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NONLINEAR PARABOLIC EQUATIONS WITH THE MIXED NONLINEAR AND NONSTATIONARY BOUNDARY CONDITIONS

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This paper deals with the initial boundary value problem for the nonlinear parabolic equation

$$\frac{\partial u}{\partial t}+Au+b_0(x,u)=f(x,t),\quad x\in\Omega,\quad t\in(0,T) \tag{1}$$

$(T<\infty)$, where A is a nonlinear elliptic operator (see Definition 1) generated by

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,u,\frac{\partial u}{\partial x}\right),$$

$\Omega\subset E^N$ is a bounded domain with Lipschitzian boundary $\partial\Omega$, $x\equiv(x_1,\ldots,x_N)$ and

$$\frac{\partial u}{\partial x}\equiv\left(\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_N}\right).$$

We consider nonlinear boundary conditions of the form

$$\begin{aligned}\frac{\partial u}{\partial t}&=-\frac{\partial u}{\partial \nu}-b_1(x,u)\quad\text{for}\quad x\in\Gamma_1,\quad t\in(0,T)\\ 0&=-\frac{\partial u}{\partial \nu}-b_2(x,u)\quad\text{for}\quad x\in\Gamma_2,\quad t\in(0,T),\end{aligned}\tag{2}$$

where $\frac{\partial u}{\partial \nu}$ is defined by

$$\frac{\partial u}{\partial \nu}=\sum_{i=1}^{N}a_i\left(x,u,\frac{\partial u}{\partial x}\right)\cos(\mu,x_i)\quad\text{for}\quad x\in\partial\Omega$$

(μ is the outward normal vector with respect to $\partial\Omega$) and Γ_1, Γ_2 are two open subsets of $\partial\Omega$ with the properties $\Gamma_1\cup\Gamma_2\cup\Lambda=\partial\Omega$, $\Gamma_1\cap\Gamma_2=\emptyset$ and $\text{mes}_{N-1}\Lambda=0$. All results hold true also in the cases $\Gamma_1=\emptyset$ or $\Gamma_2=\emptyset$.

The initial condition is of the form

$$u(x,0)=\varphi(x)\quad\text{for}\quad x\in\Omega, \tag{3}$$

where $\varphi(x)$ is sufficiently smooth (see (11)).

In § 1 the existence and uniqueness of a generalized solution (see Definition 2) is proved under monotonicity assumptions on A and $b_j(x, s)$ $(j=0, 1, 2)$. An arbitrary polynomial growth of $a_i(x, \xi)$ in $\xi \in E^{N+1}$ and $b_j(x, s)$ in $s \in E^1$ is considered. In § 2 we investigate (1)—(3) under different assumptions on A and b_j. We assume that A is a linear second order elliptic operator and b_j are of the form

$$b_0\left(t, x, u, \frac{\partial u}{\partial x}\right), \quad b_j(t, x, u) \quad (j=1, 2).$$

In this case we suppose that $b_j(t, x, \xi)$ $(j=0, 1, 2)$ are Lipschitz continuous in t and ξ. We prove the existence, uniqueness and regularity of the generalized solution which satisfies (1) for a.e. $(x, t) \in \Omega \times (0, T)$ in the classical sense. Moreover, we prove the convergence of an approximate solution $u_n(x, t)$ (see (16)) which is constructed by means of the solving of linear elliptic boundary value problems corresponding to (1), (2).

A similar boundary value problem was investigated by V. V. Barkovskij and V. L. Kulčickij in [1, 2] in the following special form: A is the Laplace operator,

$$b_j(t, x, u) = c_j(x, t)u + f_j(u, t) \quad (j=0, 1)$$

and $b_2(t, x, u) \equiv 0$, where $c_1, c_2 > 0$ and $f_j(u, t)$ $(j=0, 1)$ satisfy certain additional assumptions.

In this paper an elementary method is used based on Rothe's method developed in papers [4—8]. The results obtained can be generalized to nonlinear boundary value problems of the type (1)—(3) of higher order.

§ 1

Assumptions and definitions

For simplicity we assume that $a_i(x, \xi)$ $(i=1, ..., N)$ and $b_j(x, s)$ $(j=0, 1, 2)$ are continuous in all their variables. The growth of a_i, b_j in the variables $\xi \in E^{N+1}$, $s \in E^1$ is assumed in the form

$$|a_i(x, \xi)| \leq C(1+|\xi|^{p-1}) \quad (p>1), \quad i=1, ..., N \tag{4}$$

and

$$|b_j(x, s)| \leq C(1+|s|^{p_j-1}) \quad (p_j>1), \quad j=0, 1, 2. \tag{5}$$

In § 1 we assume that b_j are nondecreasing in s, i.e.,

$$\frac{\partial b_j(x, s)}{\partial s} > 0 \quad \text{for} \quad x \in \Gamma_j (j=1, 2), \quad x \in \Omega (j=0), \quad |s| < \infty. \tag{6}$$

Ellipticity and coerciveness of the operator A are guaranteed by the algebraic conditions

$$\sum_{i=1}^{N}(\xi_i-\eta_i)[a_i(x,\xi)-a_i(x,\eta)]\geq 0 \tag{7}$$

$$\sum_{i=1}^{N}\xi_i a_i(x,\xi)\geq C_1|\xi|^p-C_2 \tag{8}$$

for all $x\in\Omega$, $|\xi|<\infty$.

If $p_j>2$ (for certain j), then we assume

$$sb_j(x,s)\geq C_1|s|^{p_j}-C_2, \tag{9}$$

$f(x,t)$ is supposed to be Lipschitz continuous from $\langle 0,T\rangle$ into $L_2(\Omega)$, i.e.,

$$\|f(x,t)-f(x,t')\|\leq C|t-t'|. \tag{10}$$

Let us denote $r_j=\max(p_j,2)$ $(j=0,1)$ and $r_2=p_2$. We construct the space $V=W_p^1(\Omega)\cap L_{r_0}(\Omega)\cap L_{r_1}(\Gamma_1)\cap L_{r_2}(\Gamma_2)$ with the norm

$$\|\cdot\|_V=\|\cdot\|_W+\|\cdot\|_{r_0}+\|\cdot\|_{r_1}+\|\cdot\|_{r_2},$$

where $W_p^1\equiv W_p^1(\Omega)$ is the Sobolev space with the norm $\|\cdot\|_W$ and $\|\cdot\|_{r_0}$, $\|\cdot\|_{r_1}$, $\|\cdot\|_{r_2}$ are the norms of the spaces $L_{r_0}(\Omega)$, $L_{r_1}(\Gamma_1)$, $L_{r_2}(\Gamma_2)$, respectively.

Definition 1. *Let A be an operator (generally nonlinear) $A: W_p^1\to(W_p^1)^*$ ($(W_p^1)^*$ is the dual space to W_p^1) defined by the form*

$$[Au,v]=\int_\Omega\sum_{i=1}^{N}\frac{\partial v}{\partial x_i}a_i\left(x,u,\frac{\partial u}{\partial x}\right)\mathrm{d}x$$

for all $u,v\in W_p^1$.

Owing to (4) and (7) the operator A is a continuous, bounded and monotone operator.

We suppose φ from (3) to be an element of the space $V\cap L_{2p_0-2}(\Omega)\cap L_{2p_1-2}(\Gamma_1)\cap L_{2p_2-2}(\Gamma_2)$ with the properties

$$\frac{\partial\varphi}{\partial\nu}=-b_2(x,\varphi)\quad\text{(in the sense of }L_2(\Gamma_2)\text{)}; \tag{11a}$$

Green's theorem can be applied to the form $[A\varphi,v]$, i.e.,

$$[A\varphi,v]=\left(\frac{\partial\varphi}{\partial\nu},v\right)_{\partial\Omega}-(\mathcal{A}\varphi,v) \tag{11b}$$

holds for all $v\in V$, where

$$\mathcal{A}\varphi=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i\left(x,\varphi,\frac{\partial\varphi}{\partial x}\right)$$

and moreover

$$\mathcal{A}\varphi \in L_2(\Omega), \quad \frac{\partial \varphi}{\partial \nu} \in L_2(\Gamma_2). \tag{11c}$$

For simplicity we denote by $b_i(u)$ $(i=0, 1, 2)$ the nonlinear operators from $L_{p_i}(\Gamma_i)$ into $L_{q_i}(\Gamma_i)$ for $i=1, 2$ and from $L_{p_0}(\Omega)$ into $L_{q_0}(\Omega)$ for $i=0 (p_i^{-1}+q_i^{-1}=1)$, which are generated by the corresponding functions $b_i(x, u)$.

We denote $(u, v)=\int_\Omega uv \, \mathrm{d}x$, $(u, v)_{\Gamma_i}=\int_{\Gamma_i} uv \, \mathrm{d}s$ $(i=1, 2)$ and $(u, v)_{\partial\Omega} = (u, v)_{\Gamma_1} + (u, v)_{\Gamma_2}$. For simplicity we denote by $\|\cdot\|$, $\|\cdot\|_{\Gamma_1}$, $\|\cdot\|_{\Gamma_2}$ the norms in the spaces $L_2(\Omega)$, $L_2(\Gamma_1)$, and $L_2(\Gamma_2)$, respectively.

Let $u(t)$ be an abstract function from $\langle 0, T\rangle$ into V. The trace of $u(t) \in V$ (t is fixed) on $\partial\Omega$ is denoted by $u_B(t)$.

Definition 2. *Under the solution (weak) of (1)—(3) we mean an abstract function $u \in L_\infty(\langle 0, T\rangle, V)$ with properties*

1) $\dfrac{\mathrm{d}u}{\mathrm{d}t} \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$, $\dfrac{\mathrm{d}u_B}{\mathrm{d}t} \in L_\infty(\langle 0, T\rangle, L_2(\Gamma_1))$.
2) *The identity*

$$\begin{gathered}\left(\frac{\mathrm{d}u(t)}{\mathrm{d}t}, v\right) + [Au(t), v] + (b_0(u(t)), v) + \\ + \left(\frac{\mathrm{d}u_B(t)}{\mathrm{d}t}, v\right)_{\Gamma_1} + \sum_{i=1,2} (b_i(u_B(t)), v)_{\Gamma_i} = (f(t), v)\end{gathered} \tag{12}$$

holds for all $v \in V$ and a.e. $t \in (0, T)$.

