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COVERS OF DIGRAPHS

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1. Introduction

Covers of graphs and hypergraphs have been studied extensively, compare [1], [2], [6] and their references. In this paper some of these results are shown to hold also for an appropriate concept of covers of digraphs and further results are obtained. As general references for digraphs we refer to [3] or [4, chapter 16].

There are several ways to define a digraph. We have chosen one which allows multiple arcs and loops. This is called a "pseudodigraph" in [3] and a "net" in [4].

Definition 1. A *digraph* D is a quadruple (V, E, f, s) where V and E are disjoint sets (possibly infinite) and f and s are mappings from E into V. The elements of V are the *vertices* of D, the elements of E are the *arcs* of D; for an arc E the vertex fe is its *first vertex* and se is its *second vertex.* A *loop* is an arc e with $fe = se$.

A *walk* in a digraph $D = (V, E, f, s)$ is an alternating sequence $x_0, e_1, x_1, e_2, \ldots, e_n, x_n$ of vertices and arcs such that $fe_i = x_{i-1}$ and $se_i = x_i$ for $i = 1, \ldots, n$. A *semiwalk* in D is an alternating sequence $x_0, e_1, x_1, e_2, \ldots, e_n, x_n$ of vertices and edges such that either $fe_i = x_{i-1}, se_i = x_i$ or $fe_i = x_i, se_i = x_{i-1}$ for $i = 1, \ldots, n$. Thus a single vertex is considered as a trivial walk. In a *closed walk (semiwalk),* $x_0 = x_n$. A digraph D is called *(weakly) connected* or *weak* if for every pair x, y of vertices there is a semiwalk with first vertex x and last vertex y; D is called *unilaterally connected* or *unilateral* if for every two vertices x, y there exists a walk from x to y or from y to x; finally D is called *strongly connected* or *strong* if for every pair x, y of vertices there exists a walk from x to y. Following [4], C_0 denotes the class of digraphs which are not weak; C_1 all weak digraphs which are not unilateral; C_2 all unilateral digraphs which are not strong; C_3 the strong digraphs. The classes C_i, $0 \leq i \leq 3$, are called *connectivity classes.*

The maximal weak subgraphs of a digraph D are called the *(weak) components* of D; the maximal strong subgraphs are called the *strong components* of D. The components (strong components) of a digraph determine a partition of the vertex set of D.

Definition 2. Let $D_i = (V_i, E_i, f_i, s_i)$, $i = 1, 2$, be two digraphs. A *homomorphism*

h: $D_1 \to D_2$ is a mapping h: $V_1 \cup E_1 \to V_2 \cup E_2$ with $hv_1 \subset V_2$, $hE_1 \subset E_2$ and $h(fe) = f(he)$, $h(se) = s(he)$.

Definition 3. A digraph $\tilde{D} = (\tilde{V}, \tilde{E}, \tilde{f}, \tilde{s})$ is called a *cover* (with *covering projection* p) of the weak digraph $D = (V, E, f, s)$ if there exists a homomorphism p of $\tilde{D}$ onto D with the following property:

Property (*C*). For every $x' \in p^{-1}x$, $x \in V$, the restriction of p to $\{e' \in \tilde{E} | fe' = x'\}$ is a bijection onto $\{e \in E | fe = x\}$ and the restriction of p to $\{e' \in \tilde{E} | se' = x'\}$ is a bijection onto $\{e \in E | se = x\}$.

Example. Let $\tilde{D} = (\tilde{V}, \tilde{E}, \tilde{f}, \tilde{s})$ where $\tilde{V} = \boldsymbol{Z}$, $\tilde{E} = \boldsymbol{Z} \times \{0\}$ and $\tilde{f}(z, 0) = z$, $\tilde{s}(z, 0) = z + 1$ for all $(z, 0) \in \tilde{E}$. Let further $D = (V, E, f, s)$ where $V = \boldsymbol{Z}_3$, i.e., the set of residue classes of $\boldsymbol{Z}$ modulo 3, $E = \boldsymbol{Z}_3 \times \{0\}$, $f([z], 0) = [z]$ and $s([z], 0) = [z + 1]$ for all $([z], 0) \in E$. Finally define p: $\tilde{D} \to D$ by $pz := [z]$ for all $z \in \tilde{V}$ and $p(z, 0) := ([z], 0)$ for all $(z, 0) \in \tilde{E}$. Then p is a covering projection from $\tilde{D}$ onto D. In Figure 1 a picture of this situation is shown.

