Mathematica Slovaca

Jana Farková

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Mathematica Slovaca, Vol. 29 (1979), No. 1, 63--68

Persistent URL: http://dml.cz/dmlcz/136199

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ON GENERALIZED DABROUX AND CONNECTIVITY FUNCTIONS

JANA FARKOVÁ

1. Introduction and notations

It is known that the Kuratowski—Sierpinski theorem, which asserts that a function $f\colon E_1 \to E_1$ of Baire class 1 is Darboux if and only if it is a connected subset of the plane E_2 (as usually the function is identified with its graph), is not valid already in the case $f\colon E_2 \to E_1$, see [2]. This is due to the fact that already in E_2 the connected subsets form a substantially richer and more complicated system as in E_1. For this reason usually the concept of a Darboux function is modified to a given base of open subset of the space, to obtain generalizations of results from E_1 for arbitrary topological spaces.

If not specified, in the following X will be a locally connected metric space with a given base $\mathscr{B}$ of open connected subsets and all considered functions are defined on X and have real values.

We say that a function f is Darboux with respect to the base $\mathscr{B}$, shortly $\mathscr{B}$-Darboux ($f \in \mathscr{D}(\mathscr{B})$), if $f(\bar{B})$ is connected for each $B \in \mathscr{B}$.

Similarly, we say that a function f is a connectivity function with respect to the base $\mathscr{B}$, shortly $\mathscr{B}$-connectivity function ($f \in \mathscr{C}(\mathscr{B})$), if $f/\bar{B} = \{(x, f(x))\colon x \in \bar{B}\}$ is connected in $X \times E_1$ for each $B \in \mathscr{B}$.

For thus defined classes of functions the following assertion, which in some sense generalizes the Kuratowski-Sierpinski theorem, was proved in [3]:

(A) If $f\colon X \to Y$ is of Baire class 1, where X is E_n with a base $\mathscr{B}$ having some special properties and Y is a separable metric space, then $f \in \mathscr{D}(\mathscr{B}) \Leftrightarrow f \in \mathscr{C}(\mathscr{B})$.

Somewhat weaker and more general than the Darboux property is the Darboux property in the sense of Radakovič. For functions of a real variable it was introduced in [8]. In [5] it was generalized for functions on a topological space with respect to its base. In [1] these functions were studied in connection with the investigation of the uniform closure of Darboux functions.

We say that a function f is Darboux in the sense of Radakovič with respect to the base $\mathscr{B}$ ($f \in \mathscr{D}_0(\mathscr{B})$) if $\overline{f(\bar{B})}$ is connected for each $B \in \mathscr{B}$.

Similarly as the Darboux property in the sense of Radakovič, we have the following generalization or weakening of the connectivity function concept:

We say that a function f is a connectivity function in the sense of Radakovič with respect to the base $\mathscr{B}(f \in \mathscr{C}_0(\mathscr{B}))$ if $\overline{f/\bar{B}}$ is connected in $X \times E_1$ for each $B \in \mathscr{B}$.

Finally we define the classes of functions $\mathscr{D}_0$ and $\mathscr{C}_0$ as follows: $f \in \mathscr{D}_0$, or $f \in \mathscr{C}_0$ if $\overline{f(C)}$, or $\overline{f/C}$ is connected for each connected subset $C \subset X$, respectively.

Naturally the question arises what relations there are between these classes of functions, particularly, whether the analogue of assertion (A) above is valid for $\mathscr{D}_0(\mathscr{B})$ and $\mathscr{C}_0(\mathscr{B})$. In Theorem 1 we show that a similar assertion holds, even without the assumption that f is of Baire class 1, however under a special assumption on the base $\mathscr{B}$. This assumption is introduced by the next

Definition. *We say that the base $\mathscr{B}$ of X has the* (*) *property if $B_1 \cap B_2 \in \mathscr{B}$ for each B_1, $B_2 \in \mathscr{B}$, $B_1 \cap B_2 \neq \emptyset$.*

Clearly in E_n $(n > 1)$ the base of all open spheres as well as the base of all open connected subsets do not have the (*) property. On the other hand the base of all open intervals in E_n and the base of all open convex subsets have the (*) property.

$C_0(f, x)$ as usually will denote the cluster set of f at x, i.e., the set of all limit numbers of f at x ($y \in C_0(f, x) \Leftrightarrow$ there is a sequence $\{x_n\}$ such that $f(x_n) \to y$ and $x_n \to x$).

