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REMARK ON PARTIALLY ORDERED SETS, UNIVERSAL ALGEBRAS AND SEMIGROUPS

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In the paper left segments of a partially ordered set P and some subsystems of the system of all left segments of P are studied.

Results are applied by studying the system of subalgebras of a B-algebra (see Def. 3.), especially by studying the system of subalgebras of unary algebras and B-semigroups and by studying systems of ideals of semigroups.

Some results of papers [1, 2, 3, 7, 9 and 10] are generalized and completed.

The possibility of solving the above problems is given by Theorem 15, since by this Theorem the partially ordered set of all subalgebras of a B-algebra is isomorphic to the partially ordered system of all left segments of the partially ordered set of all $\mathcal{I}$-equivalence classes (see the beginning of section 2) of this B-algebra.

1. Partially ordered sets and their segments

Definition 1. ([5]) *Let $\langle P, \leqslant \rangle$ be a partially ordered set. Let S be a subset of P having the following property:*

$$\text{if} \quad \xi \in S \quad \text{and} \quad \eta \leqslant \xi, \quad \text{then} \quad \eta \in S.$$

Then S is called the left segment of P.

Right segments are defined dually.

Let $\mathcal{B}(P)$ be the boolean of P and $\mathcal{S}(P)$ the system of all left segments of P.

Theorem 1. ([5]) *$\mathcal{S}(P)$ is a complete sublattice of the boolean $\mathcal{B}(P)$.*

Lemma 1. *Subsets $H_0(\alpha) = \{\xi \in P \mid \xi \leqslant \alpha\}$ and $N_0(\alpha) = \{\xi \in P \mid \xi \ngtr \alpha\}$ are left segments of P and they are nonempty subsets.*

Subsets $H(\alpha) = \{\xi \in P \mid \xi < \alpha\}$ and $N(\alpha) = \{\xi \in P \mid \xi \ngeqslant \alpha\}$ are left segments of P.

Subsets $H_0'(\alpha) = \{\xi \in P \mid \xi \geqslant \alpha\} = P \setminus N(\alpha)$ and $N_0'(\alpha) = \{\xi \in P \mid \xi \nless \alpha\} = P \setminus H(\alpha)$ are right segments of P and they are nonempty subsets.

$H'(\alpha) = \{\xi \in P | \xi > \alpha\} = P \setminus N_0(\alpha)$ *and* $N'(\alpha) = \{\xi \in P | \xi \not\leq \alpha\} = P \setminus H_0(\alpha)$ *are right segments of P.*

We shall use the following notations:

$$\begin{aligned}
\mathcal{H}(P) &= \{H(\alpha) | \alpha \in P, H(\alpha) \neq \emptyset\} \\
\mathcal{N}(P) &= \{N(\alpha) | \alpha \in P, N(\alpha) \neq \emptyset\} \\
\mathcal{H}^*(P) &= \{H(\alpha) | \alpha \in P\} \\
\mathcal{N}^*(P) &= \{N(\alpha) | \alpha \in P\} \\
\mathcal{H}_0(P) &= \{H_0(\alpha) | \alpha \in P\} \\
\mathcal{N}_0(P) &= \{N_0(\alpha) | \alpha \in P\} \\
\mathcal{H}_0'(P) &= \{H_0'(\alpha) | \alpha \in P\}.
\end{aligned}$$

Theorem 2. ([5]) *The mapping* $\varphi: P \to \mathcal{H}_0(P)$, $\varphi(\alpha) = H_0(\alpha)$ *is a monotone isomorphism.*

Theorem 3. *The relation* $K = \cap(\mathcal{S}(P) \setminus \{\emptyset\} \neq \emptyset$ *holds iff* $\inf(P) = \alpha$ *exists. Then* $\cap(\mathcal{S}(P) \setminus \{\emptyset\}) = \{\alpha\}$ *is true.*

Proof. Let $K \neq \emptyset$. Then there exists an $\alpha \in P$ such that $\alpha \in K$. Moreover for every $\xi \in P$ we have $H_0(\xi) \supseteq K \ni \alpha$, hence $\alpha \leq \xi$ and therefore $\alpha = \inf(P)$.

If $\alpha = \inf(P)$ exists, then $\alpha \in S$ for every $S \in \mathcal{S}(P) \setminus \{\emptyset\}$, therefore $\alpha \in K$ and this means $K \neq \emptyset$.

