Mathematica Slovaca

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Mathematica Slovaca, Vol. 28 (1978), No. 4, 349--359

Persistent URL: http://dml.cz/dmlcz/136189

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ON SOME TYPES OF MAXIMAL l-SUBGROUPS OF A LATTICE ORDERED GROUP

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All lattice ordered groups dealt with in this paper are assumed to be commutative. We consider the conditions (p), (q), (h) and (β) for a lattice ordered group (for detailed definitions cf. § 1). The condition (q) is similar to a condition studied by Everett [5]. The condition (β) has been considered by Alling in [1] for the case of linearly ordered groups.

For $x \in \{p, q, h, \beta\}$ we denote by $S_x(G)$ the system of all convex l-subgroups of an l-group G that fulfil the condition (x). The system $S_x(G)$ is partially ordered under set inclusion. The class of all lattice ordered groups satisfying the condition (x) will be denoted by T_x.

§ 2 contains some auxiliary results concerning the conditions (p), (q), (h) and (β). In § 3 it is proved that for each $x \in \{p, q, h, \beta\}$ the partially ordered system $S_x(G)$ has the greatest element. From this it follows that T_x is a radical class in the sence introduced by Jakubík [7].

§ 1. Preliminaries

Let us recall some concepts, definitions and notations to be used throughout the paper. For the notations and basic concepts not introduced here, we refer to [2] and [6].

Let G be an abelian l-group. Denote by N the set of all positive integers. We say that a sequence (x_n) is in G if $x_n \in G$ for each $n \in N$. A sequence (x_n) in G is called descending if $x_n \geqq x_{n+1}$ for each $n \in N$. The concept of an increasing sequence is defined dually. Let (x_n) be a sequence in G and let $x \in G$. Suppose that there exist sequences (u_n) and (v_n) in G such that (u_n) is increasing, (v_n) is descending, $u_n \leqq x_n \leqq v_n$ for each $n \in N$ and $\vee u_n = \wedge v_n = x$. Then we shall write $x_n \to x$; we also say that (x_n) o-converges to x, or that x is an o-limit of (x_n). If (x_n) is a descending sequence and if there exists $\wedge x_n = x$, then (x_n) o-converges to x; this situation will be denoted by $x_n \downarrow x$. The meaning of $x_n \uparrow x$ is analogous. A sequence (x_n) will be called a zero sequence if $x_n \to 0$ (0 denotes the zero element of G). It is obvious that $x_n \to 0$ if and only if there exists a sequence $t_n \downarrow 0$ such that $|x_n| \leqq t_n$ $(n \in N)$. A sequence (x_n) satisfying

$$|x_n - x_m| \leqslant t_n \quad (n \in N,\ m \geqslant n)$$

for some (t_n) with $t_n \downarrow 0$ is called fundamental. Denote by $H(E)$ the set of all fundamental (zero) sequences in G. If (x_n) is o-convergent, then $(x_n) \in H$. The converse does not hold in general. If every sequence $(x_n) \in H$ is o-convergent, then G is said to be o-complete. An interval $[a, b]$ of G is called o-complete if (x_n) o-converges whenever $x_n \in [a, b]$ $(n \in N)$ and $(x_n) \in H$. Since each fundamental sequence is bounded, G is o-complete if and only if each interval of G is o-complete.

Now we describe the construction of the Cantor extension $C(G)$ of G. This construction is due to Everett [5]. Let (x_n), $(y_n) \in H$. We put $(x_n)+(y_n) = (x_n + y_n)$; further we set $(x_n) \leqslant (y_n)$ if $x_n \leqslant y_n$ for each $n \in N$. Then H turns out to be an abelian l-group and E is an l-ideal of H. The factor l-group $H/E = C(G)$ is said to be the Cantor extension of G.

The symbol $(x_n)^*$ will be used to denote the coset of $C(G)$ containing $(x_n) \in H$. The mapping $\varphi : x \mapsto (x, x, \ldots)^*$ from G into $C(G)$ is an o-isomorphism. If x and $\varphi(x)$ are identified, then every sequence $(x_n) \in H$ is o-convergent in $C(G)$ and every element of $C(G)$ is an o-limit of some sequence $(x_n) \in H$. Both symbols 0 and E will be used to denote the zero element of $C(G)$.

We say that an element $y \in G$ is an o-cluster point of a sequence (x_n) if there are sequences (u_n) and (v_n) in G such that

(*i*) $u_n \uparrow y$, $v_n \downarrow y$,

(*ii*) for each $n_0 \in N$ there exists $n \in N$, $n \geqslant n_0$ with the property $u_n \leqslant x_n \leqslant v_n$.

It is easy to prove that $y \in G$ is an o-cluster point of (x_n) if and only if y is an o-limit of a subsequence of (x_n).

In § 2 and § 3 we shall consider the following conditions for G:

(p) If $[a_n, b_n]$ $(n \in N)$ is a system of intervals of G such that $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for each $n \in N$, then $\cap [a_n, b_n]$ $(n \in N) \neq \emptyset$.