$C_0^{\bar{B}}(f, x)$, where $B \in \mathscr{B}$ and $x \in \bar{B}$, will denote the relative cluster set of f at x with respect to $\bar{B}$, which means that $y \in C_0^{\bar{B}}(f, x) \Leftrightarrow$ there is a sequence $\{x_n\}$ such that $x_n \in \bar{B}$, $x_n \to x$ and $f(x_n) \to y$.

Clearly $C_0^{\bar{B}}(f, x) = C_0(f, x)$ when $x \in B$.

2. The classes $\mathscr{D}_0(\mathscr{B})$ and $\mathscr{C}_0(\mathscr{B})$.

We prove now some properties of the classes $\mathscr{D}_0(\mathscr{B})$ and $\mathscr{C}_0(\mathscr{B})$.

It is known that for $X = E_1$ and $\mathscr{B}$ being the base of all open intervals, $f \in \mathscr{C}(\mathscr{B})$ if and only if f/E_1 is connected, i.e. if f (its graph) is connected. As the following simple example shows, for $\mathscr{C}_0(\mathscr{B})$ this is not so.

Example 1. Let $f: E_1 \to E_1$ be defined as follows: $f(x) = \sin 1/x$ if $x > 0$, $f(x) = 0$ if $x \leq 0$ and x is rational, and $f(x) = 1$ if $x < 0$ and x is irrational. Then clearly $\overline{f/E_1}$ is connected, however $\overline{f/\langle a, b\rangle}$ for $a < b \leq 0$ is not. Thus the connectivity of $\overline{f/E_1}$ does not imply $f \in \mathscr{C}_0(\mathscr{B})$.

If $f \in \mathscr{D}(\mathscr{B})$, then it is easy to see that $f(O)$ is connected for each open connected subset $O \subset X$. Similarly we immediately have the next

Proposition 1. *Let $f \in \mathscr{D}_0(\mathscr{B})$. Then $\overline{f(O)}$ is connected for each open connected subset $O \subset X$.*

The following proposition will be substantially used in the proof of Theorem 1:

Proposition 2. *Let the base $\mathscr{B}$ of X have the (*) property and let $f \in \mathscr{D}_0(\mathscr{B})$. Then $C_0^{\bar{B}}(f, x)$ is a closed interval for each $B \in \mathscr{B}$ and each $x \in \bar{B}$.*

Proof. Let $B \in \mathscr{B}$ and let $x_0 \in \bar{B}$. Put $\alpha = \inf C_0^{\bar{B}}(f, x_0)$, $\beta = \sup C_0^{\bar{B}}(f, x_0)$ and let $\alpha < \beta$ (otherwise the proof is complete). Let $\gamma \in (\alpha, \beta)$. Then there are $x_n, y_n \in \bar{B}$, $n = 1, 2, \ldots$ such that $x_n \to x_0$, $y_n \to x_0$, $f(x_n) \to \alpha$, $f(y_n) \to \beta$ and $f(x_n) < \gamma < f(y_n)$ for each $n = 1, 2, \ldots$

For each $k = 1, 2, \ldots$ take $B_k \in \mathscr{B}$ and n_k so that $x_0 \in B_k \subset O(x_0, 1/k)$ and x_{n_k}, $y_{n_k} \in B_k$, where $O(x_0, 1/k)$ is the open sphere with the centre x_0 and the radius $1/k$. Since $B_k \cap B \in \mathscr{B}$ by the (*) property of $\mathscr{B}$ and since $f \in \mathscr{D}_0(\mathscr{B})$,

$$\gamma \in \langle f(x_{n_k}), f(y_{n_k}) \rangle \subset \langle \inf_{x \in \overline{B_k \cap B}} f(x), \sup_{x \in \overline{B_k \cap B}} f(x) \rangle = f(\overline{B_k \cap B})$$

for each $k = 1, 2, \ldots$ Hence for each $k = 1, 2, \ldots$ there is an $z_k \in \overline{B_k \cap B}$ such that $|f(z_k) - \gamma| < 1/k$. Since $x_0 \in B_k \subset O(x_0, 1/k)$ for each k, $z_k \in \bar{B}$, $z_k \to x_0$ and $f(z_k) \to \gamma$. Thus $\gamma \in C_0^{\bar{B}}(f, x_0)$, what we wanted to show.