Theorem 4. *If* $K = \cap(\mathcal{S}(P) \setminus \{\emptyset\}) \neq \emptyset$, *then* $K = \cap \mathcal{H}_0(P)$.

Proof. By Theorem 3 we have $K = \{\alpha\}$, where $\alpha = \inf(P)$. Hence $H_0(\alpha) = \{\alpha\}$ and therefore $K = H_0(\alpha) \supseteq \cap \mathcal{H}_0(P)$. The converse inclusion is evident.

Theorem 5. *If* $K = \cap(\mathcal{S}(P) \setminus \{\emptyset\}) \neq \emptyset$, *then* $K = \cap \mathcal{N}_0(P)$.

Proof. We have again $K = \{\alpha\}$, where $\alpha = \inf(P)$. Hence $N_0(\alpha) = \{\alpha\}$ and therefore $K = N_0(\alpha) \supseteq \cap \mathcal{N}_0(P)$.

The converse inclusion is evident.

Theorem 6. *The relation* $K = \cap(\mathcal{S}(P) \setminus \{\emptyset\}) = \emptyset$ *holds iff* $\cap \mathcal{H}_0(P) = \emptyset$.

Proof. Clearly $\cap \mathcal{H}_0(P) = \emptyset$ implies $K = \emptyset$.

If $K = \emptyset$, then for every element $\xi \in P$ there exists an $S \in \mathcal{S}(P) \setminus \{\emptyset\}$ such that $\xi \notin S$. But every such S contains a set $H_0(\alpha)$, where $\alpha \in S$ and $\xi \notin H_0(\alpha)$. Hence for every $\xi \in P$ there exists an $H_0(\alpha)$ such that $\xi \notin H_0(\alpha)$. This implies $\xi \notin \cap \mathcal{H}_0(P)$ and therefore $\cap \mathcal{H}_0(P) = \emptyset$.

Theorem 7. *If* $K = \cap(\mathcal{S}(P) \setminus \{\emptyset\}) \neq \emptyset$, *then* $K = \cap \mathcal{H}(P)$.

Proof. Since $K = \{\alpha\}$, where $\alpha = \inf(P)$, for every $\xi \in P$, $\xi \neq \alpha$ we have $\xi \notin H(\xi)$ and $\alpha \in H(\xi) \neq \emptyset$. (It is clear that $H(\alpha) = \emptyset$.) Hence for every $\xi \neq \alpha$ we have $\xi \notin \cap \mathcal{H}(P)$, $\alpha \in \cap \mathcal{H}(P)$. Therefore $\cap \mathcal{H}(P) = \{\alpha\} = K$.

Theorem 8. *If* $K = \cap(\mathcal{S}(P) \setminus \{\emptyset\}) \neq \emptyset$, *then* $K = \cap \mathcal{N}(P)$.

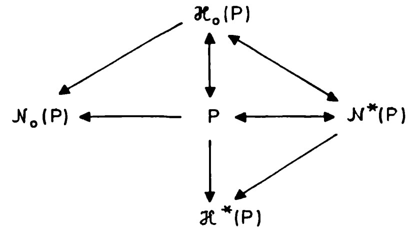
Proof. Since $K = \{\alpha\}$, where $\alpha = \inf(P)$, for every $\xi \in P$, $\xi \neq \alpha$ we have $\xi \notin N(\xi)$

and $\alpha \in N(\xi) \neq \emptyset$. (It is clear that $N(\alpha)=\emptyset$.) Hence for every $\xi \neq \alpha$ we have $\xi \notin \cap\mathcal{N}(P)$, $\alpha \in \cap\mathcal{N}(P)$. Therefore $\cap\mathcal{N}(P)=\{\alpha\}=K$.

Theorem 9. *The relation* $K=\cap(\mathcal{S}(P)\setminus\{\emptyset\})=\emptyset$ *holds iff* $\cap\mathcal{N}(P)=\emptyset$.

Proof. Clearly $\cap\mathcal{N}(P)=\emptyset$ implies $K=\emptyset$.

Let $K=\emptyset$ hold. Then $\inf(P)$ does not exist. Let $N(\xi)\neq\emptyset$, then $\xi \notin \cap\mathcal{N}(P)$ since $\xi \notin N(\xi)$. Hence $\cap\mathcal{N}(P)$ contains only elements ξ that satisfy the relation $N(\xi)=\emptyset$. But $N(\xi)=\emptyset$ implies that there exists no $\eta \in P$ such that $\eta \ngeq \xi$. This means for all $\eta \in P$ the relation $\eta \geq \xi$ is true. Hence $N(\xi)=\emptyset$ implies that $\xi=\inf(P)$. This is a contradiction. Therefore $\cap\mathcal{N}(P)=\emptyset$ must hold.

