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CLASSIFICATION AND EXTENSION BY THE TRANSFINITE INDUCTION

MICHAL ŠABO

An application of the transfinite induction to the extension of functionals defined on certain lattices is connected with a classification. The classical examples of this classification are: the Baire and Young classifications of functions and the Lebesgue classification of Borel sets [3], [4], [8], [9]. The Baire classification is characterized by adding limits of sequences, the Lebesgue and Young classifications are characterized by adding limits of monotone sequences.

Certain functionals defined on systems of sets or functions can be investigated simultaneously as functionals defined on certain lattices [2], [5].

In the first section of the paper it is shown that the obtained sublattices, closed with regard to limits, monotone limits, respectively, are the same, when both of the mentioned types of classifications are used.

The generalized classification of Baire type has been used [1]. In the present paper the generalized classification of the Young type is used to the extension of certain functionals.

The assumptions made in the present paper seem to be weaker than those in [1], however, at the end of the paper it is proved that they are equivalent. These assumptions are, of course, easier to verify. The obtained results can be used for extension of a measure and an integral [1], [2].

1. Classification

Let S be a lattice. As usually, $x \cup y$ ($x \cap y$) denotes the supremum (infimum) of x, $y \in S$. If $\{x_n\}$ is a sequence of elements of S, then $\bigcup x_n$ ($\bigcap x_n$) denotes the supremum (infimum) of the sequence $\{x_n\}$, if it exists. Denote $\lim x_n = x$ or $x_n \to x$ if $\bigcup_{n\geq 1} \bigcap_{i\geq n} x_i = \bigcup_{n\geq 1} \bigcap_{i\geq n} x_i = x$ and if all of the used elements exist. If $\{x_n\}$ is an increasing (decreasing) sequence, i.e. $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$) and $\lim x_n = x$, then we shall write

$x_n \nearrow x$ ($x_n \searrow x$). Note that for any $x = \bigcup x_n$ ($x = \bigcap x_n$) a sequence $\{x'_n\}$ can be found such that $x'_n \nearrow x$ ($x'_n \searrow x$).

Now suppose: $x_n \to x$, $y_n \to y \Rightarrow x_n \cup y_n \to x \cup y$, $x_n \cap y_n \to x \cap y$ and let C be a sublattice of S. We define:

$$(C)^* = \{x \in S, \text{ for which there is } \{x_n\}, x_n \in C, x_n \to x\}$$
$$(C)^- = \{x \in S, \text{ for which there is } \{x_n\}, x_n \in C, x_n \nearrow x\}$$
$$(C)_- = \{x \in S, \text{ for which there is } \{x_n\}, x_n \in C, x_n \searrow x\}.$$

Let A be a sublattice of S. We define three transfinite sequences as follows:

$$A_0 = B_0 = C_0 = A$$

$A_\alpha = (A_{\alpha-1})^-$, $B_\alpha = (B_{\alpha-1})_-$ if α is an odd ordinal;
$A_\alpha = (A_{\alpha-1})_-$, $B_\alpha = (B_{\alpha-1})^-$ if α is an even non-limit ordinal;

$A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ if α is a limit ordinal;

$C_\alpha = \bigcup_{\beta<\alpha} (C_\beta)^*$ for any ordinal α.

Symbolic:

$$A = A_0 \nearrow A_1 \searrow A_2 \nearrow \ldots A_\omega \nearrow A_{\omega+1} \searrow \ldots$$
$$A = B_0 \searrow B_1 \nearrow B_2 \searrow \ldots B_\omega \searrow B_{\omega+1} \nearrow \ldots$$
$$A = C_0 \to C_1 \to C_2 \to \ldots C_\omega \to C_{\omega+1} \to \ldots$$

Note that the limit ordinals are taken for even.

It is evident that $A_\alpha = A_\Omega$, $B_\alpha = B_\Omega$, $C_\alpha = C_\Omega$ (Ω is the first uncountable ordinal) for $\alpha > \Omega$ ($x_n \in A_{\alpha_n}$, $\alpha_n < \Omega \Rightarrow$ there is α_0 such that $\alpha_n < \alpha_0 < \Omega$ [6]).

Proposition 1. $A_\Omega = B_\Omega = C_\Omega$.

Proof: We prove the stronger proposition: $A_\gamma = B_\gamma = C_\gamma$ for any limit ordinal γ. Evidently, $A_\gamma \subset C_\gamma$. First, we prove $C_n \subset A_{2n}$ $n = 1, 2, 3, \ldots$ using the induction.

