. B] $\lim_{n\to\infty} m(f_n(\boldsymbol{X})) = \lim_{n\to\infty} \int f_n(t) m(x_{dt}) = \int f(t) m(x_{dt})$. From this it follows that the observable $f(\boldsymbol{X})$ is well defined. For any bounded Borel function $f$ let us set

$$f(\boldsymbol{X}) = f^+(\boldsymbol{X}) - f^-(\boldsymbol{X}).$$

It can be easily seen that $f \geq 0$ implies $f(\boldsymbol{X}) \geq 0$, $(f+g)(\boldsymbol{X}) = f(\boldsymbol{X}) + g(\boldsymbol{X})$ and $(\alpha f)(\boldsymbol{X}) = \alpha f(\boldsymbol{X})$, $\alpha \in \mathcal{R}$.

**Proposition 11.** *If $\{f_n\}$ is a sequence of bounded Borel functions such that $|f_n| \leq g$, where $g$ is a bounded Borel function and $f_n \to f$ pointwise, then $\lim_{n\to\infty} m(f_n(\boldsymbol{X})) = m(f(\boldsymbol{X}))$ for every $m \in \mathcal{M}$.*

Proof. It follows from [8, § 26, Th. D].

**Proposition 12.** *Let $\boldsymbol{X}$ be a simple semiobservable and let $g: \mathcal{U} \to \mathcal{R}$, $f: \mathcal{R} \to \mathcal{R}$ be bounded Borel functions. Then $(f \circ g)(\boldsymbol{X}) = f(g(\boldsymbol{X}))$.*

Proof. If $\boldsymbol{X}$ is simple, then by Proposition 1 there is an observable $y: \mathcal{B} \to \mathcal{L}$ such that $x_B = \chi_B(y)$, $B \in \mathcal{B}$. Then

$$m(g(\boldsymbol{X})) = \int_{\mathcal{U}} g(t) m(x_{dt}) = \int_{\mathcal{U}} g(t) m(y(dt)) = m(g(y))$$

for any $m \in \mathcal{M}$, so that

$$(f \circ g)(\boldsymbol{X}) = (f \circ g)(y) = f(g(y)) = f(g(\boldsymbol{X})).$$

We shall write $f(\boldsymbol{X}) = \int f(t) x_{dt}$.

## 6. Real semiobservables

Let $\mathcal{U} = \mathcal{R}$ and let $\mathcal{B} = \mathcal{B}(\mathcal{R})$ be the $\sigma$-algebra of Borel subsets of $\mathcal{R}$. If the logic $\mathcal{L}$ has the property $(B)$, then the $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$-semiobservables can be alternatively defined as follows.

**Proposition 13.** *Let $\mathcal{L}$ have the property $(B)$. Let $y_t \in X_1^+$, $t \in \mathcal{R}$ be such that*

*(i) $t_2 \geq t_1$ implies $m(y_{t_2} - y_{t_1}) \geq 0$ for all $m \in \mathcal{M}$,*

*(ii)* $m(y_{t-0}) = m(y_t)$ *for all* $m \in \mathcal{M}$, $t \in \mathcal{R}$,

*(iii)* $\lim_{t \to -\infty} m(y_t) = 0$, $\lim_{t \to \infty} m(y_t) = 1$ *for all* $m \in \mathcal{M}$. *Then the set* $\{y_t\colon t \in \mathcal{R}\}$ *uniquely defines a real semiobservable* $\{x_B\colon B \in \mathcal{B}\}$ *such that* $x_{(-\infty, t)} = y_t$, $t \in \mathcal{R}$.

Proof. Let us set $f_m(t) = m(y_t)$, $t \in \mathcal{M}$. For any $m \in \mathcal{M}$, $f_m(t)$ is a bounded monotone function continuous on the left and such that $f_m(-\infty) = 0$. By [8, § 43, Th. B] there exists a unique finite measure $\nu_m$ on $\mathcal{B}(\mathcal{R})$ such that $f_m(t) = \nu_m((-\infty, t))$. From $f_m(\infty) = 1$ it follows that $\nu_m$ is a probability measure. Let $m = \alpha m_1 + (1-\alpha) m_2$. Then

$$\begin{aligned} \nu_m(-\infty, t) &= \alpha m_1(y_t) + (1-\alpha) m_2(y_t) = \\ &= \alpha \nu_{m_1}((-\infty, t)) + (1-\alpha)\nu_{m_2}((-\infty, t)), \end{aligned}$$

so that

$$\nu_m(B) = \alpha \nu_{m_1}(B) + (1-\alpha)\nu_{m_2}(B),$$

where $B$ is any finite disjoint union of intervals and it can be easily seen that the set

$$\mathcal{K} = \{B \in \mathcal{B}(\mathcal{R})\colon \nu_m(B) = \alpha \nu_{m_1}(B) + (1-\alpha)\nu_{m_2}(B)\}$$

is a monotone system. Hence, $m \mapsto \nu_m$ is a convex homomorphism from $\mathcal{M}$ into the set $\mathcal{P}(\mathcal{B})$ of all probability measures on $\mathcal{B}$. By the property $(B)$, there is a unique semiobservable $\boldsymbol{X}$ such that $\nu_m(B) = m(x_B)$ for any $m \in \mathcal{M}$ and $B \in \mathcal{B}$.