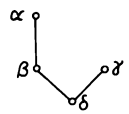


Diag. 1

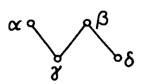
Theorem 10. *We have that* $\cap\mathcal{N}_0(P)=\{\alpha \in P \mid \alpha$ *is minimal in* $P\}$.

Proof. Let ξ be not a minimal element in P. Then there exists an $\eta \in P$ such that $\xi > \eta$. Hence $\xi \in N_0(\eta)$ and $\xi \notin \cap\mathcal{N}_0(P)$.

If α is a minimal element of P, then $\alpha \in N_0(\alpha)$. If $\xi > \alpha$, then $\alpha \in N_0(\xi)$. If ξ and α are incomparable, then $\alpha \in N_0(\xi)$ again. From this $\alpha \in \cap\mathcal{N}_0(P)$ follows and the proof is finished.



Diag. 2



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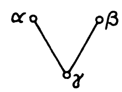


Diag. 4

Theorem 11. *We have that* $\cap\mathcal{H}(P)=\{\alpha \in P \mid \alpha$ *is minimal in P and if* $\xi \in P$ *is not minimal in P, then* $\alpha < \xi\}$.

Proof. Let ξ be not a minimal element of P. Then $\xi \notin H(\xi) \neq \emptyset$, therefore $\xi \notin \cap\mathcal{H}(P)$.

If η is a minimal element of P and there exists a $\xi \in P$ such that ξ and η are incomparable and ξ is not minimal in P, then $\eta \notin H(\xi) \neq \emptyset$, i.e. $\eta \notin \cap\mathcal{H}(P)$. Hence $\cap\mathcal{H}(P)$ can contain only minimal elements $\alpha \in P$ such that for every $\xi \in P$ that is not minimal in P the relation $\alpha < \xi$ holds.

Conversely, let $\alpha \in P$ be minimal in P and if $\xi \in P$ is not minimal in P, then let $\alpha < \xi$. For all these elements ξ we have $\alpha \in H(\xi) \neq \emptyset$. All other elements $\xi \in P$ are minimal in P and therefore $N(\xi) = \emptyset$ for all other elements $\xi \in P$. This implies $\alpha \in \cap \mathcal{H}(P)$.

Remark. $|M|$ denotes the cardinality of the set M.

Theorem 12. *The mapping* $f\colon \mathcal{H}_0(P) \to \mathcal{N}^*(P)$, $f(H_0(\alpha)) = N(\alpha)$ *is a monotone isomorphism. Hence* $|\mathcal{H}_0(P)| = |\mathcal{N}^*(P)|$.

Proof. The mapping $\varphi\colon P \to \mathcal{H}_0(P)$, $\varphi(\alpha) = H_0(\alpha)$ is a monotone isomorphism and mappings $\psi\colon P \to \mathcal{H}_0'(P)$, $\psi(\alpha) = H_0'(\alpha)$ and $\chi\colon \mathcal{H}_0'(P) \to \mathcal{N}^*(P)$, $\chi(H_0'(\alpha)) = N(\alpha)$ are monotone antiisomorphisms. This implies that $f = \chi \circ \psi \circ \varphi^{-1}$ is a monotone isomorphism.

Definition 2. *A monotone homomorphism f is called a contracting homomorphism iff* $f(\alpha) = f(\beta)$ *implies either* $\alpha = \beta$ *or* α *and* β *are incomparable.*

Theorem 13. *Mappings* $g\colon P \to \mathcal{N}_0(P)$, $g(\alpha) = N_0(\alpha)$ *and* $h\colon P \to \mathcal{H}^*(P)$, $h(\alpha) = H(\alpha)$ *are surjective contracting homomorphism.*

Proof. Let $\alpha < \beta$. From definitions we get $N_0(\alpha) \subset N_0(\beta)$, $N_0(\alpha) \neq N_0(\beta)$ and $H(\alpha) \subset H(\beta)$, $H(\alpha) \neq (\beta)$.

The last two Theorems yield the diagram 1 where $\rightarrow$ denotes a surjective contracting homomorphism and $\leftrightarrow$ denotes a monotone isomorphism.

