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Mathematica Slovaca, Vol. 28 (1978), No. 3, 277--288

Persistent URL: `http://dml.cz/dmlcz/136183`

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ON MAPPING GRAPHS AND PERMUTATION GRAPHS

WILLIBALD DÖRFLER

We consider finite simple graphs and the terminology of Harary [4] and Behzad—Chartrand [1] is used. Permutation graphs have been studied in several contexts, compare [2, 3]. For completeness we recall the definiton. Let $X=(V, E)$ be a graph with vertex-set $V=\{v_1, \ldots, v_{n'}\}$ and edge-set E and p a permutation of V. Then the *permutation graph* (X, p) is defined in the following way:

$$V(X, p)=\{v_1, \ldots, v_n, w_1. \ldots, w_n\}=V\cup V'$$

with $V'=\{w_1, \ldots, w_n\}$, $V'\cap V=\emptyset$ and

$$E(X, p)=E(X)\cup\{[w_i, w_j] \,|[v_i, v_j]\in E(X)\}\cup$$
$$\cup\{[v_i, w_j]|\; v_j=pv_i\}=E\cup E'\cup E_p$$
$$\text{with } E'=\{[w_i, w_j] \,|[v_i, v_j]\in E\} \text{ and } E_p=\{[v_i, w_j]|\; pv_i=v_j\}\,.$$

In general we shall keep this notation with the exception that an arbitrary edge of E_p will be denoted as $[x, px]$, p being considered thereby as a mapping from V onto V'. Further, if $x\in V$, then by x' we denote the corresponding element of V', i.e. if $x=v_i$, then $x'=w_i$.

This means we take a second disjoint copy X' of X and join every vertex of X' with its image under p. We are interested in the automorphisms of (X, p) and especially in the question to which extent the automorphism group $G(X, p)$ is determined by the automorphism group $G(X)$.

Definition 1. *An automorphism φ of (X, p) is called a natural automorphism of (X, p) if $\varphi V=V$ or $\varphi V=V'$ and therefore $\varphi V'=V'$ or $\varphi V'=V$. If $\varphi V=V$, $\varphi V'=V'$, then φ is called a positive natural automorphism and otherwise a negative natural automorphism.*

Remark. Clearly every permutation graph has positive natural automorphisms because the identity is one. But there are examples where no negative natural automorphisms exist.

We note some simple properties of natural automorphisms.

1) The natural automorphisms of (X, p) constitute a subgroup of the group

of automorphisms $G(X, p)$ which we will denote by $G_n(X, p)$. The positive natural automorphisms again constitute a subgroup $G_n^+(X, p)$ of $G_n(X, p)$ and $G_n^+(X, p) = G_n(X, p)$ is possible.

2) The product of two negative natural automorphisms (if they exist) is a positive natural automorphism. The product of a positive and a negative natural automorphism is a negative natural automorphism. The product of two positive natural automorphisms is again a positive natural automorphism.

3) From 2) it follows that $G_n^+(X, p)$ is a subgroup of index 2 in $G_n(X, p)$ and therefore a normal subgroup of $G_n(X, p)$ or $G_n^+(X, p) = G_n(X, p)$.

4) A positive natural automorphism φ of (X, p) induces two automorphisms φ_1 and φ_2 of X by $\varphi_1 = \varphi \mid V$ and $\varphi_2 = \varphi \mid V'$. For these two automorphisms there holds

$$\varphi_1 = p^{-1}\varphi_2 p .$$

Proof. Let $[x, px] \in E_p$, then $[\varphi x, \varphi px] \in E_p$ and therefore $\varphi px = p\varphi x$ because of the definition of E_p. Since $x \in V$, $\varphi x \in V$, $px \in V'$ this equation is equivalent to $\varphi_2 px = p\varphi_1 x$ and this holds for all $x \in V$, which proves the assertion.