(q) If (x_n) is a fundamental sequence in G and $\wedge x_n$ does exist in G, then (x_n) is o-convergent.

(h) Every bounded sequence in G possesses an o-cluster point.

(β) If α is an ordinal, A, B are nonempty linearly ordered subsets of G such that $A < B$, $\operatorname{card} A + \operatorname{card} B < \aleph_\alpha$, then there exists $g \in G$ with $A < \{g\} < B$. Here $A < B$ $(A \leqslant B)$ means that $a < b$ $(a \leqslant b)$ for each $a \in A$ and each $b \in B$. If G is linearly ordered and if it fulfils (β), then G is called an η_α-group (cf. Alling [1]).

We say that a sequence (x_n) in G converges to x if for each $0 < e \in G$ there exists $n_0 \in N$ such that $|x_n - x| < e$ for each $n \geqslant n_0$ (see [5]). An element $x \in G$ is called a cluster point of a sequence (x_n) if for each $0 < e \in G$ and each $n_0 \in N$ there exists $n \geqslant n_0$ such that $|x_n - x| < e$.

A sequence (x_n) will be called almost constant if there is $n_0 \in N$ with $x_n = x_{n_0}$ for each $n \geqslant n_0$. If G is a linearly ordered group, the o-convergence is reduced to the convergence (see [5]) and it is easily seen that the concept of an o-cluster point coincides with the concept of a cluster point. If G is an l-group that fails to be linearly ordered and if a sequence (x_n) of elements of G converges to x, then (x_n) is almost constant $(x_n = x, n \geqslant n_0)$ (cf. [5]). Therefore x is a cluster point of (x_n) if and only if for each $m \in N$ there exists $n(m) \geqslant m$ with $x_{n(m)} = x$.

Let us recall the definition of the direct (lexicographic) product of partially ordered groups (cf. [6]). Let A and B be partially ordered groups. The cartesian product G of A and B is made into a partially ordered group by putting $(a_1, b_1) \leqslant (a_2, b_2)$ if and only if $a_1 \leqslant a_2$, $b_1 \leqslant b_2$ ($a_1 < a_2$ or $a_1 = a_2$ and $b_1 \leqslant b_2$) for all $a_1, a_2 \in A$ and all $b_1, b_2 \in B$. Then G is said to be the direct (lexicographic) product of partially ordered groups A and B. We shall use the notation $G = A \times B$ $(G = A \circ B)$. By $x(A)$ $(x(B))$ we shall denote the component of $x \in G$ in the factor $A(B)$.

Since G is abelian, the notion of a convex l-subgroup of G coincides with the notion of an l-ideal of G. The additive groups of all integers, rational and real numbers (with the natural linear order) will be denoted by C, Q and R, respectively.

§ 2. The conditions (*p*), (*q*) and (*h*)

This paragraph deals with the relation between the o-completeness of G and the conditions (p), (q) and (h). Further there are investigated some relations between G and the Cantor extension $C(G)$ of G.

If $[x_n, y_n]$ $(n \in N)$ be a system of intervals of R such that $[x_n, y_n] \supseteq [x_{n+1}, y_{n+1}]$ for each $n \in N$, then $\cap[x_n, y_n]$ $(n \in N) \neq \emptyset$. The analogous statement need not hold in G.

Example 1. If $g = C \circ C$, then $\cap[(0, n); (1, -n)] = \emptyset$.

Let $[u_n, v_n]$ be a system of intervals of G with $[u_n, v_n] \supseteq [u_{n+1}, v_{n+1}]$ for each $n \in N$. Denote $K = \cap[u_n, v_n]$ $(n \in N)$.

2.1. *If* $K \neq \emptyset$ *and if*

(i) $(u_n), (v_n) \in H$,

(ii) $(u_n)^* = (v_n)^*$

hold true, then $\operatorname{card} K = 1$.

Proof. Assume that (i) and (ii) are fulfilled and let $\operatorname{card} K > 1$. Since K is a sublattice of G, there exist $x, y \in G$, $x < y$. From (ii) we get $(u_n - v_n) \in E$; hence there is a sequence $t_n \downarrow 0$ such that $0 \leqslant v_n - u_n \leqslant t_n$. Then $0 < y - x \leqslant v_n - u_n \leqslant t_n$ $(n \in N)$. This is a contradiction, because $\wedge t_n$ $(n \in N) = 0$.

2.2. *If* $K = \{x\}$, *then* $\wedge v_n = \vee u_n = x$ $(n \in N)$.

Proof. We see that $x \leqslant v_n$. Assume that $y \in G$ such that $x \leqslant y \leqslant v_n$ $(n \in N)$. Since

$y \geqslant u_n$ $(n \in N)$, we have $y \in K$. The hypothesis implies $x = y$ and so $x = \wedge v_n$ $(n \in N)$. Similarly $x = \vee u_n$ $(n \in N)$.

From 2.1 and 2.2 we obtain immediately:

2.3. *If $K \neq \emptyset$, then* card$K = 1$ *if and only if the following conditions are fulfilled:*

(*i*) $(u_n), (v_n) \in H$,

(*ii*) $(u_n)^* = (v_n)^*$.