Example 1. Let $\langle P, \leq \rangle$ be the partially ordered set given by the diagram 2. Then $H(\beta) = H(\gamma) = \{\delta\}$ and $N_0(\beta) = \{\alpha, \beta, \gamma\} \neq N_0(\gamma) = \{\alpha, \beta, \gamma, \delta\}$. Hence the relation $\{(H(\xi), N_0(\xi)) \in \mathcal{H}^*(P) \times \mathcal{N}_0(P) | \xi \in P\}$ is not a mapping of $\mathcal{H}^*(P)$ into $\mathcal{N}_0(P)$.

Example 2. (diag. 3.) Here $N_0(\alpha) = N_0(\beta) = \{\alpha, \beta, \gamma, \delta\}$ and $H(\alpha) = \{\gamma\} \neq \{\gamma, \delta\} = H(\beta)$. Hence the relation $\{(N_0(\xi), H(\xi)) \in \mathcal{N}_0(P) \times \mathcal{H}^*(P) | \xi \in P\}$ is not a mapping of $\mathcal{N}_0(P)$ into $\chi^*(P)$.

Example 3. (diag. 4.) Here $\alpha \neq \beta$, but $N_0(\alpha) = N_0(\beta) = \{\alpha, \beta, \gamma\}$ and $H(\alpha) = H(\beta) = \{\gamma\}$. Hence mappings $g\colon P \to \mathcal{N}_0(P)$, $g(\alpha) = N_0(\alpha)$ and $h\colon P \to \mathcal{H}^*(P)$, $h(\alpha) = H(\alpha)$ are not isomorphisms.

Theorem 14. $\emptyset \neq S$ *is a maximal element of* $\mathcal{S}(P) \setminus \{P\}$ *iff* $\emptyset \neq S = P \setminus \{\alpha\}$ *and* α *is a maximal element of P.*

Proof. a) $\emptyset \neq S = P \setminus \{\alpha\}$ and α is maximal in P imply $P \neq S = P \setminus \{\alpha\} = N(\alpha)$. Hence $S \in \mathcal{S}(P) \setminus \{P\}$ and S is maximal in $\mathcal{S}(P) \setminus \{P\}$.

b) $\emptyset \neq S$ is maximal in $\mathcal{S}(P) \setminus \{P\}$ implies $S \neq P$. Hence there exists an $\alpha \in P$ such that $\alpha \notin S$ i.e. $\alpha \in P \setminus S$. If $\alpha, \beta \in P \setminus S$ and $\beta \not\leq \alpha$, then $\beta \notin H_0(\alpha) \in \mathcal{S}(P) \setminus \{P\}$, $\beta \notin S$ hence $\beta \notin H_0(\alpha) \cup S = T \in \mathcal{S}(P) \setminus \{P\}$. This implies $T \supset S$, $T \neq S$, $T \neq P$, therefore S is not a maximal element in $\mathcal{S}(P) \setminus \{P\}$ and we have a contradiction that implies $\beta \leq \alpha$. Dually we get $\alpha \leq \beta$, therefore $\alpha = \beta$ and $S = P \setminus \{\alpha\}$.

If $\xi > \alpha$ for some $\xi \in S$, then $\alpha \in S = P \setminus \{\alpha\}$. From this it follows that α is a maximal element in P.

2. B-algebras and their subalgebras

Let $\langle A, F\rangle$ be a universal algebra and F the system of all operations on A.

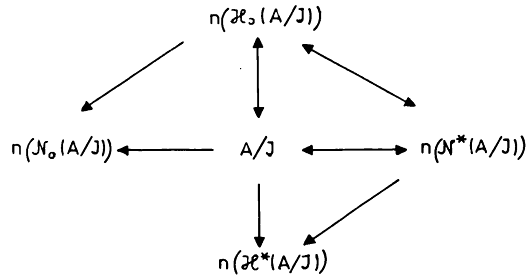
Let $\mathcal{P}(A)$ denote the set of all subalgebras of A (including the empty set).

Let $[a]$ be the principal subalgebra, generated by a.

The relation $\mathcal{J}=\{(x, y)\in A\times A\,|\,[x]=[y]\}$ is an equivalence relation on A. Let $[a]\mathcal{J}$ denote the $\mathcal{J}$-equivalence class containing a.

Let $A/\mathcal{J}=\{[a]\mathcal{J}\,|\,a\in A\}$.

Now we can introduce the following relation on $A/\mathcal{J}$: $[x]\mathcal{J}\leqslant[y]\mathcal{J}$ iff $[x]\subseteq[y]$. Then $\langle A/\mathcal{J}, \leqslant\rangle$ is a partially ordered set. (See [1, 7].)