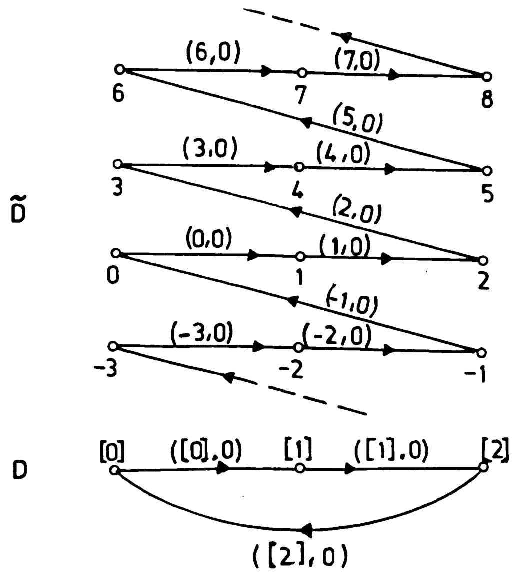

Fig. 1

In general in such figures we omit the labels and assume that vertices (arcs) of $\tilde{D}$ drawn above a vertex x (arc e) of D are mapped by p onto $x(e)$.

In the following we keep the notations introduced in Definition 3.

Proposition 1. *Let $\tilde{D}$ be a cover of the weak digraph D with covering projection p. Then all cardinalities $|p^{-1}x|$, $x \in V$, and $|p^{-1}e|$, $e \in E$, are equal.*

Proof. Let $e \in E$ be arbitrary and $x = fe$, $y = se$. From Property (C) it follows that for every $x' \in p^{-1}x$ there exists exactly one $e' \in p^{-1}e$ with $fe' = x'$. This implies $|p^{-1}x| = |p^{-1}e|$; similarly we have $|p^{-1}y| = |p^{-1}e|$ and therefore $|p^{-1}x| = |p^{-1}y|$. A simple inductive argument using the weak connectedness of D yields the assertion. □

The common cardinality of the fibres $p^{-1}x$, $x \in V$, and $p^{-1}e$, $e \in E$, is called the *multiplicity* of the cover $\tilde{D}$ of D.

It is possible to define covers of digraphs $D \in C_0$ by applying Definition 3 to each weak component of D and adding the condition that all fibres $p^{-1}x$, $x \in V$, and $p^{-1}e$, $e \in E$ have the same cardinality. This means that for each component we have the same multiplicity of the respective cover.

2. Connectivity classes

We consider the question: for which pairs (C_i, C_j), $i, j = 0, 1, 2, 3$, of connectivity classes does exist a cover $p: \tilde{D} \to D$ such that $D \in C_i$ and $\tilde{D} \in C_j$?

Theorem 1. *Let $p: \tilde{D} \to D$ be a covering projection and $D \in C_i$, $\tilde{D} \in C_j$. Then $i \geq j$.*

Proof. If $W': x'_0, e'_1, x'_1, \ldots, e'_n, x'_n$ is a walk (semiwalk) in $\tilde{D}$, then the sequence $pW' := W: px'_0, pe'_1, px'_1, \ldots, pe'_n, px'_n$ is a walk (semiwalk) in D. □

The examples in Figures 2—11 show that in fact all ten combinations $D \in C_i$, $\tilde{D} \in C_j$ with $0 \leq i, j \leq 3$, $i \geq j$, can occur.

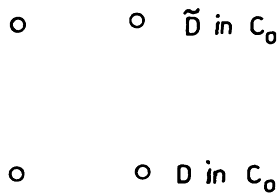

Fig. 2

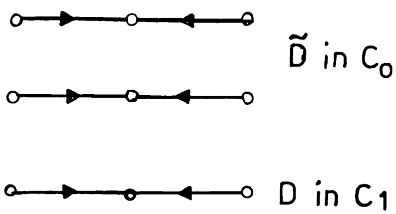

Fig. 3

In two of the examples (Fig. 9 and 10) the multiplicity of the cover $\tilde{D}$ is infinite. We show in the next theorem that this must be the case because these two combinations of connectivity classes cannot occur for finite multiplicity. The proof of the first lemma is omitted as it is immediate.

Lemma 1. *If $x, y \in V$ belong to different strong components of D, then $x' \in p^{-1}x$ and $y' \in p^{-1}y$ belong to different strong components of $\tilde{D}$.* □

Remark. The converse statement is not true as is shown by the example in Fig. 12.