2.4. *For each sequence $(x_n) \in H$ there exist sequences (u_n) and (v_n) such that (u_n) is increasing and (v_n) is descending with*

(*i*) $u_n \leqslant x_m \leqslant v_n$ $(n \in N,\ m \geqslant n)$,

(*ii*) $(u_n)^* = (v_n)^* = (x_n)^*$.

Proof. Suppose that $(x_n) \in H$. There exists a sequence (t_n) in G such that $t_n \downarrow 0$ and $|x_n - x_m| \leqslant t_n$, i.e., $-t_n \leqslant x_n - x_m \leqslant t_n$ $(n \in N,\ m \geqslant n)$. Then

$$x_n - t_n \leqslant x_m \leqslant x_n + t_n \quad (n \in N,\ m \geqslant n). \tag{1}$$

Construct sequences (u_n) and (v_n) as follows:

$$u_1 = x_1 - t_1, \quad u_n = (x_n - t_n) \vee u_{n-1} \quad (n \in N,\ n > 1),$$
$$v_1 = x_1 + t_1, \quad v_n = (x_n + t_n) \wedge v_{n-1} \quad (n \in N,\ n > 1).$$

From (1) it follows that (*i*) is valid. The sequence (u_n) is increasing and (v_n) is a descending one. Hence $[u_1, v_1] \supseteq [u_2, v_2] \supseteq \ldots$. The definition of elements u_n and v_n implies

$$x_n - t_n \leqslant u_n \leqslant x_m \leqslant v_n \leqslant x_n + t_n \quad (n \in N,\ m \geqslant n). \tag{2}$$

From (2) we obtain $0 \leqslant u_m - u_n \leqslant x_m - u_n \leqslant 2t_n$ $(n \in N,\ m \geqslant n)$. Since $2t_n \downarrow 0$, we have $(u_n) \in H$. In the same way we get $(v_n) \in H$. According to (2) we have $0 \leqslant v_n - u_n \leqslant 2t_n$ $(n \in N)$, $0 \leqslant x_n - u_n \leqslant 2t_n$ $(n \in N)$. Therefore $u_n - v_n \to 0$, $x_n - u_n \to 0$. Thus $(u_n)^* = (v_n)^*$, $(x_n)^* = (u_n)^*$ and so (*ii*) is valid.

2.5. *If G fulfils (p), then G is o-complete.*

Proof. Suppose that G fulfils (p). Let $(x_n) \in H$. Let the sequences (u_n) and (v_n) be as in 2.4. By the assumption $K = \cap [u_n, v_n]$ $(n \in N) \neq \emptyset$, hence because of 2.3 card$K = 1$. If we denote $K = \{x\}$, from 2.2 it follows $x = \wedge v_n = \vee u_n$ $(n \in N)$; hence $v_n \downarrow x$, $u_n \uparrow x$. Since $u_n \leqslant x_n \leqslant v_n$ $(n \in N)$, we have $x_n \to x$.

Example 1 shows that if G is o-complete, then G need not fulfil (p).

2.6. *G is o-complete if and only if condition (q) holds.*

Proof. Suppose that condition (q) is satisfied and let $(x_n) \in H$. According to 2.4 we can find an increasing sequence $(u_n) \in H$ and a descending sequence $(v_n) \in H$ such that $u_n \leqslant x_n \leqslant v_n$. Since $\wedge u_n = u_1$ does exist in G, the assumption implies that the sequence (u_n) is o-convergent. Consequently, $u_n \uparrow u = \vee u_n$ $(n \in N)$. By using (2) we obtain $v_n - u_n \leqslant 2t_n$; hence $0 \leqslant v_n - u \leqslant 2t_n \downarrow 0$ $(n \in N)$. Then $v_n - u \downarrow 0$, which means that $v_n \downarrow u$. We infer that $x_n \to u$; thus G is o-complete. The converse is obvious.

The condition (q) is similar to the condition

(q') If $(x_n) \in H$, then $\wedge x_n$ does exist in G.

Everett [5] has shown that condition (q') holds in G if and only if G is o-complete.

2.7. *If* $(x_n)^* \in C(G)$, $E < (x_n)^*$, *then there exists* $g \in G$, $E < g \leq (x_n)^*$.

Proof. Let $E < (x_n)^* \in C(G)$. We may suppose that $x_n \geq 0$ $(n \in N)$. By 2.4 we can find an increasing sequence $(u_n) \in H$, $u_n \leq x_n$ $(n \in N)$, $(u_n)^* = (x_n)^*$. Hence $u'_n = u_n \vee 0 \leq x_n$ $(n \in N)$. Since $(u'_n)^* = (x_n)^*$, there exists $n_0 \in N$ with $u'_{n_0} = g > 0$. From $0 < g \leq u'_n \leq x_n$ $(n \geq n_0)$ we obtain $E < g \leq (x_n)^*$.

2.8. *If* $A \neq \{E\}$ *is a convex* l*-subgroup of* $C(G)$, *then* $A \cap G \neq \{E\}$.