Diag. 5

Definition 3. *A universal algebra $\langle A, F\rangle$ is called B-algebra iff the union of an arbitrary system of its subalgebras is a subalgebra of A.*

Remark. This condition is equivalent to the following condition: the union of an arbitrary system of its principal subalgebras is a subalgebra of A.

Theorem 15. *Let $\langle A, F\rangle$ be a B-algebra. Then the mapping n: $\mathcal{S}(A/\mathcal{J})\to$ $\to\mathcal{P}(A)$, $n(S)=\cup S$ is a monotone isomorphism.*

Proof. a) Let $S\in\mathcal{S}(A/\mathcal{J})$ and $a\in\cup S$. Hence $[a]\mathcal{J}\in S$ and if $[x]\mathcal{J}\leqslant[a]\mathcal{J}$, then $[x]\mathcal{J}\in S$. This means that $[a]\subseteq\cup S$. We have $\cup S=\cup\{[a]\,|\,a\in\cup S\}$ and therefore $\cup S$ is a subalgebra of the B-algebra A.

b) Let $C\in\mathcal{P}(A)$ and let $S=\{[a]\mathcal{J}\,|\,a\in C\}$. Then $C=\cup S$ because if $a\in C$, then $[a]\mathcal{J}\subseteq C$. The set C being a subalgebra implies that if $[x]\mathcal{J}\in S$ and $[x]\mathcal{J}\geqslant[y]\mathcal{J}$, then $[y]\mathcal{J}\subseteq[x]\subseteq C$, hence $y\in C$ and therefore $[y]\mathcal{J}\in S$. This means that $S\in$ $\in\mathcal{S}(A/\mathcal{J})$ and $n(S)=\cup S=C$.

c) This mapping is clearly injective and monotone.

Remark. $n(H_0([a]\mathcal{J}))=n\ (\{[x]\mathcal{J}\,|\,[x]\mathcal{J}\leqslant[a]\mathcal{J})=\cup\{[x]\mathcal{J}\,|\,[x]\mathcal{J}\leqslant[a]\mathcal{J}\}=$ $[a]$. Hence $n(\mathcal{H}_0(A/\mathcal{J}))$ is the system of all principal subalgebras of A.

Definition 4. *If the subalgebra $n(H([a]\mathcal{J}))\neq\emptyset$, we shall call it the H-subalgebra of A.*

If the subalgebra $n(N([a]\mathcal{J}))\neq\emptyset$, we shall call it the N-subalgebra of A.

The subalgebra $n(N_0([a]\mathcal{J}))$ *will be called the* N_0*-subalgebra.*

Remark. $n(\mathcal{H}(A/\mathcal{J}))$ is the set of all H-subalgebras of A, $n(\mathcal{N}(A/\mathcal{J}))$ is the set of all N-subalgebras of A and $n(\mathcal{N}_0(A/\mathcal{J}))$ is the set of all N_0-subalgebras of A.

Definition 5. *If* $K = \cap(\mathcal{P}(A)\setminus\{\emptyset\}) \neq \emptyset$, *it is called the kernel of A.*

From Theorems 4, 5, 7, 8, 15 and 14 we get.

Theorem 16. *Let* $\langle A, F\rangle$ *be a B-algebra and K be the kernel of A. Then the following statements are true:*

a) *K is the intersection of all principal subalgebras of A.*

b) *K is the intersection of all H-subalgebras of A.*

c) *K is the intersection of all N-subalgebras of A.*

d) *K is the intersection of all* N_0*-subalgebras of A.*

Moreover we have the diagram 5.

Definition 6. $C \in \mathcal{P}(A)$ *is called a maximal subalgebra of A iff* $\emptyset \neq C \neq A$ *and there is no* $D \in \mathcal{P}(A)$ *such that* $C \subset D \subset P$, $C \neq D$, $D \neq P$.

Theorem 17. [1] *Let* $\langle A, F\rangle$ *be a B-algebra. C is a maximal subalgebra of A iff* $\emptyset \neq C = A\setminus[a]\mathcal{J}$ *and* $[a]\mathcal{J}$ *is a maximal element of* $A/\mathcal{J}$.

Remark 1. It is known that a) of Theorem 16 is true for every universal algebra A (even if it is not B-algebra). For d) of Theorem 16 see also [1, 2, 3].

Remark 2. All this is true for unary algebras studied by I. Abrhan [1, 2, 3] and for B-semigroups studied by J. Bosák [4], because they are B-algebras.