Remark 1. Owing to Green's theorem we find out from (12) easily that $u(x, t) \equiv u(t)$ is a classical solution of (1)—(3) provided $u(x, t)$ is sufficiently smooth.

Let $\mathscr{E}(\Omega)$ be the set of all functions defined on Ω having derivatives of all orders extendable continuously on $\bar{\Omega}$. By $\mathscr{D}(\Omega)$ we denote a subset of all functions from $\mathscr{E}(\Omega)$ which have support in Ω. We denote the strong convergence (in the norm) by $\rightarrow$ and the weak one by $\rightharpoonup$. By C with or without indices we denote the positive constants.

The constant C can denote also different constants in the same discussion.

A priori estimates

By means of the form

$$(\mathcal{A}_h u, v) \equiv \frac{1}{h}(u, v) + [Au, v] + (b_0(u), v) +$$

$$+\frac{1}{h}(u_B, v_B)_{\Gamma_1}+\sum_{i=1,2}(b_i(u), v_B)_{\Gamma_i}$$

for all $u, v \in V$, $h=\frac{T}{n}$ (n is a positive integer) we define an operator $\mathcal{A}_h\colon V \to V^*$ (V^* is the dual space to V). From (4)—(9) we conclude that $\mathcal{A}_h$ is a bounded, continuous and monotone operator. Due to (8), (9) we find out easily that $\mathcal{A}_h$ is coercive, i.e.,

$$(\mathcal{A}_h u, u)(\|u\|_V)^{-1} \to \infty \quad \text{for} \quad \|u\|_V \to \infty.$$

Hence using the results on monotone operators (see [3]) we find out that there exists the unique solution $u_f \in V$ of the equation $\mathcal{A}_h u = f$, for each $f \in V$.

Successively for $j=1, \ldots, n$ we construct $u_j \in V$ (they exist because of the properties of $\mathcal{A}_h$), the solutions of the equations

$$\begin{gathered}\left(\frac{u-u_{j-1}}{h}, v\right)+[Au, v]+(b_0(u), v)+\\+\left(\frac{u_B-u_{B,j-1}}{h}, v_B\right)_{\Gamma_1}+\sum_{i=1,2}(b_j(u_B), v_B)_{\Gamma_i}=(f_j, v)\end{gathered} \tag{13}$$

for all $v \in V$, where $f_j=f(jh, x)$, $u_0 \equiv \varphi$ and $h=\frac{T}{n}$.

Lemma 1. *There exist $h_0>0$ and C so that the estimates*

$$\left\|\frac{u_i-u_{i-1}}{h}\right\| \leq C, \quad \left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|_{\Gamma_1} \leq C$$

hold for all $h \leq h_0$, $i=1, \ldots, n$.

Proof. Consider (13) with $u=u_j$ for $j=i$ and $j=i-1$. Subtracting these inequalities and putting $v=(u_i-u_{i-1})h^{-1}$ we obtain

$$\begin{gathered}\left\|\frac{u_i-u_{i-1}}{h}\right\|+\frac{1}{h}[Au_i-Au_{i-1}, u_i-u_{i-1}]+\\+\frac{1}{h}(b_0(u_i)-b_0(u_{i-1}), u_i-u_{i-1})+\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|^2_{\Gamma_1}+\\\sum_{j=1,2}\frac{1}{h}(b_j(u_{B,i})-b_j(u_{B,i-1}), u_{B,i}-u_{B,i-1})_{\Gamma_j}=\\=\left(\frac{u_{i-1}-u_{i-2}}{h}, \frac{u_i-u_{i-1}}{h}\right)+\left(\frac{u_{B,i-1}-u_{B,i-2}}{h}, \frac{u_{B,i}-u_{B,i-1}}{h}\right)_{\Gamma_1}+\\+\left(f_i-f_{i-1}, \frac{u_i-u_{i-1}}{h}\right).\end{gathered}$$

Hence, owing to (6), (7) and (10) we deduce

$$\left\|\frac{u_i-u_{i-1}}{h}\right\|^2(1-C_1h)+\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|^2_{\Gamma_1}\leqslant$$
$$\leqslant\left\|\frac{u_{i-1}-u_{i-2}}{h}\right\|^2+\left\|\frac{u_{B,i-1}-u_{B,i-2}}{h}\right\|^2_{\Gamma_1}+C_2h,$$

where $h<h_0=C_1^{-1}$. From this inequality we obtain successively

$$\left(\left\|\frac{u_i-u_{i-1}}{h}\right\|^2+\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|^2_{\Gamma_1}\right)(1-C_1h)^{i-1}\leqslant$$
$$\leqslant\left\|\frac{u_1-\varphi}{h}\right\|^2+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}+C \tag{14}$$

for all $i=1,\ldots,n$. From (13) for $j=1$, $u=u_1$, $v=(u_1-\varphi)h^{-1}$ we deduce

$$\left\|\frac{u_1-\varphi}{h}\right\|^2+\frac{1}{h}[Au_1-A\varphi,u_1-\varphi]+\frac{1}{h}(b_0(u_1)-b_0(\varphi),u_1-\varphi)+$$
$$+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}+\sum_{j=1,2}\frac{1}{h}(b_j(u_{B,1})-b_j(\varphi),u_{B,1}-\varphi)_{\Gamma_j}=$$
$$=\left(f_1,\frac{u_1-\varphi}{h}\right)-\left[A\varphi,\frac{u_1-\varphi}{h}\right]-\left(b_0(\varphi),\frac{u_1-\varphi}{h}\right)+\sum_{j=1,2}\left(b_j(\varphi),\frac{u_{B,1}-\varphi}{h}\right)_{\Gamma_j}.$$

Owing to the assumptions (11a), (11b) we have

$$\left[A\varphi,\frac{u_1-\varphi}{h}\right]=\left(\frac{\partial\varphi}{\partial\nu},\frac{u_{B,1}-\varphi}{h}\right)_{\Gamma_1}+\left(\frac{\partial\varphi}{\partial\nu},\frac{u_{B,1}-\varphi}{h}\right)_{\Gamma_2}+\left(\mathcal{A}\varphi,\frac{u_1-\varphi}{h}\right)$$

and

$$\left(\frac{\partial\varphi}{\partial\nu},\frac{u_{B,1}-\varphi}{h}\right)_{\Gamma_2}+\left(b_2(\varphi),\frac{u_{B,1}-\varphi}{h}\right)_{\Gamma_2}=0.$$

Then, due to (11c), (6) and (7) we obtain successively

$$\left\|\frac{u_1-\varphi}{h}\right\|^2(1-C_1\varepsilon)+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}(1-C_2\varepsilon)\leqslant$$
$$\leqslant\lambda(\varepsilon)[\|f_1\|^2+\|\mathcal{A}\varphi\|^2+\|b_0(\varphi)\|^2+\|b_1(\varphi)\|^2_{\Gamma_1}], \tag{15}$$

where $\varepsilon>0$, $\lambda(\varepsilon)\to\infty$ for $\varepsilon\to0$ $\left(\text{because of the inequality } ab\leqslant\frac{a^2}{2\varepsilon^2}+\frac{\varepsilon^2b^2}{2}\right)$. Let us choose $\varepsilon=\frac{1}{2(C_1+C_2)}$. Then from (15), (14) and the estimate $(1-C_1h)^{i-1}\geqslant\geqslant\exp(-C_1T)$ we obtain the required result.

Lemma 2. *There exist C and $n_0>0$ such that the estimate $\|u_i\|_V\leq C$ holds for all $n\geq n_0$ and $i=1, \ldots, n$.*

Proof. Owing to Lemma 1 we have the estimates

$$\|u_i\|\leq\|\varphi\|+C \quad \text{and} \quad \|u_{B,i}\|_{\Gamma_1}\leq\|\varphi\|\Gamma_1+C$$

for all $n, i=1, \ldots, n$. Hence, from (13) for $u=u_i$, $v=u_i$ and Lemma 1 we deduce

$$[Au_i, u_i]+(b_0(u_i), u_i)+\sum_{j=1,2}(b_j(u_{B,i}), u_{B,i})_{\Gamma_j}\leq C$$

for all $n, i=1, \ldots, n$. From this estimate and (8), (9) we obtain the required result.

Now, by means of u_i $(i=1, \ldots, n)$ we construct Rothe's function $u_n(t)$:

$$u_n(t)=u_{i-1}+(t-t_{i-1})h^{-1}(u_i-u_{i-1}) \tag{16}$$

for $(i-1)h\leq t\leq ih$, $i=1, \ldots, n$. Analogously we define the step functions $x_n(t)$: $\langle 0, T\rangle\to V$, $f_n(t)$: $\langle 0, T\rangle\to L_2(\Omega)$

$$x_n(t)=u_i, \quad f_n(t)=f_i \quad \text{for} \quad (i-1)h<t\leq ih, \tag{17}$$

$i=1, \ldots, n$ and $x_n(0)=\varphi(x)$, $f_n(0)=f(0)$.

As a consequence of Lemma 1 and Lemma 2 we have the a priori estimates

$$\|u_n(t)-x_n(t)\|\leq\frac{C}{n}, \quad \|u_{B,n}(t)-x_{B,n}(t)\|_{\Gamma_1}\leq\frac{C}{n}; \tag{18}$$

$$\|u_n(t)\|_V\leq C, \quad \|x_n(t)\|_V\leq C; \tag{19}$$

$$\|u_n(t)-u_n(t')\|\leq C|t-t|', \quad \|u_{B,n}(t)-u_{B,n}(t')\|_{\Gamma_1}\leq C|t-t'| \tag{20}$$

for all n and $t, t'\in\langle 0, T\rangle$.