If $x \in C_1$, then $x = \bigcap_{n\geq 1} \bigcup_{i\geq n} x_i$, $x_i \in A$. Since $\bigcup_{i\geq n} x_i \in A_1$ and $\bigcup_{i\geq n} x_i \searrow x$, we have $x \in A_2$.

Now let $C_{n-1} \subset A_{2n-2}$ and $x \in C_n$. Then $x = \bigcap_{n\geq 1} \bigcup_{i\geq n} x_i$, $x_i \in C_{n-1}$. Thus $\bigcup_{i\geq n} x_i \in A_{2n-1}$ and $\bigcup_{i\geq n} x_i \searrow x \in A_{2n}$. This implies $C_\omega = A_\omega$, which is the first step of the transfinite induction. Further, let for the limit ordinals δ, $\delta < \gamma$ the proposition hold. If $\alpha < \gamma$, then $\alpha = \beta + n$, where β is the limit ordinal, $n < \omega$ [7]. As $C_\beta = A_\beta$, we have $C_{\beta+n} \subset A_{\beta+2n}$, $\beta + 2n < \gamma$. Therefore $C_\gamma = \bigcup_{\alpha<\gamma} C_\alpha \subset \bigcup_{\alpha<\gamma} A_\alpha = A_\gamma$. The proof of the equality $C_\gamma = B_\gamma$ is dual.

In the following sections we shall apply the classification of the type $\{A_\alpha\}_{\alpha\leq\Omega}$ to an extension of a finite functional defined on a certain lattice.

2. Assumptions

Let S be a conditionally σ-complete lattice with two further binary operations $+$, $-$ such that:

I.a. There is an element $o \in S$ such that $o = x - x$ for any $x \in S$.

b. $a_n \nearrow a,\ b_n \nearrow b \Rightarrow a_n \cap b_n \nearrow a \cap b,\ a_n + b_n \nearrow a + b$

$a_n \searrow a,\ b_n \searrow b \Rightarrow a_n \cup b_n \searrow a \cup b,\ a_n + b_n \searrow a + b$

$a_n \nearrow a,\ b_n \searrow b \Rightarrow a_n - b_n \nearrow a - b,\ b_n - a_n \searrow b - a$

Let A be a sublattice of S closed under $+$, $-$, satisfying the condition:

II. To any $x \in S$ there are $a, b \in A$ such that $a \leqslant x \leqslant b$.

Let J be a real-valued finite functional defined on A fulfilling the following conditions:

(1) $a, b \in A,\ a \leqslant b \Rightarrow J(a) \leqslant J(b)$

(2) $a, b \in A \Rightarrow J(a) - J(b) \leqslant J(a - b)$

(3) $a, b \in A \Rightarrow J(a) + J(b) = J(a \cup b) + J(a \cap b)$

(4) $a_n \in A,\ a_n \searrow o \Rightarrow \lim J(a_n) = 0.$

Note that from (2) and (4) we have: $a_n \nearrow a$ or $a_n \searrow a \Rightarrow \lim J(a_n) = J(a)$.

3. The first step

We define two transfinite sequences of sublattices of S (see Section 1.):

Symbolic:

$$A = A_0 \nearrow A_1 \searrow A_2 \nearrow A_3 \searrow \ldots A_\omega \nearrow A_{\omega+1} \searrow \ldots$$
$$A = B_0 \searrow B_1 \nearrow B_2 \searrow B_3 \nearrow \ldots B_\omega \searrow V_{\omega+1} \nearrow \ldots$$

We shall use the first sequence only to the extension.

Lemma 1. *A_α, B_α are the sublattices of S closed under $+$.*

Lemma 2. *If $\alpha < \beta$, then $A_\alpha \subset B_\beta$, $B_\alpha \subset A_\beta$.*

Lemma 3. *If $a \in A_\alpha$, $b \in B_\alpha$, then $a - b \in A_\alpha$, $b - a \in B_\alpha$.*

Proof: We use the transfinite induction. Let α be an odd ordinal, then there are $\{a_n\}$, $\{b_n\}$, $a_n \in A_{\alpha-1}$, $b_n \in B_{\alpha-1}$ such that $a_n \nearrow a$, $b_n \searrow b$. Since $a_n - b_n \in A_{\alpha-1}$, $b_n - a_n \in B_{\alpha-1}$ and $a_n - b_n \nearrow a - b$, $b_n - a_n \searrow b - a$, we have: $a - b \in A_\alpha$, $b - a \in B_\alpha$. The proof is dual if α is non-limit even ordinal. If α is a limit ordinal, the proof is trivial.

