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CORRESPONDENCE BETWEEN SUBADDITIVE INTEGRALS AND SMALL SYSTEMS

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In paper [1] there were presented two constructions. They are converse to each other and give the bijective correspondence between equivalence classes of semimeasures and small systems. In this note we shall use an analogous method for an investigation of a correspondence between equivalence classes of subadditive integrals and small systems.

Small systems were first studied in [2]. The notion of the subadditive integral was introduced for example in [3]. We shall use the following notions. Let R be the set of all real numbers, R^+ be the set of all nonnegative real numbers, Z^+ be the set of all nonnegative integers, $Z' = Z^+ \cup \{\infty\}$. For every $k \in Z^+$ we put: $\infty + k = \infty$, $k^{-\infty} = 0$. We define two functions:

$$r: R^+ \to Z' \quad r(x) = \inf \{n \in Z^+ : 2^{-n} \leqq x\},$$

$$t: Z' \to Z' \quad t(n) = \begin{cases} 0 & n = 0 \\ n-1 & 0 < n < \infty \\ \infty & n = \infty \end{cases}.$$

Definition 1. *Let (X, Σ) be a measurable space. Let M be the set of all measurable functions and $S \subset M$. A function J: $S \to R$ is called a subadditive integral if the following conditions are satisfied:*

1° *S is an additive group; $J(0) = 0$, $J(f+g) \leqq J(f) + J(g)$ for every nonnegative f, $g \in S$.*

2° *If $f \leqq g$, f, $g \in S$, then $J(f) \leqq J(g)$. For $f \in S$, $g \in M$, $|g| \leqq f$ there is $g \in S$.*

3° *If $f_n \searrow 0$, $f_n \in S$, $n = 1, 2, \ldots$, then $J(f_n) \searrow 0$.*

Remark. As an example one can consider the family S of all integrable functions (with respect to a measure μ) and the integral $J(f) = \int f d\mu$.

Definition 2. *Let K be the set of all finite measurable functions. Two subadditive integrals I, J, defined on $S \subset K$ are called equivalent if $J(f) = 0 \Leftrightarrow I(f) = 0$ for every $f \in S$.*

Construction 1. *Let J be a subadditive integral on S. For every $n \in Z^+$ we define*

$$S_n = \{f \in S: J(|f|) \leqq 3^{-1}\}.$$

Lemma 1. *The system $\{S_n\}_{n=0}^{\infty}$ obtained by the construction 1 has the following properties:*

(1) $S_0 \supset S_1 \supset \ldots \supset \{0\}$.

(2) $|f| \leqq g,\ f \in S,\ g \in S_n \Rightarrow f \in S_n,\ n \in Z^+$.

(3) $f, g, h \in S_n \Rightarrow f+g+h \in S_{t(n)},\ n \in Z^+$.

(4) $f_n \searrow 0,\ n \to \infty,\ f_n \in S,\ n \in Z^+ \Rightarrow \forall n \in Z^+\ \exists M \in Z^+$
$\forall m \geqq M: f_m \in S_n$.

Proof. Property (1) follows from the definition of $\{S_n\}_{n=0}^{\infty}$, (2) from the definition of the subadditive integral. We prove (3). If f, g, $h \in S_n$, then $J(f+g+h) \leqq J(|f+g+h|) \leqq J(|f|)+J(|g|)+J(|h|) \leqq 3.3^{-n} = 3^{-n+1}$ and from property 1° of the subadditive integral $f+g+h \in S_{t(n)}$ follows. The three properties are satisfied for $n = \infty$, too. Finally we prove (4). Definition 1 gives that for every $n \in Z^+$ there exists $M \in Z^+$ such that $J(|f_M|) \leqq 3^{-n}$. Then for every $m \geqq M$ there is $J(f_m) \leqq J(f_M) \leqq 3^{-n}$ and $f_m \in S_n$.

Definition 3. *(i) By a small system is meant a sequence $\{S_n\}_{n=0}^{\infty}$ of subsets of S having the properties (1)—(4). Put $S_\infty = \bigcap_{n=0}^{\infty} S_n$.*

(ii) Two small systems $\{S_n\}_{n=0}^{\infty}$, $\{T_n\}_{n=0}^{\infty}$ are called equivalent if $S_\infty = T_\infty$.