Lemma 3. *There exists a $u\in L_\infty(\langle 0, T\rangle, V)$ such that*

i) *$u_n(t)\to u(t)$ in $L_2(\Omega)$, $u_{B,n}(t)\to u_B(t)$ in $L_2(\Gamma_1)$ for $n\to\infty$ uniformly for $t\in\langle 0, T\rangle$;*

ii) *The (strong) derivatives $\dfrac{du(t)}{dt}$, $\dfrac{du_B(t)}{dt}$ exist for a.e. $t\in(0, T)$ and*

$$\frac{du}{dt}\in L_\infty(\langle 0, T\rangle, L_2(\Omega)), \quad \frac{du_B}{dt}\in L_\infty(\langle 0, T\rangle, L_2(\Gamma_1)).$$

Proof. The identity (13) (for $u=u_j$) can be rewritten in the form

$$\begin{aligned}&\left(\frac{d^- u_n(\tau)}{d\tau}, v\right)+[Ax_n(\tau), v]+(b_0(x_n(\tau)), v)+\\&+\left(\frac{d^- u_{B,n}(\tau)}{d\tau}, v\right)_{\Gamma_1}+\sum_{j=1,2}(b_j(x_{B,n}(\tau)), v)_{\Gamma_j}=(f_n(\tau), v)\end{aligned} \tag{21}$$

for $\tau \in (0, T)$, where $\frac{d^-}{d\tau}$ is the left hand derivative. Subtracting (21) for $n=r$ and $n=s$ and putting $v = x_r(\tau) - x_s(\tau)$ we obtain

$$\left(\frac{d^-(u_r(\tau)-u_s(\tau))}{d\tau}, u_r(\tau)-u_s(\tau)\right) + [Ax_r(\tau) - Ax_s(\tau), x_r(\tau)-x_s(\tau)] +$$

$$+ (b_0(x_r(\tau)) - b_0(x_s(\tau)), x_r(\tau) - x_s(\tau)) + \left(\frac{d^-(u_{B,r}(\tau)-u_{B,s}(\tau))}{d\tau},\right.$$

$$\left. u_{B,r}(\tau) - u_{B,s}(\tau)\right)_{\Gamma_1} + \sum_{j=1,2} (b_j(x_{B,r}(\tau)) - b_j(x_{B,s}(\tau)), x_{B,r}(\tau) - x_{B,s}(\tau))_{\Gamma_j} =$$

$$= (f_r(\tau) - f_s(\tau), x_r(\tau) - x_s(\tau)) + \left(\frac{d^-(u_r(\tau)-u_s(\tau))}{d\tau}, x_r(\tau) - u_r(\tau) - (x_s(\tau) - u_s(\tau))\right) +$$

$$+ \left(\frac{d^-(u_{B,r}(\tau)-u_{B,s}(\tau))}{d\tau}, x_{B,r}(\tau) - u_{B,s}(\tau) - (x_{B,r}(\tau) - u_{B,s}(\tau))\right)_{\Gamma_1}.$$

Let us integrate this inequality on the interval $(0, t)$. Owing to (6), (7), (18) and Lemma 1 we deduce successively

$$\frac{1}{2}\|u_r(t) - u_s(t)\|^2 + \frac{1}{2}\|u_{B,r}(t) - u_{B,s}(t)\|^2_{\Gamma_1} \leq C\left(\frac{1}{r} + \frac{1}{s}\right). \tag{22}$$

Thus, there exists a $u \in C(\langle 0, T\rangle, L_2(\Omega))$ such that $u_n(t) \to u(t)$ in $L_2(\Omega)$ for $n \to \infty$ uniformly in $t \in (0, T)$. Due to the a priori estimates (20) we have

$$\|u(t) - u(t')\| \leq C|t - t'|. \tag{23}$$

Then, owing to (19) and the reflexivity of V we conclude $u \in L_\infty(\langle 0, T\rangle, V)$ and $u_n(t) \rightharpoonup u(t)$ in V. Hence, $u_{B,n}(t) \rightharpoonup u_B(t)$ in $L_q(\partial\Omega)$ where $q = \frac{1}{p} - \frac{p-1}{p}(N-1)$ because of the imbedding $W_p^1(\Omega) \to L_q(\partial\Omega)$. From this fact and (22) we obtain $u_{B,n}(t) \to u_B(t)$ in $L_2(\Gamma_1)$ uniformly in $t \in \langle 0, T\rangle$. Moreover, from (20) the estimate

$$\|u_B(t) - u_B(t')\|_{\Gamma_1} \leq C|t - t'| \quad \text{for all} \quad t, t' \in \langle 0, T\rangle. \tag{24}$$

Owing to (23) and (24) and the result of Y. Komura (see [10]) there exist $\frac{du}{dt} \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$ and $\frac{du_B}{dt} \in L_\infty(\langle 0, T\rangle, L_2(\Gamma_1))$ and the proof is complete.

Lemma 4. *Let $u(t)$ be as in Lemma 3. Then*
i) $Au \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$
ii) $Ax_n(t) \rightharpoonup Au(t)$ *in* $L_2(\Omega)$ *for all* $t \in (0, T)$.

Proof. From (21) and Lemma 1 we obtain

$$|[Ax_n(t), v-v']| \leqslant C\|v-v'\| \quad \text{for all } n \text{ and } \quad v, v' \in \mathcal{D}(\Omega). \tag{25}$$

Thus, $Ax_n(t) \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$ and we estimate

$$\|Ax_n(t)\| \leqslant C \quad \text{for all} \quad t \in (0, T). \tag{26}$$

Hence there exists a $g_t \in L_2(\Omega)$ and a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$ (t is fixed) such that $Ax_{n_k}(t) \rightharpoonup g_t$ in $L_2(\Omega)$ (also in V^*). From the estimate

$$|[Ax_{n_k}(t), x_{n_k}(t)] - [g_t, u(t)]| \leqslant$$
$$\leqslant |[Ax_{n_k}(t) - g_t, u(t)]| + |[Ax_{n_k}(t), x_{n_k}(t) - u(t)]|,$$

Lemma 3 and (25) we deduce

$$[Ax_{n_k}(t), x_{n_k}(t)] \to [g_t, u(t)].$$

Due to the monotonicity of A we have

$$[Av - Ax_{n_k}(t), v - x_{n_k}(t)] \geqslant 0 \quad \text{for all} \quad v \in V$$

and hence passing to the limit for $k \to \infty$ we obtain $[Av - g_t, v - u(t)] \geqslant 0$ for all $v \in V$. Thus, putting $v = u(t) + \lambda w$, where $\lambda > 0$, $w \in V$ for $\lambda \to 0$ we obtain $[Au(t) - g_z, v] \geqslant 0$ for all $v \in V$ and hence $Au(t) = g_t$. From this fact Assertion ii) follows. Assertion i) follows from (26), Assertion i) and the Pettis theorem.

Existence and convergence results

Theorem 1. *The function $u(t)$ from Lemma 3 is the unique solution (see Definition 2) of the problem (1)—(3). The estimate $\|u_n(t) - u(t)\|^2 \leqslant \frac{C}{n}$ holds for all n and $t \in \langle 0, T\rangle$.*

Proof. Let us integrate (21) over $\langle 0, t\rangle$. We have

$$\begin{aligned}(u_n(t), v) - (\varphi, v) + (u_{B,n}(t), v)_{\Gamma_1} - (\varphi, v)_{\Gamma_1} + \int_0^t \Big\{[Ax_n(\tau), v] +\\ + (b_0(x_n(\tau)), v) + \sum_{j=1,2} (b_j(x_n(\tau)), v)_{\Gamma_j} - (f_n(\tau), v)\Big\}\,d\tau = 0\end{aligned} \tag{27}$$

for all $v \in V$. Since $x_n(\tau) \rightharpoonup u(\tau)$ in V for $n \to \infty$ and the imbedding $W_p^1 \to L_q(\partial\Omega)$ is compact $\left(q < \frac{1}{p} - \frac{p-1}{p}(N-1)\right)$, we have $x_{B,n}(\tau) \to u_B(\tau)$ in $L_q(\partial\Omega)$. From (5) and (19) the estimate

$$\|b_j(x_{B,n}(\tau))\|_{L_{sj}} \leqslant C$$

holds for all n, $j=0, 1, 2$, where $s_j = \frac{p_j}{p_j - 1} > 1$. From these facts we conclude $b_j(x_{B,n}(\tau)) \to b_j(u_B(\tau))$ for $n \to \infty$ in $L_1(\Gamma_j)$ for $j=1, 2$ and in $L_1(\Omega)$ for $j=0$. Hence and from the last inequality it follows that

$$(b_j(x_{B,n}(\tau)), v)_{\Gamma_j} \to (b_j(u_B(\tau)), v)_{\Gamma_j} \quad (j=1, 2)$$

$(b_0(x_n(\tau)), v) \to (b_0(u(\tau)), v)$ for all $v \in V$ and $\tau \in (0, T)$. Due to Lemma 4 and (19) we have

$$|[Ax_n(\tau), v]| \leqslant C\|v\|, \quad |(b_j(x_{B,n}(\tau)), v)| \leqslant G\|v\|_V \quad j=1, 2,$$
$$|(b_0(x_n(\tau)), v)| \leqslant C\|v\|_V$$

for all $\tau \in (0, T)$, $v \in V$. Then, using Lebesque's theorem and passing to the limit $n \to \infty$ in (27) we obtain

$$\begin{gathered}(u(t), v) - (\varphi, v) + (u_B(t), v)_{\Gamma_1} - (\varphi, v)_{\Gamma_1} + \\ + \int_0^t \Big\{[Au(\tau), v] + (b_0(u(\tau)), v) + \sum_{j=1,2} (b_j(u_B(\tau), v)_{\Gamma_j} - (f(\tau), v)\Big\} \, d\tau = 0\end{gathered} \tag{28}$$

for all $t \in (0, T)$ and $v \in V$. Hence, we deduce $u(0) = \varphi$ in $L_2(\Omega)$ and $u_B(0) = \varphi$ in $L_2(\Gamma_1)$. Differentiating (28) with respect to t, owing to Lemma 3 and Lemma 4 we conclude that $u \in L_\infty(\langle 0, T\rangle, V)$ is a solution (see (12)) of (1)—(3). The uniqueness of the solution is a consequence of the monotonicity assumptions (6) and (7). Indeed, if $u_1, u_2 \in V$ are two solutions of (1)—(3), then the inequality ($u = u_1 - u_2$) $\left(\frac{du(t)}{dt}, u(t)\right) + \left(\frac{du_B(t)}{dt}, u_B(t)\right)_{\Gamma_1} \leqslant 0$ for a.e. $t \in (0, T)$ takes place because of (6), (7) and (12). If we integrate this inequality in $(0, t)$ we obtain

$$\|u(t)\|^2 + \|u_B(t)\|^2_{\Gamma_1} \leqslant 0,$$

since $u(0) = u_B(0) = 0$. The rest of the proof follows from (22).

