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THE NUMBER OF ISOMORPHISM CLASSES OF SPANNING TREES OF A GRAPH

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In [1] B. L. Hartnell conjectures that the spanning trees of a graph containing n vertex-disjoint circuits can be partitioned into at least $n+1$ isomorphism classes. We shall prove this conjecture.

All graphs considered here are finite undirected graphs without loops and multiple edges.

First we shall define some concepts which will be used in the sequel. In [2] some concepts concerning trees are defined. For a vertex a of a tree T the mean vertex deviation is defined as

$$m_1(a)=\frac{1}{|V(T)|}\sum_{x\in V(T)} d(a,x),$$

where $V(T)$ denotes the vertex set of T (this symbol will be used also for other graphs) and $d(a,x)$ denotes the distance between the vertices a and x in T. A vertex of T with the minimal vertex deviation is called a median of T and its mean vertex deviation is called the mean vertex deviation of T. In [3] it is proved that each tree has either exactly one median, or exactly two medians which are joined by an edge. (The concept of a median is defined in [2] not only for trees, but here we shall use it only for trees.)

If v is a vertex of a tree T and e is an edge of T incident with v, then all vertices which belong to paths from v with the first edge e form a subgraph which is called a branch of T with the knag v. The branch of T with the knag v with the maximal number of vertices is called a weight branch and its number of vertices is called the weight at v. In [3] it was proved that a vertex of a tree has a minimal weight, if and only if it is a median of this tree.

We shall use the term branch also for unicyclic graphs (graphs with exactly one circuit). A branch of a unicyclic graph G is a connected component of the graph obtained from G by deleting all edges of the circuit contained in G.

By $\mathfrak{M}(n)$, where n is a positive integer, we shall denote the class of all connected graphs which contain exactly n vertex-disjoint circuits $C_1, \ldots, C_n$ and no other circuit.

We shall prove a theorem.

Theorem. *Let G be a connected graph in which the maximal number of vertex-disjoint circuits is $n \geqq 2$. Then there exist at least $n+1$ pairwise non-isomorphic spanning trees of G.*

Remark. For $n=1$ this is not true. There exist unicyclic connected graphs in which all spanning trees are isomorphic [4].

Proof. Let G be a connected graph in which the maximal number of vertex-disjoint circuits is n. Then evidently G contains a spanning subgraph belonging to $\mathfrak{M}(n)$. Each spanning tree of this subgraph is a spanning tree of G. Therefore it suffices to prove the assertion for graphs from $\mathfrak{M}(n)$. Thus let $G \in \mathfrak{M}(n)$, $n \geqq 2$. We shall use the induction according to n. Let $n=2$. The graph G contains two vertex-disjoint circuits C_1, C_2; all edges of G not belonging to them are acyclic. Let P be the path connecting a vertex of C_1 with a vertex of C_2 and not having any common edge with C_1, C_2; such a path is determined uniquely. Let v be the terminal vertex of P belonging to C_1. If C_1 has an odd length, let e be the edge of C_1 opposite to v; if C_1 has an even length, let e be an edge of C_1 incident with the vertex of C_1 opposite to v. Let f be an edge of C_1 incident with v. Evidently $e \neq f$. Let $\hat{C}_1$ (or $\hat{C}_2$) be the connected component of the graph obtained from G by deleting all edges of P, which contains C_1 (or C_2 respectively). As $\hat{C}_1$ and $\hat{C}_2$ are vertex-disjoint, at least one of them has the number of vertices not exceeding $\frac{1}{2}|V(G)|$; without loss of generality let $|V(\hat{C}_1)| \leqq \frac{1}{2}|V(G)|$. Let G' (or G'') be the graph obtained from G by deleting the edge e (or f respectively). Both G' and G'' are unicyclic graphs. First suppose that G' has at least two non-isomorphic spanning trees; let T_1, T_2 be such two trees. Without loss of generality suppose that the mean vertex deviation of T_1 is greater than or equal to that of T_2. The median of T_1 either is v, or does not belong to $\hat{C}_1$; namely $|V(\hat{C}_1)| \leqq \frac{1}{2}|V(G)|$ and therefore each vertex of $\hat{C}_1$ has a weight in T_1 greater than the weight at v. Let T_3 be the tree obtained from T_1 by adding e and deleting f. Using the weights, we can prove that T_3 has the same median (or medians) as T_1. If T_1 has only one median, let it be a; if T_1 has two medians, let a be the median of T_1 which is nearer to v. The end vertex of f distinct from v has evidently a greater distance from a in T_3 than in T_1. No vertex of G has a smaller distance from a in T_3 than in T_1. Therefore the mean vertex deviation of T_3 is greater than that of T_1 and also than that of T_2. Therefore T_1, T_2, T_3 are pairwise non-isomorphic and the assertion is true. Now suppose that all spanning trees of G' are isomorphic. If the length of C_2 is odd, then all branches of G' are isomorphic as rooted trees with the root in the vertex belonging to C_2; if it is even, so are all branches of G' at the vertices whose distance from the terminal vertex of P belonging to C_2 is even. This was proved in [4]. The graph G'' differs from G' in one branch, namely the branch which contains the vertices of C_1. If B' (or B'') is the branch of G' (or G'' respectively) containing the vertices of C_1, then they are not isomorphic as rooted trees with the root in the vertex belonging to C_2; otherwise G' and G'' would be isomorphic. Therefore G'' cannot fulfil the condition from [4] and has at least two nonisomorphic spanning trees T_1' and T_2'. Without loss of generality suppose that the mean vertex deviation