Theorem 18. *Let M be a nonempty set and let* $\langle \Pi, \leq\rangle$ *be a partially ordered set such that* Π *is a partition of M. Then there exists a B-algebra* $\langle M, F\rangle$ *such that* $\langle \Pi, \leq\rangle = \langle M/\mathcal{J}, \leq\rangle$.

Proof. For every positive integer n, for every $T, U \in \Pi$ satisfying $T \geq U$ and for every $a_1, a_2, \ldots, a_n \in T$ and $b \in U$ we define an n-ary operation f on M as follows: $f(a_1, a_2, \ldots, a_n) = b$ and $f(x_1, x_2, \ldots, x_n) = x_1$ if $(x_1, x_2, \ldots, x_n) \neq (a_1, a_2, \ldots, a_n)$. Let F be the set of all these operations then $\langle M, F\rangle$ is a universal algebra. Every principal subalgebra generated by an element $a \in T \in \Pi$ contains the set T, it contains also every set $U \in \Pi$ satisfying $U \leq T$ but it contains no other elements. This implies that the $\mathcal{J}$-equivalence classes are exactly all sets $T \in \Pi$ and the relations $\leq$ in $M/\mathcal{J}$ and in Π coincide. Moreover $\langle M, F\rangle$ is clearly a B-algebra.

Remark. If F_1 is the set of all unary operations of F, then $\langle M, F_1\rangle$ is a unary algebra, satisfying $\langle \Pi, \leq\rangle = \langle M/\mathcal{J}, \leq\rangle$.

Remark. From Theorem 18 and from Examples 1, 2 and 3 it follows that the surjective contracting homomorphisms in the diagram need not be isomorphisms and the relations $\{(H([\xi]\mathcal{J}), N_0([\xi]\mathcal{J})) \in (\mathcal{H}^*(A/\mathcal{J})) \times n(\mathcal{N}_0(A/\mathcal{J}))|[\xi]\mathcal{J} \in A/\mathcal{J}\}$ and $\{(N_0([\xi]\mathcal{J}), H([\xi]\mathcal{J})) \in n(\mathcal{N}_0(A/\mathcal{J})) \times n(\mathcal{H}^*(A/\mathcal{J}))|[\xi]\mathcal{J} \in A/\mathcal{J}\}$ need not be mappings.

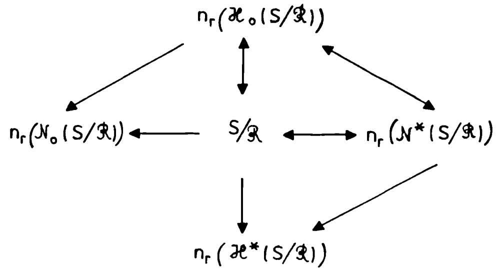
3. Semigroups and their ideals

Let **S** be a semigroup.

Let $\mathscr{R}(\mathbf{S})(\mathscr{L}(\mathbf{S}))[\mathscr{Y}(\mathbf{S})]$ denote the system of all right (left) [two-sided] ideals of **S** (including the empty set).

Let $R(a)$ $(L(a))$ $[J(a)]$ be the principal right (left) [two-sided] ideal generated by a.

The relations (Green's relations [6, 8]) $\mathscr{R}=\{(x,y) \in \mathbf{S}\times\mathbf{S} | R(x)=R(y)\}$, $\mathscr{L}=\{(x,y) \in \mathbf{S}\times\mathbf{S} | L(x) = L(y)\}$ and $\mathscr{Y}=\{(x,y) \in \mathbf{S}\times\mathbf{S} | J(x) = J(y)\}$ are equivalence relations on **S**. Let $R_a(L_a)[J_a]$ denote the $\mathscr{R}(\mathscr{L})[\mathscr{Y}]$-equivalence class containing a. Let $\mathbf{S}/\mathscr{R} = \{R_a | a \in \mathbf{S}\}$, $\mathbf{S}/\mathscr{L} = \{L_a | a \in \mathbf{S}\}$ and $\mathbf{S}/\mathscr{Y} = \{J_a | a \in \mathbf{S}\}$.