Lemma 2. *Let $p: \tilde{D} \to D$ be a covering projection of finite multiplicity and $W: x_0$, e_1, x_1, e_2, ..., e_n, x_n a closed walk in D. Then for every $x_0' \in P^{-1}x_0$ there exists a unique closed walk W' in $\tilde{D}$ containing x_0' with $pW' = W$.*

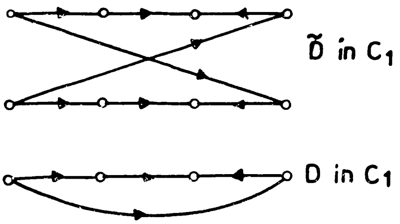

Fig. 4

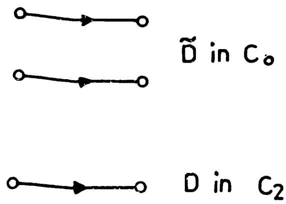

Fig. 5

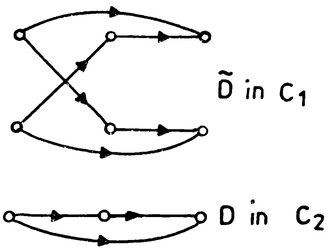

Fig. 6

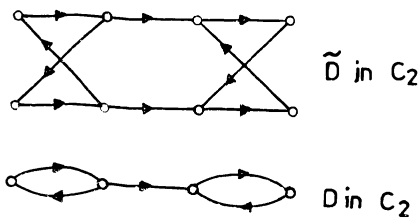

Fig. 7

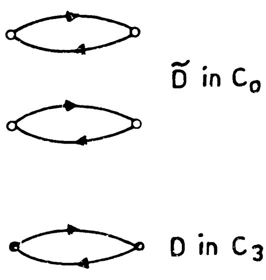

Fig. 8

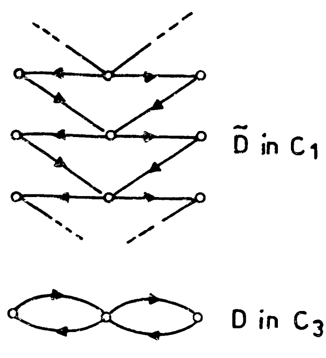

Fig. 9

Proof. Choose an arbitrary $x_0' \in p^{-1}x_0$. Let e_1' be the unique arc in $p^{-1}e_1$ with $\tilde{f}e_1' = x_0'$ and let $x_1' = \tilde{s}e_1'$; elet then e_2' be the unique arc in $p^{-1}e_2$ with $\tilde{f}e_2' = x_1'$ and let $x_2' = \tilde{s}e_2'$. Proceed in this way until all arcs in the walk W have been used. Then a vertex $x_0'' \in p^{-1}x_0$ has been reached. If $x_0''=x_0'$ then take as W' the obtained walk in $\tilde{D}$. Otherwise start again from x_0'' as above from x_0'. Since the multiplicity is finite after a finite number of repetitions, one must reach x_0' and W' is the resulting walk. □

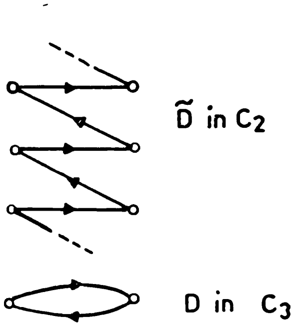

Fig. 10

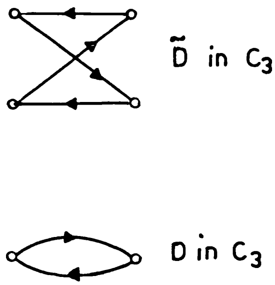

Fig. 11

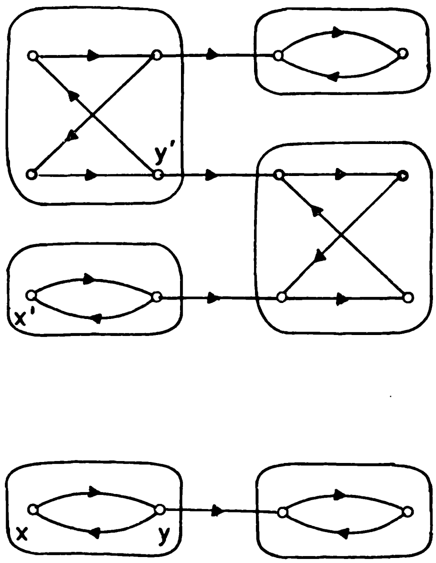

Fig. 12

Remark. Since a strong digraph contains a closed walk which runs through all vertices, Lemma 2 immediately implies that: If S is a strong component of D and $\tilde{S} \subset p^{-1}S$ is a strong component of $\tilde{D}$, then $p\tilde{S}=S$.