Construction 2. *Let $\{S_n\}_{n=0}^{\infty}$ be a small system. For every $f \in S$ we define*

$$h(f) = \sup\{n \in Z^+: f \in S_n\},$$

$$\delta(f) = 2^{-h(f)},$$

$$I(f) = \inf\left\{\sum_{i=1}^{n} \delta(f_i);\ f \leqq \sum_{i=1}^{n} f_i,\ f_i \in S,\ n \in Z^+\right\}.$$

Lemma 2. *(i) The function δ is a non decreasing function for every nonnegative $f \in S$.*

(ii) For every $a \in R^+$ and $f \in S$ there holds the implication

$$\delta(f) \leqq a \Rightarrow f \in S_{r(a)}.$$

Proof. *(i) If $f \leqq g$, then $\{n: f \in S_n\} \supset \{n: g \in S_n\}$,* $\sup\{n: f \in S_n\} \geqq \sup\{n: g \in S_n\}$ *and $\delta(f) \leqq \delta(g)$.*

(ii) Since $h(f) \in Z^+$, it is $\delta(f) = 2^{-h(f)} \leqq a < 2^{-r(a)+1}$. It follows $h(f) \geqq r(a)$, i.e. $f \in S_{r(a)}$ (cf. construction 2).

Remark. For $f \in S$ there is $\delta(f) = 0 \Leftrightarrow f \in S_\infty$.

Theorem. *(i) Function I obtained by construction 2 is a subadditive integral.*

(ii) If J is a subadditive integral on S and I is the subadditive integral obtained by constructions 1 and 2, then I and J are equivalent.

Proof. *(i)* S is an additive group. By Lemma 1 function I has the properties 1°, 2°, 3° of the subadditive integral.

(ii) To prove the equivalence between I and J we use the inequality $\delta(f) \leqq 2I(f)$ for every $f \in S$. We show that for every finite covering $|f| \leqq \sum_{i=1}^{n} f_i$, $f_i \in S$ there is

$$a = \sum_{i=1}^{n} \delta(f_i) \geqq \frac{1}{2} \delta(f),$$

i.e.

$$\delta(f) \leqq 2a.$$

If $a = \infty$ or $n = 1$, it is evident.

Hence let $a < \infty$, $n \geqq 2$ and $|f| \leqq \sum_{i=1}^{n} f_i$. We use the induction by n. We consider two cases.

a) $\delta(f_i) < \frac{a}{2}$ for $i = 1, 2, \ldots, n$.

We put

$$k = \max \left\{ j: \sum_{i=1}^{j-1} \delta(f_i) < \frac{a}{2} \right\}.$$

It is $1 < k < n$,

$$\sum_{i=1}^{k-1} \delta(f_i) < \frac{a}{2}, \quad \sum_{i=1}^{k} \delta(f_i) \geqq \frac{a}{2}, \quad \text{i.e.} \quad \sum_{i=k+1}^{n} \delta(f_i) \leqq \frac{a}{2}.$$

By using the inductive assumption we have

$$\delta \left(\sum_{i=1}^{k-1} f_i \right) \leqq 2 \sum_{i=1}^{k-1} \delta(f_i) < a,$$

$$\delta \left(\sum_{i=k+1}^{n} f_i \right) \leqq 2 \sum_{i=k+1}^{n} \delta(f_i) \leqq a$$

and

$$\delta(f_k) \leqq \sum_{i=1}^{n} \delta(f_i) = a.$$

Using Lemma 2 *(ii)* we get

$$\sum_{i=1}^{k-1} f_i, \; f_k, \; \sum_{i=k+1}^{n} f_i \in S_{r(a)},$$

thus by Lemma 1 (property (2) and (3)) there is

$$|f| \leqq \sum_{i=1}^{n} f_i \in S_{t(r(a))}, \quad f \in S_{t(r(a))}.$$

From construction 2 it follows

$$h(f) \geqq t(r(a)) \geqq r(a) - 1, \quad \delta(f) \leqq 2^{-(r(a)-1)} \leqq 2a.$$

b) There exists $i \in \{1, 2, \ldots, n\}$ such that $\delta(f_i) \geqq \frac{a}{2}$. We can suppose $i = n$. Then

$$\sum_{i=1}^{n-1} \delta(f_i) \leqq \frac{a}{2}, \quad \delta\left(\sum_{i=1}^{n-1} f_i\right) \leqq a, \quad \delta(f_n) \leqq a.$$

Now we can proceed as in the case a). There is $I(f) \leqq \delta(f)$, $\delta(f) \leqq 2I(f)$. It follows $I(f) = 0 \Leftrightarrow \delta(f) = 0$. We shall prove that $\delta(f) = 0$ is equivalent to $J(f) = 0$. If $\delta(f) = 0$, then $h(f) = \infty$, which implies $f \in \bigcap_{n=0}^{\infty} S_n$. $J(|f|) \leqq 3^{-n}$ follows for every $n \in Z^+$, i.e. $J(f) = 0$. The converse implication can be proved analogously. Therefore $I(f) = 0$ is equivalent to $J(f) = 0$, concluding the proof.

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СВЯЗЬ МЕЖДУ СУБАДДИТИВНЫМИ ИНТЕГРАЛАМИ И МАЛЫМИ СИСТЕМАМИ

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Резюме

В статье исследуется связь между эквиваленцией субаддитивных интегралов и малыми системами функций.