5) If φ_1, φ_2 are two automorphisms of X and $\varphi_1 = p^{-1}\varphi_2 p_2$, then a positive natural automorphism of (X, p) can be defined in the following way

$$\varphi x = \varphi_1 x \quad \text{for} \quad x \in V$$

$$\varphi x = \varphi_2 x \quad \text{for} \quad x \in V'.$$

Proof. We only have to show that any edge of E_p is again mapped onto such an edge. Let $[x, px] \in E_p$ and consider $[\varphi x, \varphi px]$. Since $x \in V$, $px \in V'$, we have $[\varphi x, \varphi px] = [\varphi_1 x, \varphi_2 px] = [p^{-1}\varphi_2 px, \varphi_2 px]$ the last equality following from the assumption. But this implies that $[\varphi x, \varphi px] \in E_p$.

We can sumarize 4) and 5) by

Theorem 1. *If φ_1 is an automorphism of X, then there exists a positive natural automorphism φ of (X, p) with $\varphi \mid V = \varphi_1$ iff $p\varphi_1 p^{-1}$ is an automorphism of X. If φ exists, then it is uniquely determined by φ_1 and all positive natural automorphisms are generated in this way.*

We can therefore identify the positive natural automorphisms with the ordered pairs $(\varphi_1, p\varphi_1 p^{-1})$ with $\varphi_1 \in G(X)$, $p\varphi_1 p^{-1} \in G(X)$, $\varphi_1 = \varphi \mid V$, $p\varphi_1 p^{-1} = \varphi \mid V'$ and these pairs with the product taken componentwise, i.e.

$$(\varphi_1, p\varphi_1 p^{-1})\,(\psi_1, p\psi_1 p^{-1}) = (\varphi_1\psi_1, p\varphi_1\psi_1 p^{-1})$$

form a group isomorphic with $G_n^+(X, p)$, which is isomorphic with a subgroup of $G(X)$.

Corollary. *The group $G_n^+(X, p)$ of positive natural automorphisms considered* *as a permutation group on V is isomorphic with the group* $G(X)\cap p^{-1}G(x)p$.

We mention that $p\in G(X)$ implies that the group of positive natural automorphisms is isomorphic with $G(X)$. Since with few exceptions $G(X)$ is not a normal subgroup of the symmetric group on V in general the group $G_n^+(X,p)$ will be isomorphic with a proper subgroup of $G(X)$.

Proof of the Corollary. Let $\varphi_1\in G(X)\cap p^{-1}G(X)p$, then $\varphi_1\in G(X)$ and $\varphi_1=p^{-1}\varphi_2 p$ with $\varphi_2\in G(X)$ such that $p\varphi_1 p^{-1} = \varphi_2\in G(X)$ and therefore to the pair $(\varphi_1, p\varphi_1 p^{-1})$ there corresponds a positive natural automorphism in the above described way.

We continue with similar considerations about negative natural automorphisms.

6) A negative natural automorphism φ induces two isomorphisms $\varphi_1':X\to X'$ with $\varphi_1'=\varphi\,|\,V: V\to V'$ and $\varphi_2':h'\to X$ with $\varphi_2'=\varphi\,|\,V': V'\to V$. If $\varphi_1'(v_i)=$ $=w_j$, then let $\varphi_1(v_i)=v_j$ and if $\varphi_2'(w_j)=v_i$ then let $\varphi_2(v_j)=v_i$. Let $[x, px]\in E_p$ and therefore $[\varphi x, \varphi px]\in E_p$. Now $[\varphi x, \varphi px] = [(\varphi_1 x)', \varphi_2' px]$ and because of $\varphi_1' x\in V'$, $\varphi_2' px\in V$ we must have

$$p\varphi_2' px=\varphi_1' x,\ \text{i.e.}\ p\varphi_2 p=\varphi_1.$$

7) Let conversely φ_1, φ_2 be two automorphisms of X with $\varphi_1=p\varphi_2 p$. Then a negative natural automorphism of (X,p) can be defined by

$$\varphi x=(\varphi_1 x)' \quad \text{for} \quad x\in V$$

$$\varphi x'=\varphi_2 x \quad \text{for} \quad x'\in V'.$$

Proof. We only have to consider edges in E_p. Let $[x, px]\in E_p$; then $[\varphi x, \varphi px]$ $= [(\varphi_1 x)', \varphi_2 px] = [\varphi_2 px, p(\varphi_2 px)]\in E_p$, which shows φ to be an automorphism.