In the special case when X is locally compact and the base is such that each $B \in \mathscr{B}$ is relatively compact in X we obtain more, namely the following generalization of Theorem 3.1. from [1]. (In [1] $X = E_1$.)

Proposition 3. *Let the base $\mathscr{B}$ of X have the (*) property and let each $B \in \mathscr{B}$ be relatively compact in X. Then the following conditions are equivalent:*

1) $f \in \mathscr{D}_0(\mathscr{B})$,

2) $C_0^{\bar{B}}(f, x)$ is a closed interval in E_1 for each $B \in \mathscr{B}$ and each $x \in \bar{B}$, and

3) $\bigcup_{x \in \bar{B}} C_0^{\bar{B}}(f, x) = \langle \inf_{x \in \bar{B}} f(x), \sup_{x \in \bar{B}} f(x) \rangle$ for each $B \in \mathscr{B}$.

Proof. 1) $\Rightarrow$ 2) by Proposition 2.

2) $\Rightarrow$ 3). Let $B \in \mathscr{B}$. Put $A = \bigcup_{x \in \bar{B}} C_0^{\bar{B}}(f, x)$ and $I = \langle \inf_{x \in \bar{B}} f(x), \sup_{x \in \bar{B}} f(x) \rangle$. First we show that $\bar{A} = I$. Suppose that $\bar{A} \neq I$. Then there is a non-empty open interval $(a, b) \subset I - \bar{A}$. Denote $B_1 = \{x : x \in \bar{B}, f(x) \leqslant a\}$ and $B_2 = \{x : x \in \bar{B}, f(x) \geqslant b\}$. Clearly $B_1 \neq \emptyset \neq B_2$ and $\bar{B} = B_1 \cup B_2$. Since $\bar{B}$ is connected, without loss of generality we may suppose that $\bar{B}_1 \cap B_2 \neq \emptyset$. Let $x_0 \in \bar{B}_1 \cap B_2$. Then $f(x_0) \geqslant b$ and there is a sequence $x_n \in B_1$, $n = 1, 2, \ldots$ such that $x_n \to x_0$. From the sequence $\{f(x_n)\}$ take a convergent subsequence $\{f(x_{n_k})\}$. Then $y = \lim_{k \to \infty} f(x_{n_k}) \leqslant a$. In this way $C_0^{\bar{B}}(f, x_0)$ contains the point $f(x_0) \geqslant b$, as well as the point $y \leqslant a$, which contradicts to the facts that $C_0^{\bar{B}}(f, x_0)$ is a closed interval and $C_0^{\bar{B}}(f, x_0) \cap (a, b) = \emptyset$. Thus $\bar{A} = I$.

Let now $\gamma \in I$. Then for each $n = 1, 2, \ldots$ there are $x_n \in \bar{B}$ and $y_n \in C_0^{\bar{B}}(f, x_n) \cap (\gamma - 1/n, \gamma + 1/n)$. Since $\bar{B}$ is compact by assumption we may suppose without loss of generality that the sequence $\{x_n\}$ is convergent. Let $x_0 = \lim_{n \to \infty} x_n$. Then $x_0 \in \bar{B}$.

Further for each n there must exist $z_n \in O(x_n, 1/n) \cap \bar{B}$ such that $f(z_n) \in (\gamma - 1/n, \gamma + 1/n)$. Hence $z_n \to x_0$, $f(z_n) \to \gamma$ and therefore $\gamma \in A$. Thus $A = \bar{A} = I$.

3) $\Rightarrow$ 1). Let $B \in \mathcal{B}$. We have to show that $\overline{f(\bar{B})} = I$. Let $\gamma \in I$. Then by 3) there is an $x \in \bar{B}$ such that $\gamma \in C_0^{\bar{B}}(f, x)$, Hence for each $\varepsilon > 0$ there is an $z \in \bar{B}$ such that $f(z) \in (\gamma - \varepsilon, \gamma + \varepsilon)$.

Since the base of open spheres in E_n does not have the (*) property, this proposition cannot be applied to this case. However, a similar assertion for this special case was proved in [7].

Proposition 4. *Let $\mathcal{B}$ be a base of X. Then $\mathcal{C}_0(\mathcal{B}) \subset \mathcal{D}_0(\mathcal{B})$.*

Proof. Let $f \in \mathcal{C}_0(\mathcal{B})$, let P_1 be the projection of $X \times E_1$ onto E_1 and let $B \in \mathcal{B}$. Clearly $\overline{P_1(f/\bar{B})} \subset \overline{P_1(\overline{f/\bar{B}})} \subset \overline{\overline{P_1(f/\bar{B})}}$. Since $\overline{P_1(f/\bar{B})} = \overline{f(\bar{B})}$ and since $\overline{f/\bar{B}}$ is connected, $\overline{f(\bar{B})}$ is connected. Thus $f \in \mathcal{D}_0(\mathcal{B})$.