Actually, the following regularity properties for $u(t)$ can be proved:

Lemma 5. *Let $u(t)$ be the solution of (1)—(3) and $u_n(t)$ be as in (16). Then*

i) *$Au(t)$ and the weak derivatives $\frac{du}{dt}$, $\frac{du_B}{dt}$ are defined for all $t \in (0, T)$ and are weakly continuous in t in the space $L_2(\Omega)$, $L_2(\Gamma_1)$, respectively.*

ii) *The estimate*

$$\left\|\frac{du(t)}{dt}\right\| + \left\|\frac{du_B(t)}{dt}\right\|_{\Gamma_1} \leqslant C$$

takes place for all $t \in (0, T)$.

iii) *The identity (12) holds for all $t \in (0, T)$.*

iv) $\frac{d^- u_n(t)}{dt} \rightharpoonup \frac{du}{dt}$ *in* $L_2(\Omega)$, $\frac{d^- u_{B,n}(t)}{dt} \rightharpoonup \frac{du_B(t)}{dt}$ *in* $L_2(\Gamma_1)$ *for all* $t \in (0, T)$ *if* $n \to \infty$.

Proof. From (19) and $x_n(t) \rightharpoonup u(t)$ in V we obtain

$$\|u(t)\|_V \leq C \quad \text{for all} \quad t \in (0, T). \tag{29}$$

Let $t_n \to t$ for $n \to \infty$, t_n, $t \in (0, T)$. Using the argument from Lemma 4 we prove the weak continuity of $Au(t)$ (instead of $x_n(t)$ we consider $u(t_n)$). From (23), (24) and (29) we deduce easily $(b_0(u(t_n)), v) \to (b_0(u(t)), v)$ and $(b_j(u_B(t_n)), v)_{\Gamma_j} \to (b_j(u_B(t)), v)_{\Gamma_j}$ for all $v \in V$ by the same arguments used in the proof of Theorem 1 (instead of $x_n(t)$ we consider $u(t_n)$). Thus, from the continuity of $[Au(t), v]$, $(b_0(u(t)), v)$ and $(b_j(u_B(t)), v)_{\Gamma_j}$ $(j = 1, 2)$ in t for all $v \in V$ we deduce

$$(u(t), v) + (u_B(t), v)_{\Gamma_1} \in C^1((0, T)) \tag{30}$$

for all $v \in V$ because of (28). On the other hand from (28) for $v \in \mathscr{D}(\Omega)$ we deduce $(u(t), v) \in C^1((0, T))$ and the estimate $\left|\frac{d}{dt}(u(t), v)\right| \leq C\|v\|$ holds for all $v \in \mathscr{D}(\Omega)$. Thus, $(u(t), v) \in C^1((0, T))$ for all $v \in V$ and hence $(u_B(t), v)_{\Gamma_1} \in C^1((0, T))$ for all $v \in V$ because of (30). From this fact the existence of the weak derivatives $\frac{du}{dt}$, $\frac{du_B}{dt}$ follows for all $t \in (0, T)$. Differentiating (28) with respect to t we find out that (12) holds for all $t \in (0, T)$ and thus Assertion iii) is proved. From (12) and (21) we conclude that

$$\left(\frac{d^- u_n(t)}{dt}, v\right) \to \left(\frac{du(t)}{dt}, v\right) \quad \text{for all} \quad v \in \mathscr{D}(\Omega).$$

Hence, owing to Lemma 1 we obtain that

$$\left\|\frac{du(t)}{dt}\right\| \leq C \quad \text{for all} \quad t \in (0, T).$$

From these facts and from (12), (21) and Lemma 1 we deduce similarly

$$\left\|\frac{du_B(t)}{dt}\right\| \leq C \quad \text{for all} \quad t \in (0, T) \quad \text{and} \quad \frac{d^- u_{B,n}(t)}{dt} \rightharpoonup \frac{du_B(t)}{dt}$$

for all $t \in (0, T)$. Thus, Assertions ii) and iv) are proved. From the continuity of $\left(\frac{du(t)}{dt}, v\right)$ and $\left(\frac{du_B(t)}{dt}, v\right)_{\Gamma_1}$ in $t \in (0, T)$ for all $v \in V$ and from the estimates in ii) the rest of Assertion i) follows.

If the operator A is strongly monotone, then we can prove more regularity properties of $u(t)$ and stronger convergence of $\{u_n(t)\}$ to $u(t)$.

We assume the algebraic condition for strong monotonicity in the form

$$\sum_{i=1}^{N}[a_i(x, \xi) - a_i(x, \eta)](\xi_i - \eta_i) \geq C|\xi - \eta|^p \tag{7a}$$

for all $\xi, \eta \in E^{N+1}$.

Theorem 2. *If (7a) holds instead of (7), then the estimates*
i) $\|x_n(t) - u(t)\|_W \leq Cn^{-(1/2p)}$
ii) $\|u_n(t) - u(t)\|_W \leq Cn^{-(1/2p)}$
iii) $\|u(t) - u(t')\|_W \leq C|t - t'|^{1/p}$
take place for all $n \geq n_0$ *and* $t, t' \in \langle 0, T\rangle$, *where* $u(t)$ *is the solution of (1)—(3) and* $u_n(t)$, $x_n(t)$ *are from (16) and (17), respectively.*

Proof. Subtracting (21) and (12) for $v = x_n(t) - u(t)$ we obtain

$$[Ax_n(t) - Au(t), x_n(t) - u(t)] \leq \left\|\frac{\mathrm{d}^-(u_n(t) - u(t))}{\mathrm{d}t}\right\| \|x_n(t) - u(t)\| +$$

$$+ \left\|\frac{\mathrm{d}^-(u_{B,n}(t) - u_B(t))}{\mathrm{d}t}\right\|_{\Gamma_1} \|x_{B,n}(t) - u_B(t)\|_{\Gamma_1} + \max_{0 \leq t \leq T} \|f(t)\| \, \|x_n(t) - u(t)\|$$

for all n, $t \in (0, T)$ because of Lemma 5 and (6). Owing to Lemma 1, (10), (18), Theorem 1 and (7a) we conclude

$$\|x_n(t) - u(t)\|_W^p \leq Cn^{-(1/2)}$$

and Assertion i) is proved. Due to (7a) we find out easily that the estimate

$$\frac{1}{h} \|u_i - u_{i-1}\|_W^p \leq C \quad \text{for all} \quad n, i = 1, \ldots, n$$

can be proved (see the proof of Lemma 1). Thus we have the estimate

$$\|x_n(t) - u_n(t)\|_W \leq Cn^{-(1/p)}.$$

From this and from Assertion i) Assertion ii) follows. Similarly from (11) and Lemma 5 we have

$$[Au(t) - Au(t'), u(t) - u(t')] \leq \left\|\frac{\mathrm{d}(u(t) - u(t'))}{\mathrm{d}t}\right\| \|u(t) - u(t')\| +$$

$$+ \left\|\frac{\mathrm{d}(u_B(t) - u_B(t'))}{\mathrm{d}t}\right\|_{\Gamma_1} \|u_B(t) - u_B(t')\|_{\Gamma_1} + L|t - t'| \, \|u(t) - u(t')\|$$

and hence using Lemma 5, (23) and (24) Assertion iii) follows. The construction of an approximate solution $u_n(t)$ of the problem (1)—(3) is interesting from the numerical point of view, too. However, in practice we can construct only an approximation $\tilde{u}_n(t)$ of $u_n(t)$ since only some approximations of the elements u_i

$(i=1, \ldots, n)$ can be obtained. Now, the problem of the convergence of $\tilde{u}_n(t)$ to $u(t)$ will be investigated.

Let $z \in V$ and let $u \equiv u[z]$ be the solution of the problem

$$\frac{u-z}{h} + Au + b_0(x, u) = f(x, t) \tag{1'}$$

$$\begin{aligned} u + h\frac{\partial u}{\partial \nu} &= z - hb_1(x, u) \quad \text{on} \quad \Gamma_1 \\ \frac{\partial u}{\partial \nu} &= -b_2(x, u) \quad \text{on} \quad \Gamma_2 \end{aligned} \tag{2'}$$

By $\tilde{u}[z]$ we denote an approximate solution of this problem. We construct u_i successively for $i=1, \ldots, n$ putting $z = \tilde{u}_{i-1}$ and $\tilde{u}_i = \tilde{u}[z]$, where $\tilde{u}_0 = \varphi$. By means of $\tilde{u}_i$ (instead of u_i) we construct $\tilde{u}_n(t)$ (see (16)). Let us denote

$$(\|u_z - \tilde{u}_z\|^2 + \|u_{B,z} - \tilde{u}_{B,z}\|^2_{\Gamma_1})^{1/2} = \varrho(u[z], \tilde{u}[z]).$$

Theorem 3. *Let $u(t)$ be as in Theorem 1. Then*

i) $(\|u_i - \tilde{u}_i\| + \|u_{B,i} - \tilde{u}_{B,i}\|_{\Gamma_1})^{1/2} \leq \sum_{k=0}^{i} \varrho(u[\tilde{u}_k], \tilde{u}[\tilde{u}_k])$

ii) *If $\varrho(u[\tilde{u}_i], \tilde{u}[\tilde{u}_i]) \leq \delta$ for all $i = 0, 1, \ldots, n-1$ and $\delta = O(n^{-(3/2)})$, then*

$$\|u_n(t) - u(t)\|_{C(\langle 0, T\rangle, L_2(\Omega))} = O(n^{-(1/2)})$$

iii) *If $\varrho(u[\tilde{u}_i], \tilde{u}[\tilde{u}_i]) \leq \delta$ for all $i = 0, 1, \ldots, n-1$, $\delta = O(n^{-(3/2)})$ and A is strongly monotone, then*

$$\|\tilde{u}_n(t) - u(t)\|_{C(\langle 0, T\rangle, W_p^1(\Omega))} = O(n^{-(1/2p)})$$