**Definition 9.** *Let* $\mathcal{U} = \langle a, b \rangle$, $-\infty < a < b < \infty$, $\mathcal{B}$ *be the* $\sigma$*-algebra of all Borel subsets of* $\mathcal{U}$. *Let* $\mathcal{L}$ *have the property* $(A)$. *Then*

$$e(\boldsymbol{X}) = \int_{\mathcal{U}} \lambda x_{d\lambda}$$

*will be called the mean value of the semiobservable* $\boldsymbol{X} \in \mathcal{X}_{(\mathcal{U}, \mathcal{B})}$.

Clearly, the same observable $e(\boldsymbol{X})$ can be the mean value for several semiobservables $\boldsymbol{X}$.

**Definition 10.** *For any* $m \in \mathcal{M}$ *we shall call* $m[e(\boldsymbol{X})]$ *the mean value of the semiobservable* $\boldsymbol{X}$ *in the state* $m$ *and*

$$\sigma_m^2(\boldsymbol{X}) = \int (\lambda - m[e(\boldsymbol{X})])^2 m(x_{d\lambda}) = \int \lambda^2 m(x_{d\lambda}) - m[e(\boldsymbol{X})]^2$$

*will be called the dispersion of* $\boldsymbol{X}$ *in the state* $m$.

From Proposition 12 it follows that if $\boldsymbol{X}$ is a simple semiobservable, then $e(\boldsymbol{X})^2 = \int \lambda^2 x_{d\lambda}$.

**Proposition 14.** *Let* $\boldsymbol{X}$ *be a simple semiobservable and* $\boldsymbol{Y}$ *be any semiobservable on the Hilbert space logic* $\mathcal{L}(\mathcal{H})$. *Let the mean values of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *coincide, i.e.*

$$\varrho[e(\boldsymbol{X})] = tr[\varrho \int \lambda x_{d\lambda}] = tr[\varrho \int \lambda y_{d\lambda}] = \varrho[e(\boldsymbol{Y})] = \bar{\lambda}_\varrho$$

*for any state* $\varrho$ *on* $\mathcal{L}(\mathcal{H})$. *Then for all* $\varrho$,

$$\sigma_\varrho^2(\mathbf{X}) = tr\left[\varrho\int_{\langle a,b\rangle}(\lambda-\bar{\lambda}_\varrho)^2 x_{d\lambda}\right] \leqslant tr\left[\varrho\int_{\langle a,b\rangle}(\lambda-\bar{\lambda}_\varrho)^2 y_{d\lambda}\right] = \sigma_\varrho^2(\mathbf{Y}).$$

Proof. It is enough to prove it for $\varrho = P_{[\varphi]}$, $\varphi \in \mathscr{H}$. By the Naimark theorem [1, IX, p. 393] if $\mathbf{Y} = \{y_t\colon t \in \langle a, b\rangle\}$ is a semiobservable on $\mathscr{L}(\mathscr{H})$, then there is a Hilbert space $\mathscr{H}^+$ such that $\mathscr{H} \subset \mathscr{H}^+$ and a simple semiobservable $E_t^+$ on $\mathscr{H}^+$ such that for any $\varphi \in \mathscr{H}$, $y_t\varphi = P^+E_t^+\varphi$, where $P^+$ is the projector on $\mathscr{H}$. Then for any $\varphi \in \mathscr{H}$,

$$\int_{\langle a,b\rangle}\lambda^2(y_{d\lambda}\varphi,\varphi) = \int_{\langle a,b\rangle}\lambda^2(P^+E_{d\lambda}^+\varphi,\varphi) = \int_{\langle a,b\rangle}\lambda^2(E_{d\lambda}^+\varphi, P^+\varphi) =$$

$$= \int_{\langle a,b\rangle}\lambda^2(E_{d\lambda}^+\varphi,\varphi) = \left(\left(\int_{\langle a,b\rangle}\lambda E_{d\lambda}^+\right)^2\varphi,\varphi\right) = \left\|\int_{\langle a,b\rangle}\lambda E_{d\lambda}^+\varphi\right\|^2 \geqslant \left\|P^+\int_{\langle a,b\rangle}\lambda E_{d\lambda}^+\varphi\right\|^2 =$$

$$= \left(\int_{\langle a,b\rangle}\lambda P^+E_{d\lambda}^+\varphi, \int_{\langle a,b\rangle}\lambda P^+E_{d\lambda}^+\varphi\right) = \left(\left(\int_{\langle a,b\rangle}\lambda y_{d\lambda}\right)^2\varphi,\varphi\right).$$