Our main result is the following

Theorem 1. *Let the base $\mathcal{B}$ of X have the (*) property. Then $\mathcal{D}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{B})$.*

Proof. By Proposition 4 it is enough to show that $\mathcal{D}_0(\mathcal{B}) \subset \mathcal{C}_0(\mathcal{B})$. Let $f \in \mathcal{D}_0(\mathcal{B})$ and suppose that $f \notin \mathcal{C}_0(\mathcal{B})$. Then there is a $B \in \mathcal{B}$ such that $\overline{f/\bar{B}} = A_1 \cup A_2$, where $A_1 \neq \emptyset \neq A_2$, and $A_1 \cap \bar{A}_2 = \bar{A}_1 \cap A_2 = \emptyset$. Put $B_1 = \{x \in \bar{B}, (x, f(x)) \in A_1\}$ and $B_2 = \{x \in \bar{B}, (x, f(x)) \in A_2\}$.

Clearly $\bar{B} = B_1 \cup B_2$, and $B_1 \neq \emptyset \neq B_2$ ($B_1 = \emptyset \Rightarrow f/\bar{B} \subset A_2 \Rightarrow \overline{f/\bar{B}} \subset \bar{A}_2 \Rightarrow A_1 = \emptyset$). Since $\bar{B}$ is connected, without loss of generality we may suppose that there is a point $x_0 \in \bar{B}_1 \cap B_2$. But then $(x_0, f(x_0)) \in A_2$ and there is a sequence $x_n \in B_1$, $n = 1, 2, \ldots$ such that $x_n \to x_0$. Let $\{x_{n_k}\}$ be such a subsequence of $\{x_n\}$ that the sequence $\{f(x_{n_k})\}$ is convergent and put $y = \lim_{k \to \infty} f(x_{n_k})$. Then $y \in C_0^{\bar{B}}(f, x_0)$. Since $x_{n_k} \in B_1$ for each $k = 1, 2, \ldots$, $(x_{n_k}, f(x_{n_k})) \in A_1$, and therefore $\lim_{k \to \infty} (x_{n_k}, f(x_{n_k})) = (x_0, y) \in \bar{A}_1 = A_1$. Hence $y \neq f(x_0)$ ($\bar{A}_1 \cap A_2 = \emptyset$).

Since the base $\mathcal{B}$ has the (*) property $\langle \min(f(x_0), y), \max(f(x_0), y) \rangle \subset C_0^{\bar{B}}(f, x_0)$ by Proposition 2, hence $G = \{(x_0, v) : v \in \langle \min(f(x_0), y), \max(f(x_0), y) \rangle\} \subset \overline{f/\bar{B}} = A_1 \cup A_2$.

Since $G \cap A_1 \neq \emptyset \neq G \cap A_2$, and since G is connected, A_1 and A_2 cannot be separated, a contradiction. The theorem is proved.

Remark. If $X = E_1$ and $\mathcal{B}$ is the base of open intervals in E_1, then clearly $\mathcal{B}$ has the (*) property, hence $\mathcal{D}_0(\mathcal{B}) = \mathcal{C}_0(\mathcal{B})$. But then $\mathcal{D}_0(\mathcal{B}) \cap \mathcal{B}_1 = \mathcal{C}_0(\mathcal{B}) \cap \mathcal{B}_1$, where $\mathcal{B}_1$ is the first Baire class. Hence we have the analog of the Kuratowski—Sierpinski theorem, which asserts that $\mathcal{D}(\mathcal{B}) \cap \mathcal{B}_1 = \mathcal{C}(\mathcal{B}) \cap \mathcal{B}_1$.