Diag. 6

Now we can introduce the following relations on $\mathbf{S}/\mathscr{R}$, $\mathbf{S}/\mathscr{L}$ and $\mathbf{S}/\mathscr{Y}$:

$$\begin{aligned} R_x \leq R_y &\quad \text{iff} \quad R(x) \subseteq R(y), \\ L_x \leq L_y &\quad \text{iff} \quad L(x) \subseteq L(y), \\ J_x \leq J_y &\quad \text{iff} \quad J(x) \subseteq J(y). \end{aligned}$$

Then $\langle \mathbf{S}/\mathscr{R}, \leq \rangle$, $\langle \mathbf{S}/\mathscr{L}, \leq \rangle$ and $\langle \mathbf{S}/\mathscr{Y}, \leq \rangle$ are partially ordered sets (see [6, 8]).

Theorem 15′. *Let* **S** *be a semigroup. Then the mappings*

$$\begin{aligned} n_r&\colon \mathscr{S}(\mathbf{S}/\mathscr{R}) \to \mathscr{R}(\mathbf{S}),\ n_r(S) = \cup S, \\ n_l&\colon \mathscr{S}(\mathbf{S}/\mathscr{L}) \to \mathscr{L}(\mathbf{S}),\ n_l(S) = \cup S \quad \textit{and} \\ n_j&\colon \mathscr{S}(\mathbf{S}/\mathscr{Y}) \to \mathscr{Y}(\mathbf{S}),\ n_j(S) = \cup S \end{aligned}$$

are monotone isomorphisms.

The proof is similar to the proof of Theorem 15. *It is based on the fact that the union of an arbitrary system of right (left) [two-sided] ideals is a right (left) [two-side] ideal.*

Remark. $n_r(H_0(R_a)) = R(a)$, $n_l(H_0(L_a)) = L(a)$ and $n_j(H_0(J_a)) = J(a)$. Hence $n_r(\mathscr{H}_0(\mathbf{S}/\mathscr{R}))$ $(n_l(\mathscr{H}_0(\mathbf{S}/\mathscr{L})))$ $[n_j(\mathscr{H}_0(\mathbf{S}/\mathscr{Y}))]$ is the system of all principal right (left) [two-sided] ideals of **S**.

Definition 7. *If the right (left) [two-sided] ideal* $n_r(H(R_a)) \neq \emptyset$, $(n_l(H(L_a)) \neq \emptyset)$ $[n_j(H(J_a)) \neq \emptyset]$, *we shall call it the* $H_r(H_l)$ $[H_j]$*-ideal of* $\mathbf{S}$.

If the right (left) [two-sided] ideal $n_r(N(R_a)) \neq \emptyset$ $(n_l(N(L_a)) \neq \emptyset)$ $[n_j(N(J_a)) \neq \emptyset]$, *we shall call it the* $N_r(N_l)$ $[N_j]$*-ideal of* $\mathbf{S}$.

The right (left) [twosided] ideal $n_r(N_0(R_a))$ $(n_l(N_0(L_a)))$ $[n_j(N_0(J_a))]$ *will be called the* $N_{0r}(N_{0l})$ $[N_{0j}]$*-ideal of* $\mathbf{S}$.

Remark. n_r $(\mathscr{H}(\mathbf{S}/\mathscr{R}))$ $(n_l(\mathscr{H}(\mathbf{S}/\mathscr{L})))$ $[n_j(\mathscr{H}(\mathbf{S}/\mathscr{Y}))]$ is the set of all the $H_r(H_l)$ $[H_j]$-ideals of $\mathbf{S}$, $n_r(\mathscr{N}(\mathbf{S}/\mathscr{R}))$ $(n_l(\mathscr{N}(\mathbf{S}/\mathscr{L})))$ $[n_j(\mathscr{N}(\mathbf{S}/\mathscr{Y}))]$ is the set of all the N_r (N_l) $[N_j]$-ideals of $\mathbf{S}$ and n_r $(\mathscr{N}_0(\mathbf{S}/\mathscr{R}))$ $(n_l(\mathscr{N}_0(\mathbf{S}/\mathscr{L})))$ $[n_j(\mathscr{N}_0(\mathbf{S}/\mathscr{Y}))]$ is the set of all the $N_{0r}(N_{0l})$ $[N_{0j}]$-ideals of $\mathbf{S}$.

Definition 8. *If* $K_r = \cap(\mathscr{R}(\mathbf{S})\backslash\{\emptyset\} \neq \emptyset$ $(K_l = \cap(\mathscr{L}(\mathbf{S}\backslash\{\emptyset\}) \neq \emptyset)$ $[K_j = \cap(\mathscr{Y}(\mathbf{S})\backslash\{\emptyset\}) \neq \emptyset]$, *it is called the right (left) [two-sided] kernel of* $\mathbf{S}$.