Proof. Using our notation we denote $\bar{u}_i = u[\tilde{u}_{i-1}]$, $i = 1, \ldots, n$ (the solution of (1'), (2') for $z = \tilde{u}_{i-1}$). Thus, the identity

$$\begin{aligned} &\left(\frac{\bar{u}_j - \tilde{u}_{j-1}}{h}, v\right) + [A\bar{u}_j, v] + (b_0(\bar{u}_j), v) + \\ &+ \left(\frac{\bar{u}_{B,j} - \tilde{u}_{B,j-1}}{h}, v\right)_{\Gamma_1} + \sum_{i=1,2} (b_i(\bar{u}_{B,j}), v)_{\Gamma_i} = (f_j, v) \end{aligned} \tag{13'}$$

holds for all $v \in V$. From (13) and (13') for $u = u_j$ and $v = u_j - \bar{u}_j$ we obtain

$$\begin{aligned} &\left(\frac{u_j - \bar{u}_j}{h}, u_j - \bar{u}_j\right) + [Au_j - A\bar{u}_j, u_j - \bar{u}_j] + (b_0(u_j) - b_0(\bar{u}_j), u_j - \bar{u}_j) + \\ &+ \left(\frac{u_{B,j} - \bar{u}_{B,j}}{h}, u_{B,j} - \bar{u}_{B,j}\right)_{\Gamma_1} + \sum_{i=1,2} (b_i(u_{B,j}) - b_i(\bar{u}_j), u_j - \bar{u}_j)_{\Gamma_i} = \end{aligned}$$

$$= \left(\frac{u_{j-1}-\tilde{u}_{j-1}}{h}\ ,\ u_j-\tilde{u}_j\right)+\left(\frac{u_{B,j-1}-\tilde{u}_{B,j-1}}{h}\ ,\ u_{B,j-1}-\tilde{u}_{B,j-1}\right)_{\Gamma_1};$$

hence

$$\|u_j-\tilde{u}_j\|^2+\|u_{B,j}-\tilde{u}_{B,j}\|^2_{\Gamma_1}\leqslant\|u_{j-1}-\tilde{u}_{j-1}\|^2+\|u_{B,j-1}-\tilde{u}_{B,j-1}\|^2_{\Gamma_1}$$

because of the monotonicity properties of A, b_0, b_i $(i=1,2)$. The last inequality and the triangular inequalities imply

$$(\|u_j-\tilde{u}_j\|^2+\|u_{B,j}-\tilde{u}_{B,j}\|^2_{\Gamma_1})^{1/2}\leqslant(\|\tilde{u}_j-u_j\|^2+\|\tilde{u}_{B,j}-u_{B,j}\|^2_{\Gamma_1})^{1\ 2}+$$
$$+\varrho(u[\tilde{u}_j],\tilde{u}[\tilde{u}_j])\leqslant(\|u_{j-1}-\tilde{u}_{j-1}\|^2+\|u_{B,j}-\tilde{u}_{B,j-1}\|^2_{\Gamma_1})^{1/2}+\varrho(u[\tilde{u}_j],\tilde{u}[\tilde{u}_j]).$$

From this reccurent inequality we deduce Assertion i).

Assertion ii) is a consequence of Assertion i), Theorem 1 and of the inequality

$$\|\tilde{u}_n(t)-u(t)\|\leqslant\|u_n(t)-u(t)\|+\|u_n(t)-\tilde{u}_n(t)\|$$

Assertion iii) is a consequence of Assertion ii) (for the details see the proof of Theorem 2).

§2.

In this section we consider the boundary value problem (1)—(3) under the following assumptions:

A is a linear elliptic operator of the form

$$Au=-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

where $a_{ij}\in C^{0,1}(\bar{\Omega})$ and

$$\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geqslant C_E|\xi|^2\quad\text{for all}\quad\xi\in E^N. \tag{31}$$

Instead of the operator $b_0(u)$ we consider the more general operator $b_0\left(t,u,\frac{\partial u}{\partial x}\right)$ which is generated by the function $b_0\left(t,x,u,\frac{\partial u}{\partial x}\right)$. Instead of the operators $b_j(u)$ $(j=1,2)$ we consider operators of the form $b_j(t,u)$ which are generated by the functions $b_j(t,x,u)$. We assume that b_0, b_1, b_2 are continuous in all their variables and moreover

$$|b_j(t,x,s)-b_j(t',x,s')|\leqslant C(|t-t'|+|s|\ |t-t'|+|s-s'|) \tag{32}$$

for all $t,t'\in(0,T)$, $x\in\Omega$ and $|s|,|s'|<\infty$ $(s,s'\in E^1$ for $j=1,2$ and $s,s'\in E^{N+1}$ for $j=0)$.

In this section we construct the Rothe function $u_n(t)$ (see (16)) by means of the elements $u_i(i=1,\ldots,n)$ which solve the following linear problems

$$\left(\frac{u-u_{i-1}}{h}, v\right)+[Au, v]+\left(b_0\left(t_i, u_{i-1}, \frac{\partial u_{i-1}}{\partial x}\right), v\right)+$$
$$+\left(\frac{u_B-u_{B,i-1}}{h}, v\right)_{\Gamma_1}+\sum_{j=1,2}(b_j(t_i, u_{B,i-1}), v)_{\Gamma_j}=(f_i, v) \tag{33}$$

for all $v \in V$, corresponding to the linear elliptic boundary value problems

$$\frac{u-u_{i-1}}{h}+Au+b_0\left(t_i, u_{i-1}, \frac{\partial u_{i-1}}{\partial x}\right)=f_i$$

$$u+h\frac{\partial u}{\partial v}=u_{i-1}-hb_1(t_i, u_{i-1}) \quad \text{on} \quad \Gamma_1$$

$$\frac{\partial u}{\partial v}=-b_2(t_i, u_{i-1}) \quad \text{on} \quad \Gamma_2$$

where

$$\frac{\partial u}{\partial v}=\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(\mu, x_i) \quad \text{and} \quad v \in W_2^1(\Omega).$$

Thus our scheme (33) is interesting from the numerical point of view. However, the existence and uniqueness of the solution $u(t)$ and the convergence of $u_n(t)$ to $u(t)$ will be proved under a certain additional assumption. We shall assume

$$\left|\frac{\partial b_2(t, x, s)}{\partial s}\right| \leqslant C_0 < \frac{C_E}{C_I^2} \quad \text{for all} \quad t \in \langle 0, T\rangle, x \in \Gamma_2, |s|<\infty, \tag{34}$$

where C_E is from (31) and C_I is the smallest constant in the imbedding inequality $\|v\|_{L_2(\partial\Omega)} \leqslant C_I\|v\|_W$. The conditions (11a) and (11b) are satisfied if we assume

$$\varphi \in W_2^2(\Omega) \quad \text{and} \quad \frac{\partial \varphi}{\partial v}=-b_2(0, x, \varphi) \quad \text{for} \quad x \in \Gamma_2. \tag{35}$$

In this section (4), (31), (32), (10), (34) and (35) will be assumed.

A priori estimates

Lemma 6. *There exist C_1, C_2 and $h_0>0$ such that the estimate*

$$\left\|\frac{u_i-u_{i-1}}{h}\right\|^2+\frac{1}{h}\|u_i-u_{i-1}\|_W^2+\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|_{\Gamma_1}^2 \leqslant$$
$$\leqslant C_1+C_2\sum_{j=1}^{i} h\|u_i\|_W^2$$

holds for all $h<h_0$, $i=1, \ldots, n$.

Proof. From (33) similarly as in § 1 we deduce

$$\left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + \left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_1}^2 + \frac{C_E}{h} \| u_i - u_{i-1}\|_W^2 \leq \tag{36}$$

$$\leq \frac{1}{2}\left\|\frac{u_{i-1} - u_{i-2}}{h}\right\|^2 + \frac{1}{2}\left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + \frac{1}{2}\left\|\frac{u_{B\ i-1} - u_{B\ i-2}}{h}\right\|_{\Gamma_1}^2 +$$

$$+\frac{1}{2}\left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_1}^2 + \left\|b_0\left(t_i, u_{i-1}, \frac{\partial u_{i-1}}{\partial x}\right) - b_0\left(t_{i-1}, u_{i-2}, \frac{\partial u_{i-2}}{\partial x}\right)\right\| \cdot$$

$$\cdot \left\|\frac{u_i - u_{i-1}}{h}\right\| + \sum_{j=1,2} \|b_j(t_i, u_{B,i-1}) - b_j(t_{i-1}, u_{B,i-2})\|_{\Gamma_j} \cdot$$

$$\cdot \left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_j} + \|f_i - f_{i-1}\| \left\|\frac{u_i - u_{i-1}}{h}\right\| + \frac{C_E}{h} \|u_i - u_{i-1}|^2.$$