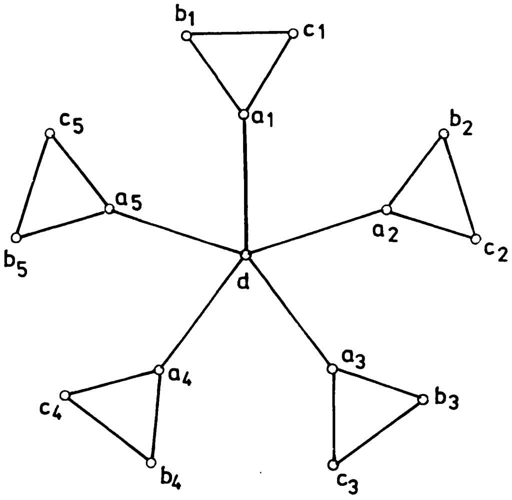

Fig.1

of T_1' is smaller than or equal to that of T_2'. Let T_3' be the tree obtained from T_1' by adding e and deleting f. Analogously as in the preceding case we prove that the mean vertex deviation of T_3' is smaller than that of T_1' and that of T_2' and the trees T_1', T_2', T_3' are pairwise non-isomorphic.

Now suppose that the assertion is true for $n = k-1$, where $k \geqq 3$, and let G be a graph from $\mathfrak{M}(k)$. Let $\mathcal{T}(G)$ be the set of all circuits C of G with the property that in the graph obtained from G by deleting all vertices and edges of C, only one connected component contains circuits; evidently $\mathcal{T}(G) \neq \emptyset$. To each $C \in \mathcal{T}(G)$ there exists exactly one edge $e(C)$ which is incident with a vertex of C and does not belong to C and separates C from all the other circuits of G. By $\hat{C}$ we denote the connected component of the graph obtained from G by deleting the edge which contains C. Evidently there exists at least one circuit $C \in \mathcal{T}(G)$ such that $|V(\hat{C})| \leqq \frac{1}{2}|V(G)|$. Let u be the vertex of C incident with $e(C)$. If C has an odd length, let h_2 be the edge of C opposite to u; if C has an even length, let h_1 be an edge of C incident with the vertex of C opposite to u. Let G_1 be the graph obtained from G by deleting h_1. We have $G_1 \in \mathfrak{M}(k-1)$. By the induction assumption G_1 has at least k pairwise non-isomorphic spanning trees; these trees are also spanning trees of G. Let T be such a spanning tree of G_1 which has a maximal mean vertex deviation.

The median of T either is u, or does not belong to C; this can be proved analogously as in the first part of the proof. Let h_2 be an edge of C incident with u; evidently $h_2 \neq h_1$. Let T' be the tree obtained from T by adding h_1 and deleting h_2; it is a spanning tree of G. Analogously as in the first part of the proof we can prove that the mean vertex deviation of T' is greater than that of T and that of any of the above mentioned k spanning trees of G_1. Therefore T' is not isomorphic to any of those k spanning trees. We have proved that G has at least $k+1$ pairwise non-isomorphic spanning tres, q.e.d.

We shall prove that the estimate of Theorem cannot be improved. Let a positive integer $n \geqq 2$ be given. Let $H_1, \ldots, H_n$ be pairwise disjoint triangles, let the vertices of H_i be a_i, b_i, c_i for $i = 1, \ldots, n$. Let d be a vertex not belonging to any H_i. Let G be the graph obtained by joining d by edges with all the vertices a_i for $i = 1, \ldots, n$.

Let T_0 be the spanning tree of G obtained by deleting all edges b_ic_i for $i = 1, \ldots, n$. Further, for $j = 1, \ldots, n$ let T_j be the spanning tree of G obtained by deleting the edges a_ib_i for $i = 1, \ldots, j$ and the edges b_ic_i for $i = j+1, \ldots, n$. The spanning trees $T_0, T_1, \ldots, T_n$ are pairwise non-isomorphic and it is easy to prove that each spanning tree of G is isomorphic to one of them. An example of such a graph for $n = 5$ is in Fig. 1.

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ЧИСЛО КЛАССОВ ИЗОМОРФИЗМА ДЕРЕВЬЕВ – ОСТОВОВ ГРАФА

Богдан Зелинка

Резюме

Пусть G – граф, в котором максимальное число вершинно непересекающихся контуров равно $n \geqslant 2$. Потом существует не менее чем $n+1$ попарно неизоморфных деревьев – остовов графа G. Эта теорема является решением проблемы Б. Л. Хартнелла.