Lemma 3. *Let* $p\colon \tilde{D} \to D$ *be a covering projection of finite multiplicity. If* S *is a strong component of* D *and if* $\tilde{S}_1, \tilde{S}_2$ *are two different strong components of* $p^{-1}S$, *then there is no arc in* $\tilde{D}$ *connecting a vertex of* $\tilde{S}_1$ *with a vertex of* $\tilde{S}_2$.

Proof. Let $e' \in p^{-1}e$, e an arc of S, be an arbitrary arc in $p^{-1}S$. Then there is a closed walk W in S containing e. From Lemma 2 follows the existence of a closed walk W' in $\tilde{D}$ containing e' and therefore e' belongs to a strong component of $p^{-1}S$. □

Theorem 2. *Let $\tilde{D}$ be a cover of D of finite multiplicity. If $D \in C_3$ then $\tilde{D} \in C_0$ or $\tilde{D} \in C_3$.*

Proof. Let $D \in C_3$ and $\tilde{D} \notin C^3$. Then $\tilde{D}$ contains at least two strong components. By Lemma 3 there are no arcs connecting vertices in different strong components of $\tilde{D}$ and therefore $\tilde{D} \in C_0$. □

If one does not consider the (more or less trivial) case $\tilde{D} \in C_0$ then for $D \in C_2$ $\tilde{D}$ may be in C_2 or in C_1 whereas for $i=1, 3$ $D \in C_i$ iff $\tilde{D} \in C_i$. The next theorem decides which of the two possible cases for finite $D \in C_2$ occurs.

Definition 4. A strong component S of a weak digraph D in $C_1 \cup C_2$ is called a *sending component* if there is no arc e in D with $fe \notin S$, $se \in S$. Similarly a strong component is called a *receiving component* if there is no arc e in D with $fe \in S$, $se \notin S$.

Remark. Every finite digraph in $C_1 \cup C_2$ contains sending and receiving components; we have $D \in C_2$ iff these components are uniquely determined. For a proof see [4].

Theorem 3. *Let $D \in C_2$ be finite and $\tilde{D} \notin C_0$ be a cover of D of finite multiplicity. Denote by S the unique sending component of D and by R the unique receiving component of D. If $p^{-1}S \in C_0$ or $p^{-1}R \in C_0$ then $\tilde{D} \in C_1$. Otherwise $\tilde{D} \in C_2$.*

Proof. Without loss of generality let $p^{-1}S \in C_0$. Then $p^{-1}S$ contains two different strong components $\tilde{S}_1$, $\tilde{S}_2$, such that no arc of $\tilde{D}$ connects a vertex of $\tilde{S}_1$ with a vertex of $\tilde{S}_2$. But $\tilde{S}_1$ and $\tilde{S}_2$ are sending components of $\tilde{D}$, because S is a sending component of D. Therefore there is no walk connecting a vertex of $\tilde{S}_1$ with a vertex of $\tilde{S}_2$, i.e., $\tilde{D} \in C_1$. For the converse assume $p^{-1}S \notin C_0$, so that $p^{-1}S$ must be strong because of Theorem 2. Lemma 1 implies that $p^{-1}S$ is a strong component of $\tilde{D}$. Furthermore $p^{-1}S$ is the unique sending component of $\tilde{D}$. Similarly follows that $p^{-1}R$ is the unique receiving component of $\tilde{D}$. From this we now have $\tilde{D} \in C_2$. □

Unsolved Problem. Simple examples show that all possible combinations of $D \in C_i$ and $\tilde{D} \in C_j$, $0 \leq i, j \leq 3$, $i \geq j$, can occur for infinite multiplicity. It appears rather difficult to find conditions characterizing each of the possible cases.

3. Operations on digraphs

In this section we consider several operations on digraphs which transform covers into covers.

Definition 5. Let $D=(V, E, f, s)$ be a digraph. Then the *line digraph* $L(D)=(V_L, E_L, f_L, s_L)$ is defined as follows: $V_L=E$, $E_L=\{(e_1, e_2)|e_1, e_2 \in E, se_1=fe_2\}$ and $f_L(e_1, e_2)=e_1$, $s_L(e_1, e_2)=e_2$ for all $(e_1, e_2)\in E_L$.

