Michal Greguš, Jr.

The existence of a solution of a nonlinear boundary value problem

Mathematica Slovaca, Vol. 30 (1980), No. 2, 127--132

Persistent URL: http://dml.cz/dmlcz/136235

Math. Slovaca 30, 1980, No. 2, 127—132

THE EXISTENCE OF A SOLUTION OF A NONLINEAR BOUNDARY VALUE PROBLEM

MICHAL GREGUŠ jr.

In this paper a nonlinear singular boundary value problem with the third order differential operator DL^2 is studied, where $D=\frac{\mathrm{d}}{\mathrm{d}x}$, $L=x\frac{\mathrm{d}}{\mathrm{d}x}$. Hence the equation is of the form

$$x^2y'''+3xy''+y'=f(x,y,y'),\quad x\in(0,1\rangle \tag{1}$$

and the conditions which are considered are either

$$\lim_{x\to0^+} xy(x)=0,\quad \sup_{x\in(0,1\rangle}|xy'(x)|<\infty,\quad y(1)=y'(1)=0, \tag{I}$$

or

$$\left|\lim_{x\to0^+} y(x)\right|<\infty,\quad \sup_{x\in(0,1\rangle}|y'(x)|<\infty,\quad y(1)=0. \tag{II}$$

This problem is a generalization of a problem studied in [5]. The existence of a solution is investigated with the help of the Green function, constructed on the basis of the theory and results given in [4], Tichonoff fixed point theorem [1] and Ascoli-Arzelá's theorem [3].

Part I. Let us consider the problem (1)—(I). The Green function of this problem is

$$\mathrm{G}(x,t)=\begin{cases}-\frac{1}{2}\ln^2t+\ln t\ln x & 0<x\leqq t\leqq1,\\ \frac{1}{2}\ln^2x & 0<t\leqq x\leqq1.\end{cases}$$

The function $\mathrm{G}(x,t)$ has the following properties:

$$\mathrm{G}(x,t)\geqq0 \text{ for each } (x,t),\quad \mathrm{G}(1,t)=\mathrm{G}(x,1)=0,\quad \lim_{x\to0^+}x\,\mathrm{G}(x,t)=0,$$
$$\mathrm{G}_x(x,t)\leqq0 \text{ for each } (x,t),\quad \mathrm{G}_x(1,t)=\mathrm{G}_x(x,1)=0,$$
$$\lim_{x\to0^+}x\,\mathrm{G}_x(x,t)=\ln t,\quad\text{and for } x\neq t\text{: } \mathrm{G}_{xx}(x,t)\geqq0,\quad \mathrm{G}_{xx}(x,1)=0,$$
$$\lim_{x\to0^+}x^2\,\mathrm{G}_{xx}(x,t)=-\ln t.$$

When g is a real continuous function on $\langle 0, 1\rangle$ such that $|g(x)| \leqq \frac{m}{3}$, $x \in (0, 1\rangle$, $m > 0$, then the function $y(x)$, $y(x) = \int_0^1 G(x, t)g(t)\,dt$ satisfies the inequalities $|xy(x)| \leqq m$, $|xy'(x)| \leqq m$ for $x \in (0, 1\rangle$ and is a solution of the linear differential equation $DL^2y = g(x)$. Moreover, $y(x)$ satisfies the conditions (I).

Suppose that the function $f(x, u, v)$ is continuous and bounded by $m/3$ on $(0, 1\rangle \times \mathscr{R} \times \mathscr{R}$ and consider the space $\mathscr{X}_1$ of all real continuous functions with continuous first derivatives on $(0, 1\rangle$ and with the finite norm $\|y\|^* = \sup\limits_{x \in (0, 1\rangle} \{|xy(x)|, |xy'(x)|\}$, $y \in \mathscr{X}_1$. On an arbitrary compact interval $\mathscr{C} \subset (0, 1\rangle$ we can define the seminorm $p_{\mathscr{C}}(y) = \sup\limits_{x \in \mathscr{C}} \{|xy(x)|, |xy'(x)|\}$, $y \in \mathscr{X}_1$. The convergence in $\mathscr{X}_1$ with the topology defined by these seminorms is the uniform convergence on each compact set $\mathscr{C}$. The system $\{p_{\mathscr{C}_n}\}_{n=1}^{\infty}$, $\mathscr{C}_n = \left\langle \frac{1}{n}, 1 \right\rangle$, $n \in \mathscr{N}$ is a countable family of seminorms on $(0, 1\rangle$, satisfying Hausdorff's axiom of separation [3]. The system $\{\{y \in \mathscr{X}_1 | p_{\mathscr{C}_n}(y) < \varepsilon\}\}_{n=1}^{\infty}$ is a subbase of neighbourhoods of the point zero (i.e. of $y \equiv 0$ on $(0, 1\rangle$). $\mathscr{X}_1$ with this topology is a complete space.