Proof. If $A \subseteq G$, the assertion is obvious. Suppose that $A \nsubseteq G$. Then there exists $E < (x_n)^* \in A$, $(x_n)^* \notin G$. In fact, because G is an l-subgroup of $C(G)$, we infer $A \subseteq G$, if each positive element from A belongs to G. With respect to 2.7 there is $g \in G, E < g \leq (x_n)^*$. The convexity A in $C(G)$ implies $g \in A$ and thus $g \in A \cap G$.

2.9. *If G is a linearly ordered group and* (x_n) *is a sequence in G, the following conditions are equivalent:*

(i) *For each* $0 < e \in G$ *there exists* $n_0 \in N$ *such that* $|x_n - x_m| < e$ $(n \in N, m \geq n \geq n_0)$,

(ii) $(x_n) \in H$.

Proof. Suppose that *(ii)* is valid. There exists a sequence (t_n) with $t_n \downarrow 0$ and $|x_n - x_m| \leq t_n$ $(n \in N, m \geq n)$. In view of [5] a sequence (a_n) in a linearly ordered group o-converges to a if and only if (a_n) converges to a. Thus for each $0 < e \in G$ there exists $n_0 \in N$ such that $t_n < e$ $(n \geq n_0)$ and so *(i)* is true. Conversely, let *(i)* hold true. If (x_n) is an almost constant sequence, it is easily seen that *(ii)* is valid. Let (x_n) be a sequence which is not almost constant. Then for each $n \in N$ there exists $m \geq n$ with $|x_n - x_m| \neq 0$. If $0 < e_1 \in G$, then according to *(i)* there exists the least number $n_1 \in N$ such that $|x_n - x_m| < e_1$ $(n \in N, m \geq n \geq n_1)$. Let $p \in N$ be the least number with the properties $p > n_1$ and $|x_{n_1} - x_p| \neq 0$. For $e_2 = |x_{n_1} - x_p| < e_1$ there exists the least $n_2 \in N$ such that $|x_n - x_m| < e_2$ $(n \in N, m \geq n \geq n_2)$. In the same way we can find n_3, and so on. Clearly, $n_1 < n_2 < n_3 < \ldots$. Let us form a sequence (u_n) by putting: $u_1 = u_2 = \ldots = u_{n_1 - 1} = e_1$, $u_{n_1} = u_{n_1+1} = \ldots = u_{n_2-1} = e_1$, $u_{n_2} = u_{n_2+1} = \ldots = u_{n_3-1} = e_2$, ... The sequence (u_n) is descending and $u_n \geq 0$ $(n \in N)$. Now we show that $\wedge u_n = 0$. If $x \in G$, $x \leq u_n$ $(n \in N)$, then $x \leq 0$. Assume that $x > 0$. By *(i)* there exists $n_0 \in N$ such that $|x_n - x_m| < x$ $(n \in N, m \geq n \geq n_0)$. Further, there are r, $s \in N$ $r \geq s \geq n_0$ such that $u_r = |x_r - x_s| < x$, a contradiction. Hence $u_n \downarrow 0$ and $|x_n - x_m| \leq u_n$ $(n \in N, m \geq n \geq n_1)$. Therefore $(x_n) \in H$.

2.10. *Let (i) and (ii) be as in 2.9. Assume that an* l*-group G contains at least one* o*-convergent sequence which is not almost constant. If (ii) implies (i), then G is a linearly ordered group.*

Proof. Suppose that G is an l-group such that condition *(ii)* implies *(i)*. Assume that G is not linearly ordered. Then there are $0 < a$, $b \in G$, $a \wedge b = 0$. According to the assumption there exists a sequence (x_n) in G such that $x_n \to x$ and for each $n_0 \in N$ we can find $n > n_0$, with $x_n \neq x$. Then there exists a sequence $t_n \downarrow 0$, $t_n > 0$ $(n \in N)$ satisfying $|x_n - x| < t_n$ $(n \in N)$. We have $(t_n) \in H$, hence (t_n) fulfils (i). Therefore there is $m_1 \in N$ such that $t_n - t_m < a$, whenever $m \geqslant n \geqslant m_1$. Similarly there is $m_2 \in N$ such that $t_n - t_m < b$, whenever $m \geqslant n \geqslant m_2$. If $m_3 = \max\{m_1, m_2\}$, then $0 \leqslant t_n - t_m \leqslant a \wedge b = 0$ for each pair n, m with $m \geqslant n \geqslant m_3$. Since (t_n) is not almost constant, we have a contradiction.

If (ii) implies (i), but each o-convergent sequence in an l-group G is almost constant, the assertion need not hold (example: $G = C \times C$).