Theorem 4. *If p: $\tilde{D}\to D$ is a covering projection, then there exists a covering projection p_L: $\tilde{D}_L\to D_L$.*

Proof. Let $D=(V, E, f, s)$ and $\tilde{D}=(\tilde{V}, \tilde{E}, \tilde{f}, \tilde{s})$ such that $\tilde{D}_L=(\tilde{V}_L, \tilde{E}_L, \tilde{f}_L, \tilde{s}_L)$. We define $p_L: \tilde{D}_L\to D_L$ by: $p_L|\tilde{V}_L := p|\tilde{E}$, $p_L(e_1', e_2') := (pe_1', pe_2')$ for $(e_1', e_2')\in \tilde{E}_L$. First we show that p_L is a homomorphism: $p_L\tilde{f}_L(e_1', e_2') = p_Le_1' = pe_1' = f_L(pe_1', pe_2') = f_Lp_L(e_1', e_2')$ for all $(e_1', e_2')\in \tilde{E}_L$. Similarly for the second vertices. Next we show that Property (C) holds. Let $e\in V_L=E$ be arbitrary and $e'\in p_L^{-1}e$. Assume there exist different elements (e', e_1') and (e', e_2') in $\tilde{E}_L$ with $p_L(e', e_1')=p_L(e', e_2')$. From this we have $(pe', pe_1') = (pe', pe_2')$ and therefore $pe_1'=pe_2'$. Since $\tilde{f}e_1'=\tilde{f}e_2'$ this contradicts p being a covering projection. By this we have shown that p_L is injective on the set $\{(e', e_1')|(e', e_1')\in \tilde{E}_L\}$; similarly p_L is injective on the set $\{(e_1', e')|(e_1', e')|(e_1', e')\in \tilde{E}_L\}$. It remains to show that p_L restricted to these sets is onto $\{(e, e_1)|(e, e_1)\in E_L\}$ and onto $\{(e_1, e)|(e_1, e)\in E_L\}$, resp. This follows from the fact that p restricted to $\{e_1'|\tilde{f}e_1'=\tilde{s}e'\}$ and to $\{e_1'|\tilde{s}e_1'=\tilde{f}e'\}$, respectively, is onto $\{e_1|fe_1=se\}$ and onto $\{e_1|se_1=fe\}$, respectively. □

Remark. From the definition of p_L follows that the covers $\tilde{D}$ of D and $\tilde{D}_L$ of D_L have the same multiplicity.

Definition 6. Let $D_i=(V_i, E_i, f_i, s_i)$, $i=1, 2$, be two digraphs. Then the *direct product* $D_1\otimes D_2$ is the digraph $(V_1\times V_2, E_1\times E_2, f_1\otimes f_2, s_1\otimes s_2)$ with

$$(f_1\otimes f_2)(e_1, e_2)=(f_1e_1, f_2e_2),\ (s_1\otimes s_2)(e_1, e_2)=(s_1e_1, s_2e_2)$$

This operation is also called *Kronecker product* or *conjunction.*

Example. See Fig. 13.

Theorem 5. *Let p_i: $\tilde{D}_i\to D_i$ be a covering projection for $i=1, 2$. Then there exists a covering projection $p_1\otimes p_2$: $\tilde{D}_1\otimes\tilde{D}_2\to D_1\otimes D_2$.*

Proof. Let $\tilde{D}_i=(\tilde{V}_i, \tilde{E}_i, \tilde{f}_i, \tilde{s}_i)$, $i=1, 2$. Define $p_1\otimes p_2$: $\tilde{D}_1\otimes\tilde{D}_2\to D_1\otimes D_2$ componentwise, i.e.,

$$(p_1\otimes p_2)(x_1', x_2') := (p_1x_1', p_2x_2') \quad \text{for} \quad (x_1', x_2')\in \tilde{V}_1\times\tilde{V}_2$$

and

$$(p_1\otimes p_2)(e_1', e_2') := (p_{1}{}_1', p_2e_2') \quad \text{for} \quad (e_1', e_2')\in \tilde{E}_1\times\tilde{E}_2.$$

The following calculation shows that $p_1\otimes p_2$ is a homomorphism:

$$\begin{aligned}(p_1\otimes p_2)((\tilde{f}_1\otimes\tilde{f}_2)(e_1', e_2'))&=(p_1\otimes p_2)(\tilde{f}_1e_1', \tilde{f}_2e_2')=\\ =(p_1(\tilde{f}_1e_1'), p_2(\tilde{f}_2e_2'))&=(f_1(p_1e_1'), f_2(p_2e_2'))=(f_1\otimes f_2)(p_1e_1', p_2e_2')=\\ &=(f_1\otimes f_2)((p_1\otimes p_2)(e_1', e_2')) \quad \text{for} \quad (e_1', e_2')\in\tilde{E}_1\times\tilde{E}_2.\end{aligned}$$