Let us take a closed ball $\mathscr{F}$ with a radius $R \geqq m$, i.e.: $\mathscr{F} = \{y \in \mathscr{X}_1 |\, \|y\|^* \leqq R\}$. The set $\mathscr{F}$ is closed, bounded and convex in the topology defined by the system of seminorms $\{p_{\mathscr{C}_n}\}_{n=1}^{\infty}$. It is convenient to consider the operator T: $\mathscr{F} \to \mathscr{F}$ determined by

$$Ty(x) = \int_0^1 G(x, t)f(t, y(t), y'(t))\,dt, \quad y \in \mathscr{X}_1.$$

T is continuous if for any y_0, each $\varepsilon > 0$ and $n \in \mathscr{N}$, there exists $\delta_0 > 0$ and $n_0 \in \mathscr{N}$, $n_0 = n$ such that $p_{\mathscr{C}_{n_0}}(y - y_0) < \delta_0$ implies $p_{\mathscr{C}_n}(Ty - Ty_0) < \varepsilon$.

The function f is uniformly continuous on any compact set $\mathscr{C}_{n_0} \times \mathscr{I}_k \times \mathscr{I}_k$, where $\mathscr{I}_k = \langle -k, k\rangle$, $k \in \mathscr{N}$. Therefore if we choose $\delta > 0$ sufficiently small and $|y(t) - y_0(t)| < \delta$, $|y'(t) - y_0'(t)| < \delta$, then $|f(t, y(t), y'(t)) - f(t, y_0(t), y_0'(t))| < \frac{\varepsilon}{3}$, $t \in \mathscr{C}_{n_0}$. For δ there exists $\delta_0 > 0$ such that if the functions y, y_0 satisfy $p_{\mathscr{C}_{n_0}}(y - y_0) < \delta_0$, where $\delta_0 = \delta \min\limits_{x \in \mathscr{C}_{n_0}} \{x\} = \frac{\delta}{n_0}$, then $|y(t) - y_0(t)| < \delta$, $|y'(t) - y_0'(t)| < \delta$. But then for $x \in \mathscr{C}_{n_0} = \mathscr{C}_n$ we have $x|Ty(x) - Ty_0(x)| \leqq \varepsilon$ and $x|(Ty(x))' - (Ty_0(x))'| \leqq \varepsilon$. Hence T is continuous.

To show the relative compactness of $T(\mathscr{F})$ we use Ascoli-Arzelá's theorem. Since for $x \in (0, 1\rangle$

$$0 \leqq \int_0^1 G(x, t)\,dt < \infty \quad \text{and} \quad 0 \geqq \int_0^1 G_x(x, t)\,dt > -\infty,$$

the sets $T(\mathscr{F})$ and $[T(\mathscr{F})]'$ are equibounded on $\mathscr{C}_n$. Equicontinuity follows from the fact that f is bounded on $\mathscr{C}_n$ and $G(x, t)$ and $G_x(x, t)$ are uniformly continuous functions on $\mathscr{C}_n$.

Hence there follows from Tichonoff theorem the existence of $y \in \mathscr{F}$ such that $Ty = y$. Since y satisfies the conditions (I), we have

Theorem 1. *Let $f(x, u, v)$ be bounded and continuous on $(0, 1\rangle \times \mathscr{R} \times \mathscr{R}$. Then there exists a solution $y(x)$ of the boundary value problem*

$$x^2y''' + 3xy'' + y' = f(x, y, y'), \quad x \in (0, 1\rangle$$

$$y(1) = y'(1) = 0, \quad \lim_{x\to 0^+} xy(x) = 0$$

such that $\sup_{x \in (0, 1\rangle} |xy'(x)| < \infty$.

Remark 1. If we consider more closely the solution $y(x)$ of problem (1)—(I), we can see that on every compact set $\mathscr{C} \subset (0, 1\rangle$ $y(x)$, together with its derivatives y', y'', y''' is a bounded function, and $\lim_{x\to 0^+} x^2y'(x) = 0$ and $\sup_{x \in (0, 1\rangle} |x^2y''(x)| < \infty$.

If we considered the space $\mathscr{X}_2$ of functions with continuous second derivatives and the finite norm

$$\|y\|_* = \sup_{x \in (0, 1\rangle} \{|x^iy(x)|, |x^jy'(x)|, |x^2y''(x)|\},$$

$i = 1$ or 2, $j = 1$ or 2, and with the system of seminorms on compact subsets of $(0, 1\rangle$ defined in a similar way as before, respectively, we could prove the existence of the solution of equation (1) with the right-hand side equal to $f(x, y, y', y'')$, f continuous and bounded. Then the proof proceeds as follows:

(a) For $m > 0$ and $|g(x)| \leqq \frac{m}{4}$ we have $|x^2y''(x)| \leqq m$.

