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# Z-SUBGROUPS OF ORDERED GROUPS

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In this paper the concept of a $z$-subgroup of a lattice-ordered group is generalized for an ordered group (henceforth a $po$-group). Properties of $z$-subgroups are investigated for the case of a 2-isolated regular $po$-group with the property (II).

A $po$-group $G=(G, +, \leqq)$ will be called *2-isolated* if the holds: If $a \in G$ satisfies $a \geqq -a$, then $a \geqq 0$. A $po$-group $G$ is said to be *regular* if the existence of inf $(a, b)$ in $G^+$ implies the existence of inf $(a, b)$ in $G$ for $a, b \in G^+$. Way say that a $po$-group $G$ has the property (II) if for each $a \in G$ there exists $a \vee -a = \sup_G (a, -a)$. (See also [3].)

We shall denote the set of all directed convex subgroups of a $po$-group $G$ by $\Gamma(G)$, the set of all convex subsemigroups with 0 of $G^+$ by $\bar{\Gamma}(G)$. In [2, Theorems 2.1, 2.2, 2.3] it is proved that $\Gamma(G)$, $\bar{\Gamma}(G)$ ordered by the inclusion form isomorphic complete lattices and that the infimum in $\bar{\Gamma}(G)$ is determined by the intersection.

Let $G$ be a $po$-group, $a_1, \ldots, a_n \in G$. We denote $U(a_1, \ldots, a_n) = \{x \in G;\ a_i \leqq x$ for all $i=1, \ldots, n\}$, $L(a_1, \ldots, a_n) = \{y \in G;\ y \leqq a_i$ for all $i=1, \ldots, n\}$. For any element $x \in G$ we write $|x| = U(x, -x)$. $x, y \in G$ will be called *disjoint* (notation $x\delta y$) if there exist $x_1 \in |x|, y_1 \in |y|$ such that $x_1 \wedge y_1 = 0$. ($x_1 \wedge y_1$ means $\inf_G (x_1, y_1)$.) For $\Phi \neq A \subseteq G$ we denote $A^\delta = \{x \in G;\ a\delta x$ for all $a \in A\}$ For $x \in G$ we write $x^\delta = \{x\}^\delta$. If $A^\delta \neq \Phi$, then $A^\delta$ will be called a *$\delta$-polar* of the set $A$. (See [3].) $A^{\delta\delta}$ means $(A^\delta)^\delta$. If $A^\delta \neq \Phi$, then $A \subseteq A^{\delta\delta}$, $A^\delta = A^{\delta\delta\delta}$. For $A^\delta \neq \Phi$, $B^\delta \neq \Phi$, $A \subseteq B$ implies $B^\delta \subseteq A^\delta$. By [3, Proposition 2.5], any $\delta$-polar of a 2-isolated $po$-group with the property (II) is an element of $\Gamma(G)$. Moreover, the set of all $\delta$-polars of a 2-isolated regular $po$-group with the property (II) ordered by the inclusion is a complete Boolean algebra and the infimum is formed by the intersection. ([3, Theorem 2.6].)

Finally, if $G$ is a $po$-group, then $M \in \Gamma(G)$ will be called a *r-subgroup* of $G$ if for any $a \in M$, $b \in G$, $a^\delta = b^\delta$ implies $b \in M$. (For an $l$-group see e.g. [1].)

**Theorem 1.** *Let $G$ be a 2-isolated regular po-group with the property (II), $M \in \Gamma(G)$. Then the $z$-subgroup generated by $M$ is $\bar{M} = \bigvee_{a \in M} a^{\delta\delta}$.*

Proof. Let $M \in \Gamma(G)$, $x$, $y \in \bar{M}$. Then by [3, Lemma of Proposition 2.8] there exist elements $a$, $b \in M^+$ such that $x \in a^{\delta\delta}$, $y \in b^{\delta\delta}$, hence $x - y \in a^{\delta\delta} \vee b^{\delta\delta} = (a+b)^{\delta\delta}$, therefore $x - y \in \bar{M}$, and so $\bar{M}$ is a subgroup of $G$.

Let $a \in \bar{M}$ (i.e. $a \in b^{\delta\delta}$, where $b \in M$), $|x| \supseteq |a|$. Then $x \vee -x \leqq a \vee -a$, and thus $x^{\delta} \supseteq a^{\delta}$, hence $x^{\delta\delta} \subseteq a^{\delta\delta}$. Therefore $x \in x^{\delta\delta} \subseteq a^{\delta\delta} \subseteq b^{\delta\delta}$ holds, and this means $x \in \bar{M}$. Hence, by [3, Lemma 2 of Proposition 2.5], we obtain $\bar{M} \in \Gamma(G)$.

Now, let $x \in \bar{M}$, $y \in G$, $x^{\delta} = y^{\delta}$. Then there exists $a \in M$ such that $x \in a^{\delta\delta}$, thus $y \in y^{\delta\delta} = x^{\delta\delta} \subseteq a^{\delta\delta}$ holds, and consequently $y \in \bar{M}$. Therefore $\bar{M}$ is a $z$-subgroup of $G$.

Let us show that $\bar{M}$ is the smallest $z$-subgroup of $G$ containing $M$. Let us suppose that for a $z$-subgroup $Z$ of $G$ there holds $M \subseteq Z$. If $0 \leqslant a \in M$, $0 \leqq x \in a^{\delta\delta}$ (i.e. $x \in \bar{M}^+$), then by [3, Proposition 2.8] $(x+a)^{\delta} = x^{\delta} \cap a^{\delta} = a^{\delta}$, hence $x + a \in Z$. Consequently $0 \leqq x \leqq x + a$, $x + a \in Z$, therefore by the convexity of $Z$ we obtain $x \in Z$. This implies $\bar{M}^+ \subseteq Z^+$, thus also $\bar{M} \subseteq Z$.