From 6) and 7) we get

Theorem 2. *If φ_1 is an automorphism of X, then there exists a negative natural automorphism φ of (X,p) with $\varphi x=(\varphi_1 x)'$ for all $x\in V$ iff $p^{-1}\varphi_1 p^{-1}$ is an automorphism of X too. If φ exists, then it is uniquely determined by φ_1 and all negative natural automorphisms are generated in this way.*

Corollary 1. *There exists a negative natural automorphism of (X,p) iff $G(X)\cap pG(X)p\neq\emptyset$.*

Corollary 2. *If p is of order two, i.e. if* $p^2=\text{id}$, *then there always exist negative natural automorphisms.*

Proof. If $p^2=\text{id}$, then $G(X)\cap pG(X)p$ contains at least $p^2=\text{id}$ and a negative natural automorphism φ_0 is defined by $\varphi_0 x=x'$, $\varphi_0 x'=x$ (compare 7) with $\varphi_1=\varphi_2=\text{id}$).

Corollary 3. *If for the permutation p of V and the group of automorphisms G(X) of the graph X there holds* $pG=Gp$ *and* $p^2=\text{id}$, *then G is isomorphic with* *a normal subgroup of index two in the group $G_n(X, p)$ of natural automorphisms of (X, p).*

Proof. From Theorem 1 and its Corollary we have that G is isomorphic with the group of positive natural automorphisms of (X, p), an isomorphism being the mapping $\varphi \to (\varphi, p\varphi p) = \psi$ where the first component of this pair is the restriction of φ to V and the second component the restriction of φ to V'. From $p^2 = \mathrm{id}$ we have that there exist negative natural automorphisms. This together implies the assertion.

Remark. In general under the assumptions of Corollary 3 $G_n(X, p)$ will not be isomorphic to the direct product of $G(X)$ and Z_2 (the cyclic group of order two). For this we need an additional assumption.

Corollary 4. *Suppose the conditions of Corollary 3 are met and additionally let p belong to the commutator of G(X) in the symmetric group on V. Then $G_n(X, p)$ is isomorphic with the direct product of $G(X)$ and Z_2.*

Proof. From the assumptions it easily follows that $\{\mathrm{id}, \varphi_0\}$, with φ_0 the negative natural automorphism φ_0 as given in Corollary 2, is a normal subgroup of $G_n(X, p)$, too and any positive natural automorphism commutes with φ_0. This implies the assertion.

We now turn to the question under what circumstances a permutation graph (X, p) can have only natural automorphisms.

Definition 2. *Let $X = (V, E)$ be a graph and $V = A \cup B$ be the partition of the vertex set corresponding to a cut-set $F \subset E$ of X. The cut-set F is called a simple cut-set if each vertex of A and each vertex of B is incident with at most one edge of F.*

Theorem 3. *If $X = (V, E)$ is connected and does not possess a simple cut-set, then for every permutation p of V the permutation graph (X, p) has only natural automorphisms. If X is not connected or if X has a simple cut-set, then for a suitable permutation p the permutation graph (X, p) has also nonnatural automorphisms.*

Proof. Let X be connected without simple cut-sets and suppose that $\varphi \in G(X, p)$ is a nonnatural automorphism. Then $\varphi V \cap V \neq \emptyset$ and also $\varphi V \cap V' \neq \emptyset$ and $V = \varphi^{-1}(\varphi V \cap V) \cup \varphi^{-1}(\varphi V \cap V') = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$. Now from the definition of a permutation graph it is clear that the edges of E_p connecting vertices of $\varphi V \cap V$ with vertices of $\varphi V \cap V'$ are a simple cut-set in the graph induced by φV and therefore V_1, V_2 determine a simple cut-set of X, a contradiction.