(b) $0 \leqq \int_0^1 G_{xx}(x, t)\, dt = \frac{1}{x^2} < \infty$ for $x \in (0, 1\rangle$.

(c) In the proof of the equicontinuity of $[T(\mathscr{F})]''$ we cannot proceed as in the proof of Theorem 1, because $G_{xx}(x, t)$ has a jump for $x = t$. However, it is possible to prove it otherwise:

Let us take an arbitrary $\varepsilon > 0$ and an arbitrary function $y'' \in [T(\mathscr{F})]''$. For $x_1 \leqq x_2 \in \mathscr{C}_n$ we have $|y''(x_1) - y''(x_2)| \leqq$

$$\Big| \int_0^{x_1} (x_1^{-2}(1 - \ln x_1) - x_2^{-2}(1 - \ln x_2)) f(t, y(t), y'(t), y''(t))\, dt +$$

$$\int_{x_1}^{x_2} (x_1^{-2} \ln t) f(t, y(t), y'(t), y''(t))\, dt -$$

$$\int_{x_1}^{x_2} x_1^{-2}(1 - \ln x_2) f(t, y(t), y'(t), y''(t))\, dt +$$

$$\left|\int_{x_2}^{1} (-x_1^{-2}+x_2^{-2})(\ln t)f(t, y(t), y'(t), y''(t))\, dt\right| \leqq$$
$$\frac{m}{4}|x_1^{-2}(1-\ln x_1)-x_2^{-2}(1-\ln x_2)|x_1-\frac{m}{4}x_1^{-2}|x_2\ln x_2-x_2-$$
$$x_1\ln x_1+x_1|+\frac{m}{4}|x_2^{-2}(1-\ln x_2)|\,|x_2-x_1|+\frac{m}{4}|x_2^{-2}-x_1^{-2}|\,|1-$$
$$x_2\ln x_2+x_2|\leqq\frac{m}{4}|P_1(x_1)-P_1(x_2)|+\frac{m}{4}n^2|P_2(x_1)-P_2(x_2)|+$$
$$\frac{m}{4}n^2(1+n)|P_3(x_1)-P_3(x_2)|+\frac{m}{4}(2+n)|P_4(x_1)-P_4(x_2)|,$$

where

$$P_1(x)=x^{-2}(1-\ln x),\ P_2(x)=x\,(\ln x-1),\ P_3(x)=x,\ P_4(x)=x^{-2}.$$

From the uniform continuity of P_i, $i=1, 2, 3, 4$ on $\mathscr{C}_n$ there follows the existence of a $\delta>0$ such that for $|x_1-x_2|<\delta$ we have $|P_i(x_1)-P_i(x_2)|<\frac{n^{-2}}{2+n}\frac{4}{m}\varepsilon$, $i=1, 2, 3, 4$. Now it is easy to show that for $|x_1-x_2|<\delta$, $x_1<x_2\in\mathscr{C}_n$ $|y''(x_1)-y''(x_2)|<\varepsilon$ is true. This result can be formulated in the form of the following existence theorem:

Theorem 2. *Let $f(x, u, v, w)$ be bounded and continuous on $(0, 1\rangle\times\mathscr{R}\times\mathscr{R}\times\mathscr{R}$. Then there exists a solution $y(x)$ of the boundary value problem*

$$x^2y'''+3xy''+y'=f(x, y, y', y''),\quad x\in(0, 1\rangle$$
$$y(1)=y'(1)=0,\quad \lim_{x\to0^+} xy(x)=0,\quad \lim_{x\to0^+} x^2y'(x)=0$$

such that $\sup_{x\in(0,1\rangle}|x^2y''(x)|<\infty$.

Part II. Let us consider now the problem (1)—(II). Its Green function is

$$H(x, t)=\begin{cases}-\frac{1}{2}\ln^2 t & x\leqq t,\\ -\ln t\ln x+\frac{1}{2}\ln^2 x & x\geqq t.\end{cases}$$

Note that $H(x, t)=-G(t, x)$. Other properties of $H(x, t)$ are:

$$H(x, t)\leqq 0,\quad \lim_{x\to0^+} H(x, t)=-\tfrac{1}{2}\ln^2 t,\quad \text{and}\quad H_x(x, t)\geqq 0,$$
$$\lim_{x\to0^+} H_x(x, t)=0.$$

The solution of the differential equation $DL^2y=h(x)$, where $h(x)$ is a real continuous function on $(0, 1\rangle$, is

$$y(x)=\int_0^1 H(x, t)h(t)\, dt.$$

If $|h(x)|<\frac{m}{4}$, m positive, $x\in(0,1\rangle$, then $|y(x)|\leqq m$, $|y'(x)|\leqq m$ and y satisfies the boundary condition (II), too.