Let $G$ be a *po*-group. Then $S \in \bar{\Gamma}(G)$ will be called a *$z$-subsemigroup* of $G^+$ if $x^{\delta} = y^{\delta}$ implies $y \in S$ for each $x \in S$, $y \in G^+$. We denote the set of all $z$-subsemigroups of $G^+$ by $\bar{\mathscr{Z}}(G)$, the set of all $z$-subgroups of $G$ by $\mathscr{Z}(G)$.

In [2, Theorem 2.1] it is proved that the mapping $\varphi\colon \Gamma(G) \to \bar{\Gamma}(G)$ given by $A\varphi = A^+$ for each $A \in \Gamma(G)$ is an isomorphism between the sets $\Gamma(G)$ and $\bar{\Gamma}(G)$ ordered by the inclusion and that $S\varphi^{-1} = \langle S \rangle$ for each $S \in \bar{\Gamma}(G)$, where $\langle S \rangle$ is the subgroup of $G$ generated by $S$.

**Theorem 2.** *Let $G$ be a 2-isolated regular po-group with the property (II). If $M \in \mathscr{Z}(G)$, then $M\varphi \in \bar{\mathscr{Z}}(G)$ and if $S \in \bar{\mathscr{Z}}(G)$, then $S\varphi^{-1} \in \mathscr{Z}(G)$.*

Proof. a) Let $M \in \mathscr{Z}(G)$, $x \in M^+$, $y \in G^+$, $x^{\delta} = y^{\delta}$. Then $y \in M \cap G^+ = M^+$, therefore $M\varphi \in \bar{\mathscr{Z}}(G)$.

b) Let $S \in \bar{\mathscr{Z}}(G)$, $u \in \langle S \rangle$, $v \in G$, $u^{\delta} = v^{\delta}$. Then $(u \vee -u)^{\delta} = (v \vee -v)^{\delta}$. Hereby $u \vee -u \in \langle S \rangle^+ = S$, $v \vee -v \in G^+$, thus $v \vee -v \in S$. And since $-(v \vee -v) \leqq v \leqq v \vee -v$, there holds (by the convexity of $\langle S \rangle$) $v \in \langle S \rangle$. Therefore $S\varphi^{-1} \in \mathscr{Z}(G)$.

**Theorem 3.** *If $G$ is a 2-isolated regular po-group with the property (II), then $\mathscr{Z}(G)$ and $\bar{\mathscr{Z}}(G)$ form isomorphic complete lattices that are closed $\wedge$-subsemilattices of $\Gamma(G)$ and $\bar{\Gamma}(G)$, respectively.*

Proof. Let $S_i \in \bar{\mathscr{Z}}(G)$ $(i \in I)$, $S = \bigcap_{i \in I} S_i$. If $x \in \langle S \rangle$, then $x^{\delta\delta} \subseteq \langle S_i \rangle$ for each $i \in I$, and since $x^{\delta\delta}$ and $\langle S_i \rangle$ belong to $\Gamma(G)$, $(x^{\delta\delta})^+ \subseteq S_i$ for each $i \in I$. Hence $(x^{\delta\delta})^+ \subseteq S$. But this means that $x^{\delta\delta} \subseteq \langle S \rangle$, thus $\langle S \rangle \in \mathscr{Z}(G)$. Therefore $S \in \bar{\mathscr{Z}}(G)$. And since $G^+ \in \bar{\mathscr{Z}}(G)$, then $\bar{\mathscr{Z}}(G)$ is a complete lattice. The rest is evident.

In [3, Corollary 1 of Proposition 1.2] it is proved that for each positive element $a$ of a 2-isolated $po$-group $G$, the smallest directed convex subgroup of $G$ containing $a$ is $C(a) = \{x \in G; |x| \supseteq |na| \text{ for a positive integer } n\}$. If $G$ has also the property (II) and if $b$ is an arbitrary element of $G$, then each directed convex subgroup $B$ of $G$ containing $b$ contains $b \vee -b$, too, therefore $B \supseteq C(b \vee -b)$. It follows that the smallest directed convex subgroup $C(b)$ containing $b$ is equal to $C(b \vee -b)$.

**Theorem 4.** *Let G be a 2-isolated regular po-group with the property (II). Then the following are equivalent:*

(1) $\Gamma(G) = \mathscr{Z}(G)$.

(2) $C(a) = a^{\delta\delta}$ for each $a \in G$.

(3) $C(a) = C(b)$ if and only if $a^{\delta\delta} = b^{\delta\delta}$ for each $a, b \in G$.

Proof. $1 \Rightarrow 2$: Let $a \in G$. By the assumption $C(a) \in \mathscr{Z}(G)$, hence $a^{\delta\delta} \subseteq C(a)$. And since $a^{\delta\delta} \in \Gamma(G)$, $C(a) \subseteq a^{\delta\delta}$ always holds.

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 1$: Let $M \in \Gamma(G)$, $a \in M$, $b \in G$, $a^{\delta} = b^{\delta}$. Then $a^{\delta\delta} = b^{\delta\delta}$, thus $b \in C(b) = C(a) \subseteq M$.

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## Z-ПОДГРУППЫ УПОРЯДОЧЕННЫХ ГРУПП

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### Резюме

В статье обобщено понятие $z$-подгруппы из теории решёточно упорядоченных групп для любых упорядоченных групп. В частности, здесь показаны некоторые основные свойства $z$-подгрупп в случае 2-изолированных регулярных упорядоченных групп, в которых существует sup $(a, -a)$ для длюбого элемента $a$.