Similarly one shows that for

$$(e_1', e_2') \in \tilde{E}_1 \times \tilde{E}_2,$$

$$(p_1 \otimes p_2)((\tilde{s}_1 \otimes \tilde{s}_2)(e_1', e_2')) = (s_1 \otimes s_2)((p_1 \otimes p_2)(e_1', e_2')).$$

Immediately from the definition we see that $p_1 \otimes p_2$ is onto. Finally we show Property (C) for $p_1 \otimes p_2$. There is a bijection between the set A of arcs in $D_1 \otimes D_2$ with first (second) vertex (x_1, x_2) and the cartesian product of the set B of arcs in D_1 with first (second) vertex x_1 and the set C of arcs in D_2 with first (second) vertex

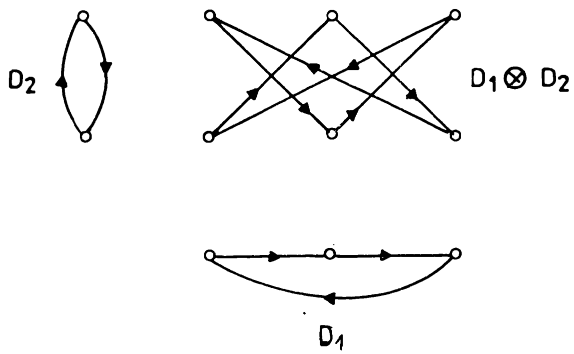

Fig. 13

x_2. An analogous statement holds for $\tilde{D}_1 \otimes \tilde{D}_2$ and a vertex (x_1', x_2') with sets $\tilde{A}$, $\tilde{B}$, $\tilde{C}$. If now $p_i x_i' = x_i$, $i = 1, 2$, then p_1 is a bijection from $\tilde{B}$ onto B and p_2 is a bijection from $\tilde{C}$ onto C and so $p_1 \otimes p_2$ a bijection from $\tilde{A}$ onto A. This is Property (C) for $p_1 \otimes p_2$. □

Definition 7. Let $D_i = (V_i, E_i, f_i, s_i)$, $i = 1, 2$, be two digraphs, where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$. The *cartesian product* $D_1 \times D_2$ is the digraph $(V_1 \times V_2, V_1 \times E_2 \cup E_1 \times V_2, f_1 \times f_2, s_1 \times s_2)$ with

$$\begin{aligned}
(f_1 \times f_2)(x_1, e_2) &= (x_1, f_2 e_2) \quad \text{for} \quad (x_1, e_2) \in V_1 \times E_2 \\
(f_1 \times f_2)(e_1, x_2) &= (f_1 e_1, x_2) \quad \text{for} \quad (e_1, x_2) \in E_1 \times V_2 \\
(s_1 \times s_2)(x_1, e_2) &= (x_1, s_2 e_2) \quad \text{for} \quad (x_1, e_2) \in V_1 \times E_2 \\
(s_1 \times s_2)(e_1, x_2) &= (s_1 e_1, x_2) \quad \text{for} \quad (e_1, x_2) \in E_1 \times V_2
\end{aligned}$$

Example. See Fig. 14.

Theorem 6. *Let p_i: $\tilde{D}_i \to D_i$ be a covering projection for $i = 1, 2$. Then there exists a covering projection $p_1 \times p_2$: $\tilde{D}_1 \times \tilde{D}_2 \to D_1 \times D_2$.*

Proof. Let $D_i = (V_i, E_i, f_i, s_i)$, $i = 1, 2$. Define $p_1 \times p_2$ as follows:

$$\begin{aligned}
(p_1 \times p_2)(x_1', x_2') &:= (p_1 x_1', p_2 x_2') \quad \text{for} \quad (x_1', x_2') \in \tilde{V}_1 \times \tilde{V}_2 \\
(p_1 \times p_2)(x_1', e_2') &:= (p_1 x_1', p_2 e_2') \quad \text{for} \quad (x_1', e_2') \in \tilde{V}_1 \times \tilde{E}_2 \\
(p_1 \times p_2)(e_1', x_2') &:= (p_1 e_1', p_2 x_2') \quad \text{for} \quad (e_1', x_2') \in \tilde{E}_1 \times \tilde{V}_2.
\end{aligned}$$