By a suitable application of the inequality $ab \leq \frac{a^2\varepsilon^2}{2} + \frac{b^2}{2\varepsilon^2}$ ($\varepsilon > 0$) and due to (32) we obtain

$$\left\|b_0\left(t_i, u_{i-1}, \frac{\partial u_{i-1}}{\partial x}\right) - b_0\left(t_{i-1}, u_{i-2}, \frac{\partial u_{i-2}}{\partial x}\right)\right\| \left\|\frac{u_i - u_{i-2}}{h}\right\| \leq$$

$$\leq C_1 h + C_2 h \left\|\frac{u_i - u_{i-1}}{h}\right\|^2 + C_3 h \|u_{i-1}\|_W^2 + C_4 h \left\|\frac{u_{i-1} - u_{i-2}}{h}\right\|^2 + C_d \frac{1}{h} \|u_{i-1} - u_{i-2}\|_W^2,$$

where $C_d = \frac{1}{2}(C_E - C_I^2 C_0)$ (see (34)). Similarly we estimate

$$\|b_1(t_i, u_{B,i-1}) - b_1(t_{i-1}, u_{B,i-2})\|_{\Gamma_1} \left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_1} \leq$$

$$\leq C_1 h + C_2 h \left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_1}^2 + C_3 h \|u_{B,i-1}\|_{\Gamma_1}^2 + \frac{C_4}{h} \|u_{B,i-1} - u_{B,i-2}\|_{\Gamma_1}^2 \leq$$

$$\leq C_1 h + C_2 h \left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_1}^2 + Ch \|u_{i-1}\|_W^2 + \frac{C_d}{h} \|u_{i-1} - u_{i-2}\|_W^2$$

because of the imbedding $W_2^1(\Omega) \to L_2(\partial\Omega)$. Owing to (32), (34) and the imbedding $W_2^1(\Omega) \to L_2(\partial\Omega)$ we conclude

$$\|b_2(t_i, u_{B,i-1}) - b_2(t_{i-1}, u_{B,i-2}\|_{\Gamma_2} \left\|\frac{u_{B,i} - u_{B\ i-1}}{h}\right\|_{\Gamma_2} \leq$$

$$\leq C_1 h + C_2 h \|u_{B,i-1}\|_{\Gamma_2}^2 + C_0 \left\|\frac{u_{B,i} - u_{B,i-1}}{h}\right\|_{\Gamma_2} \|u_{B,i-1} - u_{B,i-2}\|_{\Gamma_2} \leq$$

$$\leq C_1 h + C_3 h \|u_{i-1}\|_W^2 + \frac{C_I^2 C_0}{2h} \|u_i - u_{i-1}\|_W^2 + \frac{C_I^2 C_0}{2h} \|u_{i-1} - u_{i-2}\|_W^2.$$

From the estimates obtained and from (36) we have

$$(1-C_1h)\left[\left\|\frac{u_i-u_{i-1}}{h}\right\|^2+\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|^2_{\Gamma_1}+\frac{C}{h}\|u_i-u_{i-1}\|^2_W\right]\leqslant$$
$$\leqslant(1-C_2h)\left[\left\|\frac{u_{i-1}-u_{i-2}}{h}\right\|^2+\left\|\frac{u_{B,i-1}-u_{B,i-2}}{h}\right\|^2_{\Gamma_1}+\right.$$
$$\left.+\frac{C}{h}\|u_{i-1}-u_{i-2}\|^2_W\right]+C_3h\|u_{i-1}\|^2_W+C_4h$$

where $C=C_E-\frac{C_I^2C_0}{2}>0$ and $h<h_0=\frac{1}{C_1+C_2}$. By a successive application of this recurrent inequality we obtain

$$(1-C_1h)^{i-1}\left[\left\|\frac{u_i-u_{i-1}}{h}\right\|^2+\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|^2_{\Gamma_1}+\frac{C}{h}\|u_i-u_{i-1}\|^2_W\right]\leqslant \tag{37}$$
$$\leqslant(1-C_2h)^{i-1}\left[\left\|\frac{u_1-\varphi}{h}\right\|^2+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}\right]+\frac{C}{h}\|u_1-\varphi\|^2_W+$$
$$+C_3\sum_{j=1}^{i-1}(1-C_1h)^{j-1}h\|u_j\|^2_W+C_4\sum_{j=1}^{i-1}(1-C_1h)^{j-1}.$$

Now, from (33) for $u=u_1$, $v=\frac{u_1-\varphi}{h}$ and from (35) we conclude (see the proof of Lemma 1)

$$\left\|\frac{u_1-\varphi}{h}\right\|^2+\frac{C_E}{h}\|u_1-\varphi\|^2_W+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}\leqslant$$
$$\leqslant\|A\varphi\|\left\|\frac{u_1-\varphi}{h}\right\|+\left\|b_0\left(t_1,\varphi,\frac{\partial\varphi}{\partial x}\right)\right\|\left\|\frac{u_1-\varphi}{h}\right\|+$$
$$+\|b_2(t_1,\varphi)-b_2(0,\varphi)\|_{\Gamma_2}\left\|\frac{u_{B,1}-\varphi}{h}\right\|_{\Gamma_2}+\|b_1(t_1,\varphi)\|_{\Gamma_1}\left\|\frac{u_{B,1}-\varphi}{h}\right\|_{\Gamma_1}$$

and hence owing to (34), (35) and (32) we have

$$\left\|\frac{u_1-\varphi}{h}\right\|^2+\frac{C_E}{h}\|u_1-\varphi\|^2_W+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}\leqslant$$
$$\leqslant C_1\|\varphi\|^2_{W_2^2}+C_3h\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}+C_4\|b_1(0,\varphi)\|^2+\frac{C_E}{2h}\|u_1-\varphi\|^2_W+C_5h.$$

From this estimate we obtain

$$\left\|\frac{u_1-\varphi}{h}\right\|^2+\frac{1}{h}\|u_1-\varphi\|^2_W+\left\|\frac{u_{B,1}-\varphi}{h}\right\|^2_{\Gamma_1}\leqslant C,$$

where C is independent on n, $i=1, \ldots, n$. Thus, due to (37) we obtain the required result since there exist K_1, $K_2>0$ such that the estimates

$$K_1<(1-C_1h)^i, \quad (1-C_2h)^i<K_2$$

hold for all $h<h_0$, $i=1, \ldots, n$.

Lemma 7. *There exist C_1, C_2, $n_0>0$ such that the estimates*

i) $|[Au_i, u_i]|\leqslant C_1+C_2\sum_{j=1}^{i} h\|u_j\|_W^2+\frac{C_E}{16}\|u_{i-1}\|_W^2$

ii) $|(b_2(t_i, u_{i-1}), u_i)_{\Gamma_2}|\leqslant C_1+C_2\sum_{j=1}^{i} h\|u_j\|_W^2+\frac{C_E}{8}\|u_{i-1}\|_W^2$

hold for all $n\geqslant n_0$, $i=1, \ldots, n$.

Proof. From (33) for $u=u_i$ and $v\in\mathscr{D}(\Omega)$ we obtain

$$|[Au_i, v]|\leqslant\left\|\frac{u_i-u_{i-1}}{h}\right\|\|v\|+(C_1+C_2\|u_{i-1}\|_W)\|v\| \tag{38}$$

and hence $\|Au_i\|\leqslant\left\|\frac{u_i-u_{i-1}}{h}\right\|+C_1+C_2\|u_{i-1}\|_W$. From Lemma 6 we have

$$\|u_i\|\leqslant C_1+C_2\left(\sum_{j=1}^{i} h\|u_i\|_W^2\right)^{1/2} \tag{39}$$

for all n, $i=1, \ldots, n$. From these estimates we obtain the estimate i). Similarly from (33) for $u=u_i$, $v=u_i$ we have

$$\begin{aligned}|(b_2(t_i, u_{i-1}), u_i)_{\Gamma_2}\leqslant&\left\|\frac{u_i-u_{i-1}}{h}\right\|\|u_i\|+\\ +&\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|_{\Gamma_1}\|u_{B,i}\|_{\Gamma_1}+|[Au_i, u_i]|+\\ +&(C_1+C_2\|u_{i-1}\|_W)\|u_i\|+(C_1+C_2\|u_{i-1}\|_{\Gamma_1})\|u_i\|_{\Gamma_1}+\|f_i\|\,\|u_i\|.\end{aligned} \tag{40}$$

From Lemma 6 we have

$$\|u_i\|_{\Gamma_1}\leqslant C_1+C_2\left(\sum_{j=1}^{i} h\|u_j\|_W^2\right)^{1/2}.$$

Applying (39), (41) and the estimate i) in (41) we obtain the estimate ii).

Lemma 8. *There exist C and $n_0>0$ such that the estimates*

i) $\left\|\frac{u_i-u_{i-1}}{h}\right\|\leqslant C$, $\left\|\frac{u_{B,i}-u_{B,i-1}}{h}\right\|_{\Gamma_1}\leqslant C$

ii) $\|Au_i\|\leqslant C$

iii) $\|u_i\|_V \leq C$

iv) $\frac{1}{h}\|u_i - u_{i-1}\|_W^2 \leq C$

are valid for all $n > n_0$, $i = 1, \ldots, n$.

Proof. From (33) (for $u = u_i$, $v = u_i$), Lemma 6, Lemma 7 and from (31) we conclude

$$C_E\|u_i\|_W^2 \leq C_1 + C_2 \sum_{j=1}^{i} h\|u_j\|_W^2 + \frac{C_E}{4}\|u_{i-1}\|_W^2.$$

Hence, using the estimate

$$\frac{C_E}{4}\|u_{i-1}\|_W^2 \leq \frac{C_E}{2}\|u_i\|_W^2 + \frac{C_E}{2}\|u_i - u_{i-1}\|_W^2$$

and Lemma 6 we obtain

$$\|u_i\|_W^2 \leq C_1 + C_2 \sum_{j=1}^{i} h\|u_j\|_W^2$$

for all $i = 1, \ldots, n$, $n > n_0$. From this estimate we obtain successively $\left(0 < h < h_0 = \frac{1}{C_2}\right)$

$$\|u_1\|_W^2 \leq \frac{C_1}{1 - C_2 h}, \ldots, \|u_i\|_W^2 \leq \frac{C_1}{1 - C_2 h}\left(1 + \frac{C_2 h}{1 - C_2 h}\right)^{i-1}.$$

But $\left(1 + \frac{C_2 h}{1 - C_2 h}\right)^{i-1} \leq C$ holds for all $i = 1, \ldots, n$, $n > n_0$, where C is a suitable constant. Thus the estimate ii) is proved. The estimates i), iii) and iv) are consequences of ii), Lemma 6 and Lemma 7.

Let Ω' be a subdomain of Ω such that $\bar{\Omega}' \subset \Omega$.

Lemma 9. *There exist* $C(\Omega')$, $n_0 > 0$ *such that* $\|u_i\|_{W_2^2(\Omega')} \leq C(\Omega')$ *for all* $n > n_0$, $i = 1, \ldots, n$.