Definition 9. *A right (left) [two-sided] ideal* $R(L)[J]$, $\emptyset \neq R \neq \mathbf{S}$ $(\emptyset \neq L \neq \mathbf{S})$ $[\emptyset \neq J \neq \mathbf{S}]$ *of a semigroup* $\mathbf{S}$ *is called a maximal right (left) [two-sided] ideal of* $\mathbf{S}$ *iff there is no right (left) [two-sided] ideal* $R'(L')$ $[J']$ *of* $\mathbf{S}$ *such that* $R \subset R' \subset \mathbf{S}$, $R \neq R' \neq \mathbf{S}$ $(L \subset L' \subset \mathbf{S},\ L \neq L' \neq \mathbf{S})$ $[J \subset J' \subset \mathbf{S},\ J \neq J' \neq \mathbf{S}]$.

From Theorems 4, 5, 7, 8, 15′ and 14 we get results for right (left) [two-sided] ideals of a semigroup $\mathbf{S}$. We shall formulate these results only for right ideals.

Theorem 16′. *Let* $\mathbf{S}$ *be a semigroup and* K_r *be the right kernel of* $\mathbf{S}$. *Then the following statements are true:*

a) K_r *is the intersection of all the principal right ideals of* $\mathbf{S}$.
b) K_r *is the intersection of all the* H_r*-ideals of* $\mathbf{S}$.
c) K_r *is the intersection of all the* N_r*-ideals of* $\mathbf{S}$.
d) K_r *is the intersection of all the* N_{0r}*-ideals of* $\mathbf{S}$.

Moreover we have the diagram 6.

Theorem 17′. ([1, 10]) *Let* $\mathbf{S}$ *be a semigroup.* C *is a maximal right ideal of* $\mathbf{S}$ *iff* $\emptyset \neq C = \mathbf{S}\backslash R_a$ *and* R_a *is a maximal element of* $\mathbf{S}/\mathscr{R}$.

Remark. All these results are also true for grupoids. For another way how to obtain the results for semigroups and grupoids from results for B-algebras see [1, 3].

If $\langle A, F\rangle$ is a B-algebra, then there exists a unary algebra $\langle A, F^*\rangle$ such that $\langle A/\mathscr{I}(F), \leqslant\rangle = \langle A/\mathscr{I}(F^*), \leqslant\rangle$. This is an unpublished result of I. Abrhan and Theorem 18 is its generalization.

For Theorem 18 see also [5] Theorem II.5.6. and Exercise 5(a) following this Theorem.

REFERENCES

[1] АБРГАН, И.: О максимальных подалгебрах в унарных алгебрах. Mat. Čas., 24, 1974, 113—128.
[2] АБРГАН, И.: О минимальных множествах образующих в унарных алгебрах. Mat. Čas., 25, 1975, 305—317.
[3] АБРГАН, И.: О $\mathcal{I}$-подалгебрах в унарных алгебрах, о простых идеалах и $\mathcal{I}$-идеалах в группоидах и полугруппах. Math. Slovaca, 28, 1978, 61—80.
[4] BOSÁK, J.: B-pologrupy. Mat-fyz. Čas., 11, 1961, 32—44.
[5] COHN, P. M.: Universal algebra, Harper and Row, New York, 1965.
[6] CLIFFORD, A. H., PRESTON, G. B.: The algebraic theory of semigroups. Vol. I., Providence, R.I., 1961.
[7] GRÄTZER, G.: Universal Algebra, D. Van Nostrand, Princeton, N.I., 1968.
[8] PETRICH, M.: Introduction to semigroups, Merrill, Colombus, Ohio, 1973.
[9] SCHWARZ, Š.: Prime ideals and maximal ideals in semigroups. Czechosl. Math. J., 12, 94, 1969.
[10] SCHWARZ, Š.: Maximálne ideály a štruktúra pologrúp. Mat.-fyz. Čas., 1953, 17—39.

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ЗАМЕЧАНИЕ О ЧАСТИЧНО УПОРЯДОЧЕННЫХ МНОЖЕСТВАХ, УНИВЕРСАЛЬНЫХ АЛГЕБРАХ И ПОЛУГРУППАХ

Роберт Шулка

Резюме

Применяя частично упорядоченные множества мы доказываем, что непустое пересечение подалгебр некоторого класса универсальных алгебр содержащего класс унарных алгебр можно получить также в виде пересечения некоторых сообственных подсистем подалгебр этих универсальных алгебр.