First we show that $p_1 \times p_2$ is a homomorphism:

$$(p_1 \times p_2)((\tilde{f}_1 \times \tilde{f}_2)(x_1', e_2')) = (p_1 \times p_2)(x_1', \tilde{f}_2 e_2') =$$
$$= (p_1 x_1', p_2(\tilde{f}_2 e_2')) = (p_1 x_1', f_2(p_2 e_2')) = (f_1 \times f_2)(p_1 x_1', p_2 e_2') =$$
$$= (f_1 \times f_2)((p_1 \times p_2)(x_1', e_2')) \quad \text{for} \quad (x_1', e_2') \in \tilde{V}_1 \times \tilde{E}_2.$$

Analogously one proceeds for $(e_1', x_2') \in \tilde{E}_1 \times \tilde{V}_2$ and for $\tilde{s}_1$, $\tilde{s}_2$ instead of $\tilde{f}_1$, $\tilde{f}_2$. Surjectivity of $p_1 \times p_2$ follows from its definition. There is a bijection between the

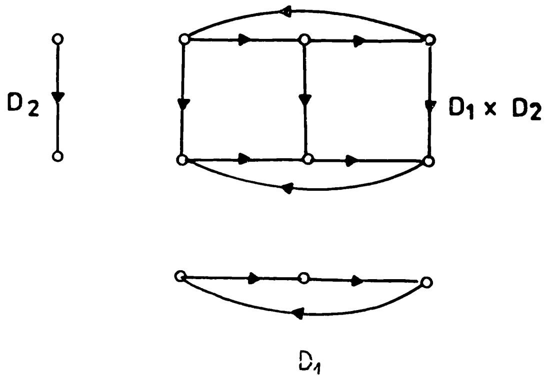

Fig. 14

set A of arcs in $D_1 \times D_2$ with first (second) vertex (x_1, x_2) and the union of the set B of arcs in D_1 with first (second) vertex x_1 and the set C of acrs in D_2 with first (second) vertex x_2. An analogous statement holds for $\tilde{D}_1 \times \tilde{D}_2$ and a vertex (x_1', x_2') with sets $\tilde{A}$, $\tilde{B}$, $\tilde{C}$. If now $p_i x_i' = x_i$, $i = 1, 2$, then p_1 is a bijection from $\tilde{B}$ onto B and p_2 is a bijection from $\tilde{C}$ onto C and so $p_1 \times p_2$ is a bijection from $\tilde{A}$ onto A. But this is Property (C) for $p_1 \times p_2$. □

Remark. If in Theorems 5 and 6 the cover $\tilde{D}_i$ of D_i has finite multiplicity n_i, $i = 1, 2$, then the constructed covers $\tilde{D}_1 \otimes \tilde{D}_2$ of $D_1 \otimes D_2$ and $\tilde{D}_1 \times \tilde{D}_2$ of $D_1 \times D_2$ have multiplicity $n_1 n_2$. This follows immediately from the definition of the respective covering projections.

4. Universal cover

In algebraic topology (compare [5]) it is well known that under certain conditions for a topological space X a universal cover U exists which has the property that every cover of X is a continuous image of U. In this section we show a similar theorem for digraphs. For this we need some definitions.

Definition 8. Let p_i: $\tilde{D}_i \to D$, $i = 1, 2$, be two covering projections onto D. A homomorphism φ: $\tilde{D}_1 \to \tilde{D}_2$ is a *cover-homomorphism* if $p_2 \circ \varphi = p_1$. A *cover-izomorphism* is a bijective cover-homomorphism.

Lemma 4. *Let p: $\tilde{D} \to D$ be a covering projection and $x_0 \in V$, $x_0' \in p^{-1}x_0$. For a semiwalk W in D with first vertex x_0 there exists a unique semiwalk W' in $\tilde{D}$ with first vertex x_0' and $pW' = W$.*

Proof. The construction of W' proceeds along similar lines as used in the proof of Lemma 2. The uniqueness of W' follows from Property (C).

Definition 9. A semiwalk W: $x_0, e_1, x_1, e_2, x_3, \ldots, e_n, x_n$ is called to be of *normal form*, if $x_0 \neq x_1$, $x_1 \neq x_2$, ..., $x_{n-1} \neq x_n$, $e_1 \neq e_2$, $e_2 \neq e_3$, ..., $e_{n-1} \neq e_n$.