Proof. The element $u_i \in V$ satisfies the identity

$$[Au, v] + \frac{1}{h}(u, v) = -\left(\frac{u_i - u_{i-1}}{h} + b_0\left(t_i, u_{i-1}, \frac{\partial u_{i-1}}{\partial x}\right) + \left(f_i, v\right)\right) \equiv (F_h^{(i)}, v),$$

i.e., u_i is the solution of the equation $Au + \frac{1}{h}u = F_h^{(i)}$ in the sense of distributions. The operator $A + \frac{1}{h}I$ (I is the identity operator) is W_2^1 elliptic (see [9]) because of (31). Thus, using the results on regularity in the interior of the domain Ω (see [9]) we obtain

$$\|u_i\|_{W_2^2(\Omega')} \leq C(\Omega')(\|u_i\|_W + \|f_h^{(i)}\|).$$

Hence, owing to Lemma 6 we obtain the required result.

By means of u_i $(i=1, \ldots, n)$ we define $u_n(t)$ and $x_n(t)$ by (16), (17) As a consequence of Lemma 8 we have the following a priori estimates

$$\left\|\frac{\mathrm{d}^- u_n(t)}{\mathrm{d}t}\right\| \leqslant C, \quad \left\|\frac{\mathrm{d}^- u_{B,n}(t)}{\mathrm{d}t}\right\|_{\Gamma_1} \leqslant C \tag{42}$$

$$\|u_n(t)\|_V \leqslant C, \quad \|x_n(t)\|_V \leqslant C \tag{43}$$

$$\|u_n(t)-x_n(t)\| \leqslant \frac{C}{n}, \quad \left\|x_n(t)-x_n\left(t-\frac{T}{n}\right)\right\| \leqslant \frac{C}{n} \tag{44}$$

$$\|x_n(t)\|_{W_2^2(\Omega')} \leqslant C(\Omega'), \quad \|u_n(t)\|_{W_2^2(\Omega')} \leqslant C(\Omega') \tag{45}$$

$$\|u_n(t)-u_n(t')\| \leqslant C|t-t'|, \quad \|u_{B,n}(t)-u_{B,n}(t')\|_{\Gamma_1} \leqslant C|t-t'|. \tag{46}$$

Now we define

$$b_{j,n}(t,x,\xi) = b_j(t_i,x,\xi) \text{ for } t_{i-1}<t\leqslant t_i,\ i=1,\ldots,n \ b_{j,n}(0,x,\xi) = b_j(0,x,\xi),$$
$$j=0,1,2 \ (\xi \in E^{N+1} \text{ for } j=0 \text{ and } \xi \in E^1 \text{ for } j=1,2).$$

Using our notation we can write

$$\left(\frac{\mathrm{d}^- u_n(t)}{\mathrm{d}t}, v\right)+\left(\frac{\mathrm{d}^- u_{B,n}(t)}{\mathrm{d}t}, v\right)_{\Gamma_1}+[Ax_n(t), v]+\left(b_{0,n}\left(t, x, x_n\left(t-\frac{T}{n}\right),\right.\right.$$
$$\left.\left.\frac{\partial x_n\left(t-\frac{T}{n}\right)}{\partial x}\right), v\right)+\sum_{j=1,2}\left(b_{j,n}\left(t, x_{B,n}\left(t-\frac{T}{n}\right)\right), v\right)_{\Gamma_j}=(f_n(t), v) \tag{47}$$

for all $\frac{T}{n} \leqslant t \leqslant T$, $v \in V$ and then we pass to the limit for $n \to \infty$ in (47).

Lemma 10. *There exists $u \in L_\infty(\langle 0, T\rangle, V)$ such that*

i) *There exists a subsequence $\{u_{n_k}(t)\}$ of $\{u_n(t)\}$ satisfying $u_{n_k}(t) \to u(t)$ in $L_2(\Omega)$, $u_{B,n_k}(t) \to u_B(t)$ in $L_2(\Gamma_1)$ for $k \to \infty$ uniformly in $t \in \langle 0, T\rangle$.*

ii) *There exist derivatives $\frac{\mathrm{d}u}{\mathrm{d}t} \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$, $\frac{\mathrm{d}u_B}{\mathrm{d}t} \in L_\infty(\langle 0, T\rangle, L_2(\Gamma_1))$.*

Proof. Owing to the compactness of the imbedding $W_2^1(\Omega)$ into $L_2(\partial\Omega)$, (43) and from the reflexivity of $W_2^1(\Omega)$ we conclude: there exist $u(t) \in L_2(\Omega)$, $g(t) \in L_2(\partial\Omega)$ (t is fixed) and a subsequence $\{u_{n_k}(t)\}$ such that $u_{n_k}(t) \to u(t)$ in $L_2(\Omega)$, $u_{B,n_k}(t) \to g(t)$ in $L_2(\partial\Omega)$. By the method of diagonalization we can find a subsequence of $\{u_n(t)\}$ (denoted again by $\{u_n(t)\}$) such that $u_n(t) \to u(t)$ in $L_2(\Omega)$ and $u_{B,n}(t) \to g(t)$ in $L_2(\Gamma_1)$ for all rational points $t \in \langle 0, T\rangle$. Then, from (46) we find out easily that $u_n(t) \to u(t)$ in $L_2(\Omega)$ and $u_{B,n}(t) \to g(t)$ in $L_2(\Gamma_1)$ for all $t \in \langle 0, T\rangle$. From the reflexivity of V and from (43) we conclude that $u(t) \in V$, $u_n(t) \rightharpoonup u(t)$ in V and $u_{B,n}(t) \to u_B(t)$ in $L_2(\partial\Omega)$. Thus $u_B(t) \equiv g(t)$. Owing to the Borel covering theorem and (46) we deduce that $u_n(t) \to u(t)$ in $L_2(\Omega)$ and

$u_{B,n}(t) \to u_B(t)$ in $L_2(\Gamma_1)$ uniformly in $t \in \langle 0, T \rangle$. From $u_n(t) \rightharpoonup u(t)$ in V and (43) we deduce the estimate

$$\|u(t)\|_V \leq C \quad \text{for all} \quad t \in \langle 0, T \rangle$$

from which $u \in L_\infty(\langle 0, T \rangle, V)$ follows and thus Assertion i) is proved. From Assertion i) and from (46) we have

$$\|u(t) - u(t')\| \leq C|t - t'|, \quad \|u_B(t) - u_B(t')\|_{\Gamma_1} \leq C|t - t'| \tag{48}$$

for all $t, t' \in \langle 0, T \rangle$. Assertion ii) follows from (48) and from the result of Y. Komura [10] similarly as in § 1.

The subsequence $\{u_{n_k}(t)\}$ from Lemma 10 and the corresponding subsequence $\{x_{n_k}(t)\}$ will be denoted by $\{u_n(t)\}$, $\{x_n(t)\}$, respectively.

Lemma 11. *Let $u(t)$ be as in Lemma 10. Then, $u(t) \in W_2^2(\Omega')$ and $x_n(t) \to u(t)$, $x_n\left(t - \frac{T}{n}\right) \to u(t)$, $u_n(t) \to u(t)$ in the norm of the space $W_2^1(\Omega')$ for all $t \in \langle 0, T \rangle$ and $\Omega', \bar{\Omega}' \subset \Omega$.*

Proof. Due to (45) and to the reflexivity of $W_2^2(\Omega')$ we have the following assertion: there exist $w_t \in W_2^2(\Omega')$ and a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$ such that $x_{n_k}(t) \rightharpoonup w_t$ in $W_2^2(\Omega')$ and hence $x_{n_k}(t) \to w_t$ in $W_2^1(\Omega')$. On the other hand $x_{n_k}(t) \to u(t)$ in $L_2(\Omega')$ because of Lemma 10 and (44). Thus, $w_t \equiv u(t)$ and also $x_n(t) \rightharpoonup u(t)$ in $W_2^2(\Omega')$, $x_n(t) \to u(t)$ in $W_2^1(\Omega')$. Similarly we prove the analogical assertion concerning the sequences $\left\{x_n\left(t - \frac{T}{n}\right)\right\}$ and $\{u_n(t)\}$ because of (43) and (44).

Theorem 3. *The function $u(t)$ from Lemma 10 is the unique solution of (1)—(3) and $u(x, t) \equiv u(t)$ satisfies (1) for a.e. $(x, t) \in \Omega \times (0, T)$ in the classical sense.*

Proof. Integrating (47) over the interval $\left(\frac{T}{n}, t\right)$ we have

$$\begin{gathered}
(u_n(t), v) - \left(u_n\left(\frac{T}{n}\right), v\right) + (u_{B,n}(t), v)_{\Gamma_1} - \left(u_{B,n}\left(\frac{T}{n}\right), v\right)_{\Gamma_1} + \\
+ \int_{T/n}^{t} \left\{[Ax_n(\tau), v] + \left(b_{0,n}\left(\tau, x_n\left(\tau - \frac{T}{n}\right), \frac{\partial x_n\left(\tau - \frac{T}{n}\right)}{\partial x}\right), v\right) + \right. \\
\left. + \sum_{j=1,2} \left(b_{j,n}\left(\tau, x_{B,n}\left(\tau - \frac{T}{n}\right)\right), v\right)_{\Gamma_j} - (f_n(\tau), v)\right\} \mathrm{d}\tau = 0
\end{gathered} \tag{49}$$

for all $v \in V$ and $t \in \left(\frac{T}{n}, T\right)$. As a consequence of Lemma 8, Lemma 10, Lemma 11, (38), (32) and the a priori estimates (42)—(46) we deduce the following assertions:

$$[Ax_n(\tau), v] \to [Au(\tau), v], \quad |[Ax_n(\tau), v]| \leq C\|v\|$$

for all $\tau \in (0, t)$ and $v \in V$;

$$b_{0,n}\left(\tau, x, x_n\left(\tau - \frac{T}{n}\right), \frac{\partial x_n\left(\tau - \frac{T}{n}\right)}{\partial x}\right) \to b_0\left(\tau, x, u(\tau), \frac{\partial u(\tau)}{\partial x}\right)$$

in $L_2(\Omega')$ and

$$\left\| b_{0,n}\left(\tau, x, x_n\left(\tau - \frac{T}{n}\right), \frac{\partial x_n\left(\tau - \frac{T}{n}\right)}{\partial x}\right)\right\| \leqq C$$