From 2.9 and 2.10 it follows

Theorem 2.1. *Assume that an l-group G contains at least one o-convergent sequence which is not almost constant. G is linearly ordered if and only if the conditions (i) and (ii) from 2.9 are equivalent.*

2.11. *If an interval $[0, a]$ is a chain in G, then $[E, a]$ is a chain in $C(G)$.*

Proof. Assume that there exist $(x_n)^*, (y_n)^* \in [E, a]$, $(x_n)^* \| (y_n)^*$. According to 2.7 there are g and h from G such that $E < g \leqslant (x_n)^*$, $E < h \leqslant (y_n)^*$. If $(x_n)^* \wedge (y_n)^* = E$, then $g \| h$ which is impossible because $[0, a]$ is a chain. Now let $(x_n)^* \wedge (y_n)^* = (z_n)^* > E$. Introduce the notations $(u_n)^* = (x_n)^* - (z_n)^* > E$, $(v_n)^* = (y_n)^* - (z_n)^* > E$. Hence $(u_n)^* \wedge (v_n)^* = E$. In a similar way as above we obtain a contradiction.

Theorem 2.2. *$C(G)$ is a linearly ordered group if and only if G is a linearly ordered group.*

Proof. Let G be a linearly ordered group. $C(G)$ being an l-group, it suffices to verify that $[E, (x_n)^*]$ is a chain for each $(x_n)^* \in C(G)$, $(x_n)^* > E$. Every fundamental sequence in G is bounded. To get this result it suffices to put $n = 1$ in (i) from 2.4. Hence an element $a \geqslant (x_n)^*$ does exist in G. By the assumption and 2.11 $[0, a]$ is a chain in $C(G)$ and so $[E, (x_n)^*]$ is a chain as well. The converse is obvious.

The system $\{a_i : i \in M\}$ of elements from G will be called disjoint if $M \neq \emptyset$, $a_i > 0$ for all $i \in M$ and $a_i \wedge a_j = 0$, whenever $i, j \in M$, $i \neq j$. Let α be a cardinal. Assume that the following condition is fulfilled in G:

$(F(\alpha))$ *If $\{a_i : i \in M\}$ is a disjoint system in G, then* $\operatorname{card} M < \alpha$.

In Conrad's paper [3] there is studied the condition $F(\aleph_0)$. The condition $(F(\alpha))$ was considered by Jakubík [8].

2.12. *The condition $(F(\alpha))$ holds in $C(G)$ if and only if it holds in G.*

Proof. Let G satisfy the condition $(F(\alpha))$ and let $S = \{a_i : i \in M\}$ be a disjoint system in $C(G)$. With respect to 2.7 for each $i \in M$ there is $g_i \in G$ with $E < g_i \leqslant a_i$. Hence $\{g_i : i \in M\}$ is a disjoint system in G and therefore $\operatorname{card} M < \alpha$. The converse is obvious.

A subset A of G is said to be a basis for G (cf. Conrad [3]) if

(i) an interval $[0, a]$ is a chain for each $0 < a \in A$, (*ii*) A is a disjoint set,

(*iii*) if $0 \leq b \in G$ such that $b \wedge a = 0$ for each $a \in A$, then $b = 0$.

2.13. *A basis* $A = \{a_i\colon i \in M\}$ *for* G *is a basis for* $C(G)$.

Proof. Let A be a basis for G. In view of 2.11 we obtain that $[E, a_i]$ is a chain in $C(G)$; and thus (*i*) is fulfilled in $C(G)$. It is clear that (*ii*) holds in $C(G)$ as well. It remains to verify only (*iii*). Let $E \leq (x_n)^* \in C(G)$, $(x_n)^* \wedge a = E$ for each $a \in A$. We have to show that $(x_n)^* = E$. Assume that $E < (x_n)^*$. According to 2.7 there exists $g \in G$, $E < g \leq (x_n)^*$. Since A is a basis for G, from $g \wedge a = 0$ it follows that $g = 0$, a contradiction.

2.14. *If* $x_n \to x$, *then* x *is the only* o*-cluster point of* (x_n).

Proof. If $x_n \to x$, then there are sequences (u_n) and (v_n) such that $u_n \uparrow x$, $v_n \downarrow x$ and

$$u_n \leq x_n \leq v_n \quad (n \in N). \tag{3}$$

Then x is an o-cluster point of (x_n). Let also $x' \in G$ be an o-cluster point of (x_n). Hence for each $n_0 \in N$ there exists $n \geq n_0$ with the property

$$u'_n \leq x_n \leq v'_n, \tag{4}$$

where $u'_n \uparrow x'$, $v'_n \downarrow x'$. Let us form a sequence $(x_{n(m)})$ $(n \in N)$ such that for each $m \in N$ we find $n(m) \in N$ with the property $u'_{n(m)} \leq x_{n(m)} \leq v'_{n(m)}$. If $m_1 < m_2$, we can choose $n(m_1) < n(m_2)$. By using (3) and (4) we get $u_{n(m)} + u'_{n(m)} \leq 2x_{n(m)} \leq v_{n(m)} + v'_{n(m)}$ $(m \in N)$. Therefore $2x_{n(m)} \to x + x'$. The assumption implies $2x_{n(m)} \to 2x$, hence $x + x' = 2x$, $x = x'$.

2.15. *If* x *is an* o*-cluster point of* $(x_n) \in H$, *then* $x_n \to x$.