Remark. In a walk of normal form, by the last condition loops are not allowed to occur. Thus in connection with covers, no restriction results since by Property (C) arcs in $p^{-1}e$ are loops if e is a loop where p is a covering projection. So there would even be no loss of generality in restricting ourselves to digraphs without loops. Finally a digraph is weak iff every two vertices are connected by a walk of normal form.

Let $D = (V, E, f, s)$ be a weak digraph. We define a digraph $U = U(D) = (V_U, E_U, f_U, s_U)$ in the following way. We choose a fixed vertex $x_0 \in V$ and take V_U as the set of all semiwalks of normal form in D with first vertex x_0. E_U consists of all ordered pairs (W_1, W_2), W_1, $W_2 \in V_U$, where W_1 results from W_2 by deleting the last vertex and last arc from W_2. Finally we define for $(W_1, W_2) \in E_U$:

$f_U(W_1, W_2) := W_1$ and $s_U(W_1, W_2) := W_2$ if the first vertex of the last arc of W_2 is the last vertex of W_1,

$f_U(W_1, W_2) := W_2$ and $s_U(W_1, W_2) := W_1$ if the first vertex of the last arc of W_2 isthe last vertex of W_2.

Remark. A simple consideration using the weak connectedness of D shows that also U is weakly connected. Further it can be seen immediately that U does not contain a closed semiwalk of normal form, i.e., U is in some sense simply connected (without nontrivial circuits).

Theorem 7. *Let D be a weak digraph and U as defined above. The digraph U is a universal cover of D in the sense that every weakly connected cover $\tilde{D}$ of D is the image of U under a cover-homomorphism φ.*

Proof. Let p: $\tilde{D} \to D$ be a covering projection including the possibility $\tilde{D} = D$. Let $x_0 \in V$ be the fixed vertex used in the definition of U and choose a fixed vertex $x_0' \in p^{-1}x_0$. The map φ is defined on V_U as follows. Let $W \in V_U$ be a semiwalk of normal form in D with first vertex x_0 and denote by W' the unique semiwalk in $\tilde{D}$ with first vertex x_0' and $pW' = W$ (see Lemma 4). Clearly W' is also of normal form. Then φW is defined as the last vertex of W' On E_U the map φ is defined in the following way. Let $(W_1, W_2) \in E_U$ and denote by $e \in E$ the last arc of W_2. If $f_U(W_1, W_2) = W_1$ then define $\varphi(W_1, W_2)$ as the unique arc $e' \in p^{-1}e$ with first vertex φW_1;if $f_U(W_1, W_2) = W_2$ then define $\varphi(W_1, W_2)$ as the unique arc $e' \in p^{-1}e$ with first vertex φW_2. So by this definition we have $\varphi(f_U(W_1, W_2)) = \tilde{f}(\varphi(W_1, W_2))$ for all $(W_1, W_2) \in E_U$ and it is immediate that also holds $\varphi(s_U(W_1, W_2)) = \tilde{s}(\varphi(W_1, W_2))$ such that φ is a homomorphism. Surjectivity of φ follows from the weak connectedness of $\tilde{D}$. Now we show Property (C) to hold for φ. Let $W \in V_U$ and $(W, W_i) \in E_U$, $i = 1, 2$, be arcs with $f_U(W, W_i) = W$. Denote by $e_i \in E$ the last arc of W_i. Assume $\varphi(W, W_1) = \varphi(W, W_2)$. From this follows $e_1 = e_2$ and since W_1, W_2 are of normal form this further implies $W_1 = W_2$. The analogous consideration with s_U instead of f_U shows the injectivity of φ on the respective sets of arcs. Surjectivity is clear, so Property (C) holds for φ. We have shown that φ: $U \to \tilde{D}$ is a covering projection. Taking $\tilde{D} = D$ and as p the identity on D we obtain that U is a cover of D ; denote the corresponding covering projection by p_U. With respect to this covering projection the homomorphism φ: $U \to \tilde{D}$ is a cover homomorphism since $p \circ \varphi = p_U$. This accomplishes the proof of the theorem.

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ПОКРЫШКИ ГРАФОВ

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Резюме

Понятие покрышки можно определить для орграфов так же, как и для графов и гиперграфов. Первая группа результатов рассматривает соотношения между классами связности орграфа и его покрывающих орграфов. Показывается, что некоторые унарные и бинарные операции, определенные на классе всех орграфов, преобразовывают покрышки в покрышки. Наконец, для слабого орграфа D доказано существование универсальной покрышки U, т.е. такой покрышки U орграфа D, что всякая слабо связная покрышка $\tilde{D}$ орграфа D является покрышково-гомоморфным образом орграфа U.