which imply that

$$\left(b_{0,n}\left(\tau, x_n(\cdot), \frac{\partial x_n(\cdot)}{\partial x}\right), v\right) \to \left(b_0\left(\tau, u(\tau), \frac{\partial u(\tau)}{\partial x}\right), v\right)$$

for all $v \in V$ and $\tau \in (0, T)$;

$$\left(b_{j,n}\left(\tau, x_{B,n}\left(\tau - \frac{T}{n}\right)\right), v\right)_{\Gamma_j} \to (b_j(\tau, u_B(\tau)), v)_{\Gamma_j} \; (j = 1, 2)$$

$$\text{and} \quad \left\| b_{j,n}\left(\tau, x_{B,n}\left(\tau - \frac{T}{n}\right)\right)\right\|_{\Gamma_j} \leq C \quad \text{for all} \quad n, \tau \in \left(\frac{T}{n}, T\right);$$

$$\left(u_n\left(\frac{T}{n}\right), v\right) \to (\varphi, v) \quad \text{and} \quad \left(u_{B,n}\left(\frac{T}{n}\right), v\right)_{\Gamma_1} \to (\varphi, v)_{\Gamma_1}$$

for all $v \in V$. On the basis of this assertion and of the Lebesque theorem we can pass to the limit $n \to \infty$ in (49). We obtain

$$(u(t), v) - (\varphi, v) + (u_B(t), v)_{\Gamma_1} - (\varphi, v)_{\Gamma_1} +$$
$$+ \int_0^t \left\{[Au(\tau), v] + \left(b_0\left(\tau, u(\tau), \frac{\partial u(\tau)}{\partial x}\right), v\right) + \right.$$
$$\left. + \sum_{j=1,2} (b_j(\tau, u_B(\tau)), v)_{\Gamma_j}\right\} d\tau = \int_0^t (f(\tau), v) \, d\tau$$

for all $v \in V$. Hence, we conclude that $u(t)$ is a solution of (1)—(3). The uniqueness of $u(t)$ can be proved similarly as in [8]. Let u_1, u_2 be two solutions of (1)—(3). Then the element $u = u_1 - u_2$ satisfies the inequality

$$\left(\frac{du(t)}{dt}, v\right) + \left(\frac{du_B(t)}{dt}, v\right)_{\Gamma_1} +$$

$$+[Au(t), v] - C_1\|u\|_W\|v\| - C_2\|u\|_{\Gamma_1}\|v\|_{\Gamma_1} - C_0\|u\|_{\Gamma_2}\|v\|_{\Gamma_2} \leq 0$$

for all $v \in V$ because of (12) and (32). Putting $u = e^{\lambda t} v$ $(\lambda > 0)$ we obtain the following inequality for v

$$\lambda\|v\|^2 + \frac{1}{2}\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2_{\Gamma_1} + \frac{1}{2}\frac{d}{dt}\|v\|^2_{\Gamma_1} +$$

$$+ C_E\|v\|^2_w - C_d\|v\|^2_w - C_1\|v\|^2 - C_2\|v\|^2_{\Gamma_1} - C_0 C_1^2\|v\|^2_w \leq 0,$$

where $C_d = C_E - C_0 C_I^2$. If $\lambda > \max(C_1, C_2)$, then we have

$$\frac{d}{dt}\|v(t)\|^2 + \frac{d}{dt}\|v_B(t)\|^2_{\Gamma_1} \leq 0.$$

Integrating this inequality over $(0, t)$ we obtain $\|v(t)\| = 0$ because of $v(0) = v_B(0) = 0$.

Since $\frac{du}{dt} \in L_\infty(\langle 0, T\rangle, L_2(\Omega))$ we deduce easily that there exists the distributive derivative $\frac{\partial u(x, t)}{\partial t} \in L_2(\Omega \times (0, T))$. Hence there exists the classical derivative $\frac{\partial u(x, t)}{\partial t}$ for a.e. $x \in \Omega$ and for a.e. $t \in (0, T)$ (see [9]). Further, from $u \in L_\infty(\langle 0, T\rangle, W_2^2(\Omega'))$ (Ω' is arbitrary with $\bar{\Omega}' \subset \Omega$) we deduce that there exist partial derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $(i, j = 1, \ldots, N)$ in the classical sense for a.e. $x \in \Omega$ and for a.e. $t \in (0, T)$. Then, from (12) for $v \in \mathcal{D}(\Omega)$ and Green's theorem we obtain that (1) is satisfied for a.e. $(x, t) \in \Omega \times (0, T)$ in the classical sense and the proof is complete.

Remark 2. As a consequence of the uniqueness of the solution we obtain that the entire sequences $\{u_n(t)\}$ and $\{x_n(t)\}$ (see (16), (17)) converge to the solution $u(t)$ of (1)—(3).

We can prove the results contained in Lemma 5 similarly as those in § 1. Instead of Theorem 2 we can prove

Theorem 4. *Let $\{x_n(t)\}$, $\{u_n(t)\}$ be as in (16), (17), respectively. Then*
i) *$x_n(t) \to u(t)$ in $W_2^1(\Omega)$ uniformly for $t \in \langle 0, T\rangle$;*
ii) *$u_n(t) \to u(t)$ in $W_2^1(\Omega)$ uniformly for $t \in \langle 0, T\rangle$;*
iii) *there exists a C such that $\|u(t) - u(t')\|_V \leq C|t - t'|$ holds for all $t, t' \in \langle 0, T\rangle$.*

Proof. From (47) and (12) for $v = x_n(t) - u(t)$ we estimate

$$C_E\|x_n(t) - u(t)\|^2_W \leq C_1\|x_n(t) - u(t)\| + C_2\|x_{B,n}(t) - u_B(t)\|_{\Gamma_1} + \\ + C_0\left\|x_{B,n}\left(t - \frac{T}{n}\right) - u_B(t)\right\|_{\Gamma_2}\|x_{B,n}(t) - u_B(t)\|_{\Gamma_2} \tag{50}$$

because of (31), (32), (34) and the estimates

$$\|f(t)\| + \left\|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right\| + \left\|\frac{\mathrm{d}^- u_n(t)}{\mathrm{d}t}\right\|_{\Gamma_1} +$$

$$+ \left\|b_{0,n}\left(t, x, x_n\left(t-\frac{T}{n}\right), \frac{\partial x_n\left(t-\frac{T}{n}\right)}{\partial x}\right)\right\| + \left\|b_0\left(t, x, u(t), \frac{\partial u(t)}{\partial x}\right)\right\| \leqslant C_1$$

and

$$\left\|\frac{\mathrm{d}u_B(t)}{\mathrm{d}t}\right\|_{\Gamma_1} + \left\|\frac{\mathrm{d}^- u_{B,n}(t)}{\mathrm{d}t}\right\|_{\Gamma_1} + \left\|b_{1,n}\left(t, x, x_{B,n}\left(t-\frac{T}{n}\right)\right)\right\|_{\Gamma_1} + \|b_1(t, x, u_B(t))\|_{\Gamma_1} \leqslant C_2$$

for all n, $t \in \langle 0, T\rangle$. Due to (43) and Lemma 8 iv) we have

$$C_0\left\|x_{B,n}\left(t-\frac{T}{n}\right) - u_B(t)\right\|_{\Gamma_2} \|x_{B,n}(t) - u_B(t)\|_{\Gamma_2} \leqslant C_0 C_I^2(\|x_n(t) - u(t)\|_W^2 +$$

$$+ \left\|x_n\left(t-\frac{T}{n}\right) - x_n(t)\right\|_W \|x_n(t) - u(t)\|_W) \leqslant C_0 C_I^2(\|x_n(t) - u(t)\|_W^2 + C\sqrt{h})$$

and hence, owing to (50) we have

$$\|x_n(t) - u(t)\|_W^2 \leqslant \frac{1}{C_d}\left(C_1\|x_n(t) - u(t)\| + C_2\|x_{B,n}(t) - u_B(t)\|_{\Gamma_1} + \frac{C_3}{\sqrt{n}}\right).$$

Assertion i) follows from this estimate, Lemma 10 and Remark 2. Assertion ii) follows from i) and the estimate

$$\|u_n(t) - u(t)\|_W^2 \leqslant 2\|x_n(t) - u(t)\|_W^2 +$$

$$+ 2\|x_n(t) - u_n(t)\|_W^2 \leqslant 2\|x_n(t) - u(t)\|_W^2 + \frac{C}{\sqrt{n}}$$

because of Lemma 8 (iv)). From (12) we deduce similarly as in § 1 the estimate

$$\begin{aligned} C_E\|u(t) - u(t')\|_W^2 &\leqslant C_1(\|u(t) - u(t')\| + \\ &+ \|u_B(t) - u_B(t')\|_{\Gamma_1}) + C_2|t - t'|\ \|u(t)\| + C_3|t - t'| + \\ &+ C_4\|u(t)\|_{\Gamma_1}|t - t'| + C_5\|u(t)\|_{\Gamma_2}|t - t'| + C_0\|u(t) - u(t')\|_{\Gamma_2}^2. \end{aligned} \tag{51}$$

Using (47) and the estimate

$$C_0\|u(t) - u(t')\|_{\Gamma_2}^2 \leqslant C_0 C_I^2\|u(t) - u(t')\|_W^2$$

in (51) we obtain the required result iii) and the proof is complete.

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НЕЛИНЕЙНЫЕ ПАРАБОЛИЧЕСКИЕ УРАВНЕНИЯ С НЕЛИНЕЙНЫМИ СМЕШАННЫМИ И НЕСТАЦИОНАРНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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Резюме

В работе рассматривается нелинейное параболическое уравнение второго порядка $u_t + Au(t) = f(t)$ в области $\Omega \times (0, T)$ с нестационарными и смешанными граничными условиями

$$u_t = -\frac{\partial u}{\partial \nu_A} + b_1(t, x, u) \quad \text{и} \quad 0 = -\frac{\partial u}{\partial \nu_A} + b_2(t, x, u)$$

на частях границы $\partial\Omega$. Доказывается существование и единнственность решения. Построено приближенное решение задачи и исследована его сходимость в отвечающих функциональных пространствах.