Corollary 1. *If $a \in A_\alpha$, $b \in A_{\alpha+1}$, then $b - a \in A_{\alpha+1}$.*

Corollary 2. *If a, $b \in A_\alpha$, where α is a limit ordinal, then $a - b \in A_\alpha$.*

Lemma 4. *Let α, β be odd ordinals. Then A_α, $B_{\beta+1}$ (B_β, $A_{\alpha+1}$) are closed under limits of increasing (decreasing) sequences.*

Proof: We shall prove the first part only. Let $a_n \nearrow a$, $a_n \in A_\alpha$. For any n there is a sequence $\{a_n^i\}_i$, $a_n^i \nearrow a_n$ $(i \to \infty)$, $a_n^i \in A_{\alpha-1}$. Put $c_k = \bigcup_{j=1}^{k} a_j^k$. Evidently $a \geqslant \bigcup c_n \geqslant \bigcup a_n = a$. Thus $c_n \nearrow a$, $c_n \in A_{\alpha-1}$.

Now we define a functional $J_1: A_1 \to R$ as follows $J_1(a) = \lim J(a_n)$, where $a_n \nearrow a$, $a_n \in A$, $a \in A_1$. We shall have to prove that the definition is correct. The existence of the limit follows from (II) and (1). We shall prove that the functional J_1 is unambigously defined in the known way [5].

Let $\{a_n\}$, $\{c_n\}$ be increasing sequences of elements of A converging to $a \in A_1$. Then $a_m \cap c_n \nearrow a_m \cap a = a_m$. Therefore $J(a_m) = \lim_n J(a_m \cap c_n) \leqslant \lim_n J(c_n)$, i.e. $\lim_m J(a_m) \leqslant \lim_n J(c_n)$. Analogously, we can prove the reverse inequality.

Theorem 1. *The functional J_1 is an extension of J satisfying* (1), (3) *and*

(2′) $a \in A_1, b \in A \Rightarrow J_1(a) - J(b) \leqslant J_1(a-b)$

(4′) $a_n \nearrow a$, $a_n \in A_1 \Rightarrow a \in A_1$ *and* $\lim J_1(a_n) = J_1(a)$.

Proof: (1): $a_n \nearrow a$, $c_n \nearrow c$, $a \leqslant c$, a_n, $c_n \in A$ $a_n' = a_n \cap c_n \nearrow a$ and $J_1(a) = \lim J(a_n') \leqslant \lim J(c_n) = J_1(c)$.

(2′), (3) — trivial.

(4′): $a_n \nearrow a$, $a_n \in A_1$. By Lemma 4, we have $a \in A_1$.

We construct the sequence $\{c_n\}$ (see the proof of Lemma 4). Then $J_1(a) = \lim J(c_n) \leqslant \lim J_1(a_n) \leqslant J_1(a)$.

Corollary 3. *If $a_n \searrow o$, $a_n \in A_1$, then* $\lim J_1(a_n) = 0$.

Proof: Let $\varepsilon > 0$. For any a_n an element $a_n^1 \in A$ can be found such that $o \leqslant a_n^1 \leqslant a_n$ and $J_1(a_n) - J(a_n^1) \leqslant \frac{\varepsilon}{2^n}$. If we put $a_n^2 = a_1^1 \cap a_2^1 \cap \ldots \cap a_n^1$, then $o \leqslant a_n^2 \leqslant a_n^1 \leqslant a_n$ and $a_n^2 \searrow o$. Hence, by (4), $\lim J(a_n^2) = 0$. Using the induction we prove

$$J_1(a_n) - J(a_n^2) \leqslant \sum_{i=1}^{n} \frac{\varepsilon}{2^i}.$$

Indeed, $J_1(a_n) - J(a_n^2) = J_1(a_n) - J(a_n^1 \cap a_{n-1}^2) = J_1(a_n) - J(a_n^1) - J(a_{n-1}^2) + J(a_n^1 \cup a_{n-1}^2)$. Using the hypothesis of the induction and the fact that $a_n^1 \cup a_{n-1}^2 \leqslant a_{n-1}$ we have:

$$J_1(a_n) - J(a_n^2) \leqslant \frac{\varepsilon}{2^n} + \sum_{i=1}^{n-1} \frac{\varepsilon}{2^i} = \sum_{i=1}^{n} \frac{\varepsilon}{2^i}.$$

Hence, $\lim J_1(a_n) \leqslant \lim J(a_n^2) + \varepsilon = \varepsilon$.