From results of [5] and [6] it follows that $\mathcal{D}(\mathcal{B})\cap\mathcal{B}_1 = \mathcal{D}_0(\mathcal{B})\cap\mathcal{B}_1$. Hence we have the following

Corollary. *Let $X=E_1$ and let $\mathcal{B}$ be the base of all open intervals in E_1. Then*

$$\mathcal{D}(\mathcal{B})\cap\mathcal{B}_1=\mathcal{D}_0(\mathcal{B})\cap\mathcal{B}_1=\mathcal{C}(\mathcal{B})\cap\mathcal{B}_1=\mathcal{C}_0(\mathcal{B})\cap\mathcal{B}_1 .$$

In this real case also clearly $\mathcal{D}_0(\mathcal{B})=\mathcal{D}_0$ and $\mathcal{C}_0(\mathcal{B})=\mathcal{C}_0$, hence $\mathcal{D}_0=\mathcal{C}_0$ by Theorem 1. The following simple example shows that for $X=E_2$ this is not true.

Example 2. Define $f\colon E_2\to E_1$ as follows: $f(x, y) = \cos x$ for $x\leq 0$, $f(x, y) = \sin 1/x$ for $x>0$. Clearly $f\in\mathcal{D}_0\cap\mathcal{B}_1$, but $f\notin\mathcal{C}_0$ (for example, the set $C = \{(x, y)\colon x>0, y=\sin 1/x\} \cup \{(0, 0)\}$ is connected, hower $\overline{f/C}$ is not connected.

The following theorem is concerned with the relationships between the classes $\mathcal{D}_0(\mathcal{B})$ and $\mathcal{D}_0$ and $\mathcal{C}_0(\mathcal{B})$ and $\mathcal{C}_0$ in general.

Theorem 2. *Let $f\in\mathcal{D}_0(\mathcal{B})$ $(f\in\mathcal{C}_0(\mathcal{B}))$ be such that $C_0(f, x) \subset \overline{f(C)}$ for each non-degenerated connected subset $C\subset X$,* with $x\in C$. *Then $f\in\mathcal{D}_0(f\in\mathcal{C}_0)$.*

We omit the proof of this theorem, since it is very similar to the proof of Theorem 2.4. from [4], which gives a sufficient condition that a function $f\in\mathcal{D}(\mathcal{B})$ maps each connected subset into a connected subset.

Let $\mathcal{A}$ denote the class of all functions $f\colon X\to E_1$ for which $C_0(f, x) \subset \overline{f(C)}$ for each non-degenerated connected $C\subset X$ and each $x\in C$. Then from Theorems 1 and 2 we immediately have the next

Corollary. *Let the base $\mathcal{B}$ of X have the (*) property. Then $\mathcal{D}_0(\mathcal{B})\cap\mathcal{A} = \mathcal{C}_0(\mathcal{B})\cap\mathcal{A} = \mathcal{D}_0\cap\mathcal{A} = \mathcal{C}_0\cap\mathcal{A}$.*

Finally we give an example of a function $f\in\mathcal{D}_0(\mathcal{B})$ such that $f\notin\mathcal{D}_0$. This example shows that the condition $C_0(f, x) \subset \overline{f(C)}$ from Theorem 2 cannot be omitted.

Example 3. Let $X=E_2$ and let $\mathcal{B}$ be the base of open intervals in E_2. Let further $\varphi\colon E_1\to E_1$ be a function which maps each non degenerated interval onto E_1. Define $f\colon E_2\to E_1$ as follows: $f(x, y) = \varphi(x))$ if $y=0$, and $f(x, y) = x$ if $y\neq 0$. Then it is easy to see that $f\in\mathcal{D}_0(\mathcal{B})$ (even $f\in\mathcal{D}(\mathcal{B})$).

Take x_0 so that $\varphi(x_0)\neq x_0$. Then $C=\{(x_0, y)\colon y\in\langle 0, 1\rangle\}$ is a non-degenerated connected subset of E_2, but $f(C) = \{x_0\} \cup \{\varphi(x_0)\}$, hence $f(C)$ is not connected. Thus $f\notin D_0$. Since for the point $(x_0, 0)\in C$ we have $C_0(f, (x_0, 0))=E_1$, the assumption $C_0(f, x) \subset \overline{f(C)}$ from Theorem 2 is really not satisfied.

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Received June 1, 1977

Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

ОБ ОБОБЩЕННЫХ ФУНКЦИЯХ: ДАРБУ И СО СВЯЗНЫМ ГРАФИКОМ

Яна Фаркова

Резюме

В этой статье вводятся и рассматриваются некоторые классы обобщенных функций: Дарбу и со связным графиком. Эти функции определены на метрическом пространстве. Свойство Дарбу и связность графика относятся к некоторой базе и кроме того понимаются в смысле Радаковича.