Proof. Let (u_n) and (v_n) be as in 2.4. By the assumption there exists a subsequence $(x_{n(m)})$ of (x_n) such that $x_{n(m)} \to x$. With respect to (2) we have $u_n \leq x_{n(m)} \leq v_n$ $(n \in N, m \geq n)$. Therefore $u_n \leq x \leq v_n$ $(n \in N)$. Thus $(u_n)^* \leq (x, x, \ldots)^* \leq (v_n)^*$ and 2.4. implies $(u_n)^* = (v_n)^* = (x, x, \ldots)^*$. Hence $u_n \uparrow x$, $v_n \downarrow x$ and by using (2) we obtain the assertion. Since every fundamental sequence is bounded, with respect to 2.15 we conclude

2.16. *If* G *fulfils* (h), *then* G *is* o*-complete.*

The converse does not hold in general.

Example 3. $G = Q \circ R$ is an o-complete l-group (see [4]). The sequence $(x_n) = \left(\left(\frac{1}{n}, 0\right)\right)$ in G is bounded but it possesses no o-cluster point. Assume that $(x, y) \in G$ is an o-cluster point of (x_n). Hence there are sequences (u_n) and (v_n) such that $u_n \uparrow (x, y)$, $v_n \downarrow (x, y)$ and for each $n_0 \in N$ there exists $n \geq n_0$ with the property $u_n \leq x_n \leq v_n$. There exists $n_1 \in N$ with the property $u_n(Q) = v_n(Q) = x$ $(n \geq n_1)$ (see [4]). If $x > 0$, then $x > \frac{1}{n_2}$ for some $n_2 \in N$. Hence $u_n > x_n$ $(n \geq n_3$

$= \max\{n_1, n_2\}$), a contradiction. If $x=0$, then $x_n > v_n$ $(n \geqslant n_1)$, again a contradiction.

2.17. *If G satisfies (h), then it satisfies (p) as well.*

Proof. Let $[u_n, v_n]$ $(n \in N)$ be a system of intervals of G such that $[u_n, v_n] \supseteq [u_{n+1}, v_{n+1}]$ for each $n \in N$. The sequence (v_n) is bounded and hence by the assumption it has an o-cluster point x. There exists a subsequence (v_p) of (v_n) with $v_p \downarrow x$. Therefore, $v_n \downarrow x$ and $u_n \leqslant x \leqslant v_n$ $(n \in N)$. This shows that $x \in \cap [u_n, v_n]$ $(n \in N)$ and (p) holds true.

If G fulfils (p), then G fails to satisfy (h); it suffices to put $G = R \circ R$. The sequence $(x_n) = \left(\left(\frac{1}{n}, 0\right)\right)$ has no o-cluster point.

2.18. *If G fulfils the condition (h), then G is archimedean.*

Proof. Assume (by way of contradiction) that G satisfies (h) and it fails to be archimedean. Then there exist $a, b \in G$, $a>0$, $b>0$ with $na<b$ $(n \in N)$. We wish to show that the bounded sequence (na) has no o-cluster point. Suppose that x is an o-cluster point of (na). Then we can find sequence (u_n) and (v_n) with $u_n \uparrow x$, $v_n \downarrow x$. For each $n_0 \in N$ there is $n \geqslant n_0$ such that $u_n \leqslant na \leqslant v_n$. We obtain $v_n > ka$ $(n, k \in N)$. Hence $na < \wedge v_n = x$ $(n \in N)$ and thus $(n+1)\, a<x$, $na<x-a$. For each $m \in N$ there exists $n \geqslant m$ such that $u_m \leqslant u_n \leqslant na < x-a$. Hence $x = \vee u_m$ $(m \in N)$ $\leqslant x-a$, a contradiction.

If G is archimedean then the condition (h) need not hold in G, for example if $G=Q$.

§ 3. The greatest l-ideals of G

In this paragraph it will be shown that for each $x \in \{p, q, h, \beta\}$ the partially ordered system $S_x(G)$ possesses the greatest element M_x.

It is easy to verify that G fulfils the condition (x) if and only if each interval of G fulfils the condition (x). Let us form the set

$M_x = \{g \in G$: the interval $[0, |g|]$ fulfils the condition $(x)\}$.

Let $x, y, c \in G$, $x \leqslant c \leqslant y$.

3.1. *If the intervals $[x, c]$ and $[c, y]$ satisfy condition (p) then the interval $[x, y]$ fulfils condition (p) as well.*