Corollary 4. *If $a_n \searrow a$; a_n, $a \in A_1$, then* $\lim J_1(a_n) = J_1(a)$.

Proof: As $a \in A_1$, there is a sequence $\{b_n\}$, $b_n \in A$, $b_n \nearrow a$ and $J_1(a) = \lim J(b_n)$.

Then $o \leqslant a_n - b_n \in A_1$ (Corollary 1) and $a_n - b_n \searrow o$. Therefore, by Corollary 3, $\lim J_1(a_n - b_n) = 0$. Since $0 \leqslant J_1(a_n) - J(b_n) \leqslant J_1(a_n - b_n)$, we have:

$$\lim J_1(a_n) = \lim J(b_n) = J_1(a).$$

4. The second step

Now we define a functional J_2: $A_2 \to R$ as follows: $J_2(a) = \lim J_1(a_n)$, where $a_n \searrow a$, $a_n \in A_1$, $a \in A_2$. The existence of the limit is clear ((II), (1)). We shall show the independence of J_2 on the choice of the sequence. It is sufficient to prove: a_n, $b_n \in A_1$, $a \in A_2$, $a_n \searrow a$, $b_n \searrow a$ imply: $\lim J_1(a_n) = \lim J_1(b_n)$.

(I.b) implies: $a_n \cup b_m \searrow a_n \cup a = a_n \in A_1$. According to Corollary 4, $\lim\limits_m J_1(a_n \cup b_m) = J_1(a_n)$. Therefore $J_1(a_n) = \lim\limits_m J_1(a_n \cup b_m) \geqslant \lim\limits_m J_1(b_m)$ i.e. $\lim\limits_n J_1(a_n) \geqslant \lim\limits_m J_1(b_m)$. We show similarly that $\lim J_1(a_n) \leqslant \lim J_1(b_m)$.

Theorem 2. *The functional J_2 is an extension of J_1 satisfying* (1), (3) *and*

(2″) $a \in A_2$, $b \in A_1 \Rightarrow J_2(a) - J_2(b) \leqslant J_2(a - b)$

(4″) $a_n \searrow a$, $a_n \in A_2 \Rightarrow a \in A_2$ *and* $\lim J_2(a_n) = J_2(a)$.

Proof: The proof of (4″) is dual to the proof of (4′) and the other assertions are easy.

5. The transfinite induction

Theorem 3. *For any α, the functional J extends to a functional J_α: $A_\alpha \to R$, such that:*

(i) $a, b \in A_\alpha$, $a \leqslant b \Rightarrow J_\alpha(a) \leqslant J_\alpha(b)$

(ii) $a \in A_\alpha$, $b \in A_{\alpha-1}$ ($b \in A_\alpha$), *α-non-limit (limit) ordinal* $\Rightarrow J_\alpha(a) - J_\alpha(b) \leqslant J_\alpha(a - b)$

(iii) $a, b \in A_\alpha \Rightarrow J_\alpha(a) + J_\alpha(b) = J_\alpha(a \cup b) + J_\alpha(a \cap b)$

(iv) If $\varepsilon > 0$, $a \in A_\alpha$, *then there is* $b \in A_2$, $b \leqslant a$ *such that* $J_\alpha(a) - J_\alpha(b) \leqslant \varepsilon$

(v) $a_n \searrow o$, $a_n \in A_\alpha \Rightarrow \lim J_\alpha(a_n) = J_\alpha(o) = 0$

(vi) $a_n \nearrow a$ ($a_n \searrow a$), $a_n \in A_\alpha$, *α-non-limit odd (even) ordinal* $\Rightarrow a \in A_\alpha$ *and* $\lim J_\alpha(a_n) = J_\alpha(a)$.