From $(\int\lambda y_{d\lambda}\varphi,\varphi) = (\int\lambda x_{d\lambda}\varphi,\varphi) = \bar{\lambda}_\varphi$ for any $\varphi \in \mathscr{H}$ it follows that $\int\lambda y_{d\lambda} = \int\lambda x_{d\lambda}$. Thus we get

$$\sigma_\varphi^2(\mathbf{Y}) = \left(\int_{\langle a,b\rangle}\lambda^2 y_{d\lambda}\varphi,\varphi\right) - \bar{\lambda}_\varphi^2 \geqslant \left(\left(\int_{\langle a,b\rangle}\lambda y_{d\lambda}\right)^2\varphi,\varphi\right) - \bar{\lambda}_\varphi^2 =$$

$$= \left(\left(\int_{\langle a,b\rangle}\lambda x_{d\lambda}\right)^2\varphi,\varphi\right) - \bar{\lambda}_\varphi^2 = \sigma_\varphi^2(\mathbf{X}).$$

From Proposition 14 it follows that simple semiobservables are characterized out of all the semiobservables giving the same mean values by the smallest possible fluctuations. This fact is mentioned in [13], without the proof.

For more general logics we get the following statement (a generalization of the Jensen inequality).

**Proposition 15.** *Let $g\colon \langle a, b\rangle \to \mathscr{R}$ be a convex function. Let $\mathbf{X} \in \mathscr{X}(\langle a, b\rangle, \mathscr{B})$ be such that $e(\mathbf{X})$ is compatible with*

$$\int_{\langle a,b\rangle} g(\lambda)x_{d\lambda}. \quad \textit{Then} \quad \int_{\langle a,b\rangle} g(\lambda)x_{d\lambda} \geqslant g(e(\mathbf{X})).$$

Proof. The convex function $g$ can be written as a supremum of a countable set of linear functions $f_n(t) = a_nt + b_n$, $a_n, b_n \in \mathscr{R}$, $t \in \langle a, b\rangle$. From this it follows that $g(x) = \sup_n f_n(x)$ for any observable $x$ on $\mathscr{L}$. From $e(\mathbf{X}) \leftrightarrow g(\mathbf{X})$ we get that there is an observable $z$ and Borel functions $u$, $v$ such that $e(\mathbf{X}) = u(z)$, and

$$g(\mathbf{X}) = \int_{\langle a,b\rangle} g(\lambda)x_{d\lambda} = v(z).$$

Then for any $m \in \mathcal{M}$,

$$\int_{\langle a,b\rangle} g(\lambda) m(x_{d\lambda}) \geqslant \int_{\langle a,b\rangle} f_n(\lambda) m(x_{d\lambda}) = a_n m(e(\boldsymbol{X})) + b_n =$$

$$= m[f_n(e(\boldsymbol{X}))], \quad \text{i.e.} \quad m(v(z)) \geqslant m[f_n(u(z))], \quad n = 1, 2, \ldots.$$

From this it follows that

$$z(\{t\colon v(t) - f_n(u(t)) < 0\}) = 0.$$

Indeed, let $a = z(\{t\colon v(t) - f_n(u(t)) < 0\}) \neq 0$, then by [6, Lemma 2,1] there is $m_0 \in \mathcal{M}$ such that $m_0(a) = 1$, but then $m_0(v(z) - f_n(u(z))) < 0$, a contradiction. Then $v(t) \geqslant f_n(u(t))$, $n = 1, 2, \ldots$ implies that $v(t) \geqslant \sup_n f_n(u(t))$, so that

$$z(\{t\colon v(t) \geqslant \sup_n f_n(u(t))\}) = 1,$$

i.e. $m(v(z)) \geqslant m\ (\sup_n f_n(u(z)))$. From this we get for any $m \in \mathcal{M}$,

$$m\left(\int_{\langle a,b\rangle} g(\lambda) x_{d\lambda}\right) \geqslant m(\sup_n f_n(e(\boldsymbol{X}))) = m[g(e(\boldsymbol{X}))].$$

Proposition 15 implies that a generalization of Proposition 14 is valid provided $e(\boldsymbol{Y}) \leftrightarrow \int \lambda^2 y_{d\lambda}$.

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*Ústav merania a meracej techniky SAV*
*Dúbravská cesta*
*885 27 Bratislava*

## СЕМИНАБЛЮДАЕМЫЕ НА КВАНТОВЫХ ЛОГИКАХ

Сылвиа Пулманнова

Резюме

В работе исследуется понятие семинаблюдаемых, являющееся обобщением понятия наблюдаемых на логике. Показаны некоторые свойства множества всех семинаблюдаемых: выпуклость, секвенциальная замкнутость в слабой топологии, исследованы экстремальные точки и введено понятие интеграла функций по семинаблюдаемым.