Proof. Let $[a_n, b_n]$ $(n \in N)$ be a system of intervals in G such that $[a_n, b_n] \subseteq [x, y]$ $(n \in N)$ and $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots$. Denote $\bar{a}_n = a_n \vee c$, $\bar{b}_n b_n \vee c$, $\bar{\bar{a}}_n = a_n \wedge c$, $\bar{\bar{b}}_n = b_n \wedge c$. Therefore $[\bar{a}_n, \bar{b}_n] \subseteq [c, y]$ $(n \in N)$, $[\bar{\bar{a}}_n, \bar{\bar{b}}_n] \subseteq [x, c]$ $(n \in N)$, $[\bar{a}_1, \bar{b}_1] \supseteq [\bar{a}_2, \bar{b}_2] \supseteq \ldots$, $[\bar{\bar{a}}_1, \bar{\bar{b}}_1] \supseteq [\bar{\bar{a}}_2, \bar{\bar{b}}_2] \supseteq \ldots$. Hence, from the assumption it follows that there exist $\bar{z} \in \cap [\bar{a}_n, \bar{b}_n]$ $(n \in N)$ and $\bar{\bar{z}} \in \cap [\bar{\bar{a}}_n, \bar{\bar{b}}_n]$ $(n \in N)$. Let n be a fixed positive integer. From $a_n - \bar{\bar{a}}_n = \bar{a}_n - c$ we get $a_n = \bar{\bar{a}}_n + (\bar{a}_n - c)$. Since $\bar{\bar{a}}_n \leqslant \bar{\bar{z}}$ and $\bar{a}_n - c \leqslant \bar{z} - c$, we have $a_n \leqslant \bar{\bar{z}} + \bar{z} - c = z$. In a similar way obtain $b_n \geqslant z$. Then $z \in \cap [a_n, b_n]$ $(n \in N)$ and the proof is finished.

3.2. *M_p is an l-ideal of G.*

Proof. Let $g, h \in M_p$. By the assumption the intervals $[0, |g|]$ and $[0, |h|]$ satisfy condition (p). Because of $[0, |h|] \simeq [|g|, |g|+|h|]$, according to 3.1 the interval $[0, |g|+|h|]$ fulfils (p). From $0 \leqslant |g+h| \leqslant |g|+|h|$ (see [6]) it follows that $[0, |g+h|]$ satisfies (p) and so $g+h \in M_p$. Since $|g| = |-g|$, M_p is a subgroup of G. From $|g \vee h| \leqslant |g| \vee |h| \leqslant |g|+|h|$ we conclude that M_p is a sublattice of G. It is easily seen that M_p is a convex subset of G and the proof is complete.

Theorem 3.1. *M_p is the greatest l-ideal of G satisfying condition (p).*

Proof. First, we prove that M_p fulfils (p). It suffices to show that every interval of M_p fulfils (p). Let $[a, b]$ we any interval of M_p. Since $0 \leqslant b-a \in M_p$, by the definition of the set M_p we obtain that $[0, b-a]$ fulfils (p) and $[0, b-a] \simeq [a, b]$ implies that (p) holds true in M_p. Now let M' be any l-ideal of G satisfying (p) and let $g \in M'$. Then $[0, |g|] \subseteq M'$ and thus $[0, |g|]$ fulfils the condition (p), hence $g \in M_p$. This shows that $M' \subseteq M_p$.

3.3. *If the intervals $[x, c]$ and $[c, y]$ are o-complete, then the interval $[x, y]$ is o-complete.*

Proof. Suppose that $(x_n) \in H$ and $x_n \in [x, y]$ $(n \in N)$. We have to prove that (x_n) is an o-convergent sequence. By [6], Chapt. V we have $|x_n \vee c - x_m \vee c| \leqslant |x_n - x_m|$ and $|x_n \wedge c - x_m \wedge c| \leqslant |x_n - x_m|$. Hence $(x_n) \in H$ implies $(x_n \vee c) \in H$ and $(x_n \wedge c) \in H$. By hypothesis $x_n \vee c \to \bar{t}$ and $x_n \wedge c \to \bar{t}$. Since

$$x_n = (x_n \vee c) + (x_n \wedge c) - c$$

for any $n \in N$ (see [6], Chapt. V), it is easy to prove that $x_n \to \bar{t} + \bar{\bar{t}} - c$.

Let us denote

$M = \{g \in G$: the interval $[0, |g|]$ is o-complete$\}$.

In a similar manner as in 3.2 the following assertion can be proved:

Theorem 3.2. *M is the greatest o-complete l-ideal of G.*

Since $M = M_q$, we have

Corollary. *M_q is the greatest l-ideal of G satisfying the condition (q).*

3.4. *If the intervals $[x, c]$ and $[c, y]$ satisfy condition (h), then the interval $[x, y]$ fulfils (h) as well.*

Proof. We intend to show that every sequence (x_n) with $x_n \in [x, y]$ $(n \in N)$ has an o-cluster point. By the assumption there exist a subsequence $(\bar{x}_{n(i)})$ of $(x_n \vee c)$ and a subsequence $(\bar{\bar{x}}_{n(j)})$ of $(x_n \wedge c)$ such that $\bar{x}_{n(i)} \to \bar{t}$ and $\bar{\bar{x}}_{n(j)} \to \bar{\bar{t}}$. Let $(n(k))$ be a subsequence of $(n(i))$ and of $(n(j))$. Evidently $\bar{x}_{n(k)} \to \bar{t}$ and $\bar{\bar{x}}_{n(k)} \to \bar{\bar{t}}$. Since $x_n = (x_n \vee c) + (x_n \wedge c) - c$ for any $n \in N$, we obtain $x_{n(k)} \to \bar{t} + \bar{\bar{t}} - c$. Thus (x_n) has an o-cluster point. Therefore the following assertion holds:

Theorem 3.3. *M_h is the greatest l-ideal of G fulfilling the condition (h).*

3.5. *If the intervals $[x, c]$ and $[c, y]$ satisfy condition (β), then the interval $[x, y]$ fulfils (β) as well.*

Proof. Let A and B be arbitrary nonempty linearly ordered sets such that $A \subset [x, y]$, $B \subset [x, y]$, $A < B$, $\operatorname{card} A + \operatorname{card} B < \aleph_\alpha$. We have to prove that there exists $z \in [x, y]$, $A < \{z\} < B$. Denote $a \vee c = \bar{a}$, $a \wedge c = \bar{\bar{a}}$, $b \vee c = \bar{b}$, $b \wedge c = \bar{\bar{b}}$ for each $a \in A$ and each $b \in B$; further, denote $\bar{A} = \{\bar{a}\colon a \in A\}$, $\bar{B} = \{\bar{b}\colon b \in B\}$, $\bar{\bar{A}} = \{\bar{\bar{a}}\colon a \in A\}$ and $\bar{\bar{B}} = \{\bar{\bar{b}}\colon b \in B\}$. We have $\operatorname{card}(\bar{A} \cap \bar{B}) \leqslant 1$ and $\operatorname{card}(\bar{\bar{A}} \cap \bar{\bar{B}}) \leqslant 1$. From $\operatorname{card} \bar{A}$, $\operatorname{card} \bar{\bar{A}} \leqslant \operatorname{card} A$ and $\operatorname{card} \bar{B}$, $\operatorname{card} \bar{\bar{B}} \leqslant \operatorname{card} B$ we obtain $\operatorname{card} \bar{A} + \operatorname{card} \bar{B} < \aleph_\alpha$ and $\operatorname{card} \bar{\bar{A}} + \operatorname{card} \bar{\bar{B}} < \aleph_\alpha$. First we shall show that if $\operatorname{card}(\bar{\bar{A}} \cap \bar{\bar{B}}) = 1$, then $\bar{A} < \bar{B}$. Let there exist $a \in A$ and $b \in B$ with $a \wedge c = b \wedge c$. We have $a \vee c < b \vee c$. This follows immediately from $A < B$ and from the distributivity of G. Let $a_1 \in A$, $b_1 \in B$, $a_1 \leqslant a$. If $b_1 \geqslant b$, then $a_1 \vee c \leqslant a \vee c < b \vee c \leqslant b_1 \vee c$, hence $a_1 \vee c < b_1 \vee c$. If $b_1 < b$, then $a_1 \vee c < b_1 \vee c$. In fact, if $a_1 \vee c < a \vee c$, then $a_1 \vee c < b_1 \vee c$ because of $a \vee c \leqslant b_1 \vee c$. If $a_1 \vee c = a \vee c$ and $a_1 \vee c = b_1 \vee c$, from $b_1 \wedge c = b \wedge c = a \wedge c$ it follows $b_1 = a$, a contradiction. The proof is analogous to that of $a_1 > a$. In a similar way we show that if $\bar{A} \cap \bar{B}$ is a one-element set, then $\bar{\bar{A}} < \bar{\bar{B}}$.

Let a be an arbitrary element of A. If $\bar{A} < \bar{B}$, then the assumption implies that there exists $\bar{z} \in [c, y]$, $\bar{A} < \{\bar{z}\} < \bar{B}$. From $\bar{\bar{A}} \leqslant \bar{\bar{B}}$ we infer that there is $\bar{\bar{z}} \in [x, c]$, $\bar{\bar{A}} \leqslant \{\bar{\bar{z}}\} \leqslant \bar{\bar{B}}$. Since $a - \bar{\bar{a}} = \bar{a} - c$, we obtain $a = \bar{\bar{a}} + (\bar{a} - c)$. From $\bar{\bar{a}} \leqslant \bar{\bar{z}}$, $\bar{a} - c < \bar{z} - c$ it follows $z = \bar{\bar{z}} + (\bar{z} - c) > a$. In a similar manner we obtain $z < b$ for each $b \in B$. We conclude that $A < \{z\} < B$. Under the assumption $\bar{\bar{A}} < \bar{\bar{B}}$ the situation is analogous.

By the same method as in 3.2 we can prove the following statement:

Theorem 3.4. *M_β is the greatest l-ideal of G fulfilling condition* (β).

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Received March 25, 1976

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НЕКОТОРЫЕ ТИПЫ МАКСИМАЛЬНЫХ l-ПОЛУГРУПП СТРУКТУРНО УПОРЯДОЧЕННОЙ ГРУППЫ

Штефан Чернак

Резюме

Пусть G коммутативная структурно упорядоченная группа. В этой статье рассматриваются условия для G кассающиеся последовательностей в G. Доказано, что существуют максимальные l-идеалы в G, удовлетворяющие одному из этих условий. Подобные условия исследовали Эверетт и Аллинг.