Proof: We are using the transfinite induction. Define: If $a \in A_\alpha$, α-non-limit odd (even) ordinal, then $J_\alpha(a) = \lim J_{\alpha-1}(a_n)$, where $a_n \in A_{\alpha-1}$, $a_n \nearrow a$ ($a_n \searrow a$). If $a \in A_\alpha$, α-limit ordinal, then $J_\alpha(a) = J_\beta(a)$, $\beta < \alpha$. The first and second step have been proved in sections 3 and 4, respectively. Let the theorem be true for any β, $\beta < \alpha$. First, we show that $J_\alpha(a)$ depends on the element a only. If $\alpha - 1$ is a limit ordinal, then the independence $J_\alpha(a)$ from the choice of a sequence $\{a_n\}$ can be proved as it was proved for J_1. The required implication:

$$a_n \nearrow a;\quad a,\, a_n \in A_{\alpha-1} \Rightarrow \lim J_{\alpha-1}(a_n) = J_{\alpha-1}(a),$$

follows from *(i), (ii), (v)* and Corollary 2.

Let $\alpha - 1$ be a non-limit even ordinal. Let $a_n \nearrow a$, $b_n \nearrow a$; a_n, $b_n \in A_{\alpha-1}$, $a \in A_\alpha$. From the definition of $J_{\alpha-1}$, one can found a sequence $\{a_n^1\}$, $a_n^1 \in A_{\alpha-2}$, $a_n^1 \geqslant a_n$ and $J_{\alpha-1}(a_n^1) - J_{\alpha-1}(a_n) \leqslant \frac{\varepsilon}{2^n}$. We construct a sequence $\{a_n^2\}$, where $a_n^2 = a_1^1 \cup a_2^1 \cup \ldots \cup a_n^1 \in A_{\alpha-2}$. Using (II), there is $c \in A$, $c \geqslant a$. Therefore, the elements a_n^1 may be chosen such that $a_n^1 \leqslant c$. Because the sequence $\{a_n^2\}$ is bounded above by c, we have $a_n^2 \nearrow \bar{a} \in A_{\alpha-2}$. Using the induction we prove: $\lim J_{\alpha-1}(a_n^2) - \lim J_{\alpha-1}(a_n) \leqslant \varepsilon$ (Similarly as in the proof of Corollary 3). Hence $b_m \leqslant a \leqslant \bar{a}$ and $\lim J_{\alpha-1}(a_n) + \varepsilon \geqslant \lim J_{\alpha-1}(a_n^2) = J_{\alpha-1}(\bar{a}) \geqslant J_{\alpha-1}(b_m)$ for any m. Since ε is arbitrary, $\lim J_{\alpha-1}(a_n) \geqslant \lim J_{\alpha-1}(b_n)$. By symmetry, we have the inverse inequality.

If α is an even non-limit ordinal, the proof is analogous to the proof of the independence of J_2 from the choice of a sequence. (The implication: $a_n \searrow a$; a, $a_n \in A_{\alpha-1} \Rightarrow \lim J_{\alpha-1}(a_n) = J_{\alpha-1}(a)$, follows from *(i), (ii), (v)*).

In the case of a limit ordinal α the definition of $J_\alpha(a)$ is evidently correct (the assumption $J_\gamma(a) \neq J_\beta(a)$ $\gamma < \beta < \alpha$ contradicts the hypothesis of the induction). The proof of *(i), (ii), (iii)* is trivial. *(iv)* — Obviously, the statement is true for $\alpha = 1, 2$. If $\alpha > 2$ is an odd ordinal and $a \in A_\alpha$, then there are $c \in A_{\alpha-1}$, $b \in A_2$, $b \leqslant c \leqslant a$ such that:

$$J_\alpha(a) - J_\alpha(c) + J_\alpha(c) - J_\alpha(b) \leqslant \varepsilon.$$

If α is a non-limit even ordinal, then there is $\{a_n\}$, $a_n \in A_{\alpha-1}$, $a_n \searrow a$. To any n we can find $a_n^1 \in A_2$, $a_n^1 \leqslant a_n$ such that $J_\alpha(a_n) - J_\alpha(a_n^1) \leqslant \frac{\varepsilon}{2^n}$. We construct a sequence $\{a_n^2\}$ (see the proof of Corollary 3), $a_n^2 \searrow \bar{a}$, $\bar{a} \in A_2$ and $J_\alpha(a) - J_\alpha(\bar{a}) \leqslant \varepsilon$. The proof of *(iv)* for limit ordinals is trivial.