Let $f(x,u,v)$ be continuous and bounded by $\frac{m}{4}$ on the set $(0,1\rangle\times\langle -K,K\rangle\times\langle -K,K\rangle$, $K\geqq m$. Consider the space $\mathscr{D}_1\left(\frac{1}{n},1\right)$ of all real functions with the first derivative continuous on $\left\langle\frac{1}{n},1\right\rangle$, $n\in\mathscr{N}$ and the norm $\|y\|_n=\sup\limits_{x\in\langle\frac{1}{n},1\rangle}\{|y(x)|,|y'(x)|\}$. The space $\mathscr{D}_1\left(\frac{1}{n},1\right)$ is a Banach space [2]. Let $\mathscr{X}$ be the space of all real functions with continuous first derivatives on $(0,1\rangle$ and the finite norm $\|y\|_\infty=\sup\limits_{x\in(0,1\rangle}\{|y(x)|,|y'(x)|\}$, $y\in\mathscr{X}$. Take a ball $\mathscr{B}=\{y\in\mathscr{X}|\ \|y\|_\infty\leqq K\}$ and define the operator $\mathrm{S}\colon\mathscr{B}\to\mathscr{B}$ by $\mathrm{S}y(x)=\int_0^1\mathrm{H}(x,t)\,f(t,y(t),y'(t))\,\mathrm{d}t$, $y\in\mathscr{X}$. S is well-defined, S is continuous (this follows from the uniform continuity of f on $\left\langle\frac{1}{n},1\right\rangle\times\langle -K,K\rangle\times\langle -K,K\rangle$) and $\mathrm{S}(\mathscr{B})$ is relatively compact (this follows from Ascoli-Arzelá's theorem, as well as the fact that $0\geqq\int_0^1\mathrm{H}(x,t)\,\mathrm{d}t=-x>-\infty$, $\int_0^1\mathrm{H}_x(x,t)\,\mathrm{d}t=1$ and that $\mathrm{H}(x,t)$ and $\mathrm{H}_x(x,t)$ are uniformly continuous on $\left\langle\frac{1}{n},1\right\rangle$). Therefore we have

Theorem 3. *Let $f(x,u,v)$ be a bounded and continuous function $(0,1\rangle\times\mathscr{R}\times\mathscr{R}$. Then there exists a solution $y(x)$ of the nonlinear boundary value problem*

$$x^2y'''+3xy''+y'=f(x,y,y'),\quad x\in(0,1\rangle$$

$$\left|\lim_{x\to0^+}y(x)\right|<\infty,\quad \sup_{x\in(0,1\rangle}|y'(x)|<\infty,\quad y(1)=0$$

such that $\sup\limits_{x\in(0,1\rangle}|xy''(x)|<\infty$.

Remark 2. It is interesting to note that in the case of conditions (II) $\lim\limits_{x\to0^+}y(x)$ exists and is finite. This fact follows from the boundedness of $y'(x)$. From the mean value theorem we then have that $y(x)$ is uniformly continuous on $(0,1\rangle$. Therefore each solution $y(x)$ of (1)—(II) is continuously extendable on $\langle 0,1\rangle$ and bounded.

On the other hand, in problem (1)—(I) difficulties arise. Generally we can say about a solution of that problem only that it is on the left end bounded by the function $\frac{1}{x}$, in other words that its "growth" is not arbitrarily large.

REFERENCES

[1] HARTMAN, P.: Ordinary differential equation, John Wiley and sons, New York—London—Sydney 1964.
[2] ŠILOV, G. J.: Matematická analýza, Alfa, Bratislava 1974.
[3] YOSIDA, K.: Functional analysis, Springer-Verlag, Berlin—Heidelberg—New York 1968.
[4] COLLATZ, L.: Problémy charakteristických hodnôt s technickými aplikáciami, SVTL, Praha 1965.
[5] ĎURIKOVIČOVÁ, M.: On the existence of solution of a singular boundary value problem, Mat. Čas., 23, 1973, 34—39.

Received November 25, 1977

Katedra matematickej analýzy
Prírodovedeckej fakulty UK
Mlynská dolina
816 31 Bratislava

СУЩЕСТВОВАНИЕ РЕШЕНИЯ ОДНОЙ НЕЛИНЕЙНОЙ КРАЕВОЙ ЗАДАЧИ

Михал Грегуш, мл.

Резюме

При помощи теоремы о неподвижной точке доказаны теоремы о существовании решения дифференциального уравнения (1) с краевыми условиями (I) или (II). Решение краевой задачи (1), (I) существует в том случае, когда функция f непрерывна и ограничена (теорема 1,2). Напротив того, решение краевой задачи (1), (II), при условии непрерывности и ограниченности f, ограничено и продолжительно на отрезок $\langle 0, 1\rangle$ (теорема 3). В статье тоже исследованы свойства функции Грина выше упомянутого уравнения (1).