(v) — Let $a_n \searrow o$, $a_n \in A_\alpha$. Using *(iv)*, we construct a sequence $\{a_n^1\}$, $a_n^1 \in A_2$, $J_\alpha(a_n) - J_\alpha(a_n^1) \leqslant \frac{\varepsilon}{2^n}$. Putting $a_n^2 = a_1^1 \cap a_2^1 \cap \ldots \cap a_n^1$, we obtain the sequence $\{a_b^2\}$, $a_n^2 \searrow o$ such that $\lim J_\alpha(a_n) - \lim J_\alpha(a_n^2) \leqslant \varepsilon$. The proof of *(vi)* is analogous to the proof of (4').

Theorem 4. *There is a lattice $\bar{A}$, $A \subset \bar{A} \subset S$, closed under the operations* $+$, $-$ *and a functional $\bar{J}$: $\bar{A} \to R$, which is the unique extension of J satisfying* (1), (2), (3) *and*

(5) *If $a_n \in \bar{A}$, $a_n \nearrow a$ or $a_n \searrow a$, then $a \in \bar{A}$ and* $\lim \bar{J}(a_n) = \bar{J}(a)$

Proof: We put $\bar{A} = A_\Omega$, $\bar{J} = J_\Omega$. It is sufficient to prove the uniqueness only. Let K: $\bar{A} \to R$ be another extension of J satisfying (1) and (5). Denote $G = \{x \in \bar{A}$, for which $K(x) = \bar{J}(x)\}$. We shall prove: $G \supset A_\alpha$, for $\alpha \leqslant \Omega$, using the transfinite induction. Let $G \supset A_\gamma$ for any γ, $\gamma < \alpha$. If $x \in A_\alpha$ then there is $\{a_n\}$, $a_n \in A_{\alpha-1}$ and $a_n \nearrow x$ or $a_n \searrow x$. We have: $\bar{J}(x) = \lim \bar{J}(a_n) = \lim K(a_n) = K(x)$. Hence $x \in G$. It implies $G \supset A_\alpha$. If α is a limit ordinal, the proof follows from the definition of A_α and the hypothesis of the induction.

6. The conclusion

The conditions required for the lattice in the present paper and those in [1] are equivalent. Namely, if $x_n \to x$ and $y_n \to y$, then it can be shown that $x_n + y_n \to x + y$, $x_n - y_n \to x - y$, $x_n \cup y_n \to x \cup y$ and $x_n \cap y_n \to x \cap y$, using (I) and the conditional σ-completness of S.

It is clear that the results of [1] and those of the present paper are equivalent, too. The apparent difference is in (5) and (P) only. Evidently (P)$\Rightarrow$(5). We shall show the reverse implication. Let $x_n \to x$, $x_n \in \bar{A}$. From the definition of $\bar{A}$ we have $x \in \bar{A}$. As $x = \bigcap_{n \geq 1} \bigcup_{i \geq n} x_i = \bigcup_{n \geq 1} \bigcap_{i \geq n} x_i$, we have: $\bigcup_{i \geq n} x_i \searrow x$, $\bigcap_{i \geq n} x_i \nearrow x$, $\bar{J}\left(\bigcup_{i \geq n} x_i\right) \geq \bar{J}(x_n) \geq \bar{J}\left(\bigcap_{i \geq n} x_i\right)$. Thus $\bar{J}(x) = \lim \bar{J}\left(\bigcap_{i \geq n} x_i\right) \leq \liminf \bar{J}(x_n) \leq \limsup \bar{J}(x_n) \leq \lim \bar{J}\left(\bigcup_{i \geq n} x_i\right) = \bar{J}(x)$ i.e. $\lim \bar{J}(x_n) = \bar{J}(x)$.

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КЛАССИФИКАЦИЯ И ПРОДОЛЖЕНИЕ МЕТОДОМ ТРАНСФИНИТНОЙ ИНДУКЦИИ

Михал Шабо

Резюме

Пусть A – подструктура σ-монотонной структуры S. Наименьшее множество над A, замкнутое относительно пределов последовательностей или монотонных последовательностей можно получить постепенным добавлением пределов последовательностей, или монотонных последовательностей. В литературе это знакомо как классификация функций Бера или классификация борелевских множеств. В первой части показываем, что в общем случае полученные структуры одинаковы. С классификацией тесно связан метод трансфинитной индукции, который применяем в частях 2–5, вместе с классификацией второго типа, к продолжению некоторого конечного функционала определенного на некоторой подструктуре A, замкнутой относительно операций $+$, $-$. В части 6 показываем, что результат работы, из которого вытекают теоремы о продолжении меры в интеграла, не зависит от выбора типа классификации.