Now suppose that by $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ a simple cut-set F of X is determined and that X is connected. Let the edges in F be $[x_1, y_1], \ldots, [x_s, y_s]$ with

$x_i \in V_1$, $y_i \in V_2$ for $i = 1, 2, \ldots, s$. Define the permutation p on V in the following way:

$$px_i = y_i,\ py_i = x_i, \qquad i = 1, \ldots, s$$

and

$$px = x \quad \text{if} \quad x \in V - \{x_1, \ldots, x_s, y_1, \ldots, y_s\}\ .$$

Then in (X, p) a nonnatural automorphism φ can be defined (for notation compare above)

$$\begin{aligned} \varphi x &= x' && \text{if} \quad x \in V_1, \\ \varphi x &= x && \text{if} \quad x \in V_2, \\ \varphi x' &= x && \text{if} \quad x' \in V_1', \\ \varphi x' &= x' && \text{if} \quad x' \in V_2'. \end{aligned}$$

That means that φ interchanges pointwise the two copies of V_1 and keeps V_2, V_2' pointwise fixed. It is an easy task to show that φ is an automorphism. If X is not connected, then it suffices to take for p the identity because then in (X, p) one can combine a positive natural automorphism of one component with a negative natural automorphism of another component to obtain a nonnatural automorphism of (X, p).

Remark 1. In general the automorphism group $G(X, p)$ because of the last theorem will contain $G_n(X, p)$ as a proper subgroup. We give an example for this where $G_n(X, p) = \{\text{id}\}$, but $G(X, p) \neq \{\text{id}\}$. (see Fig 1.)

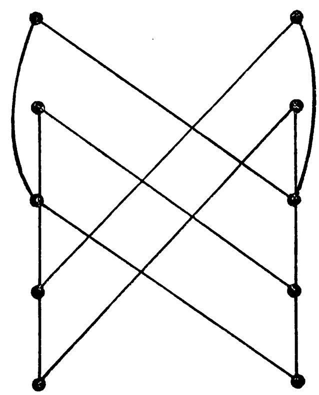

Fig. 1

Remark 2. In connection with the last theorem we should mention that two permutation graphs (X_1, p_1) and (X_2, p_2) with nonisomorphic graphs X_1, X_2 can be isomorphic if X_1 and X_2 possess simple cut-sets. An example is given in the next figure 2.

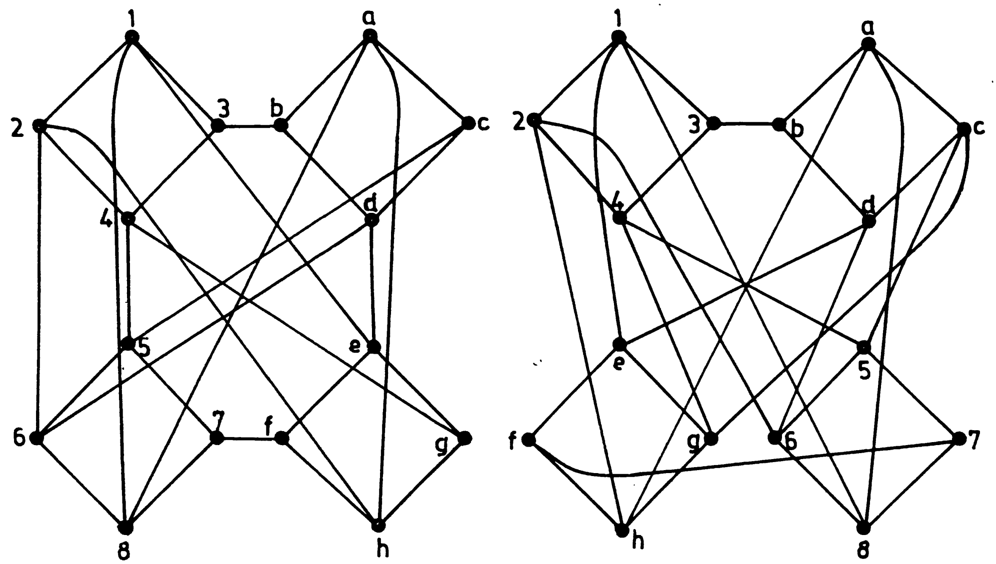

Fig. 2

This example contradicts also the possible conjecture that from $(X_1, p_1) \cong (X_2, p_2)$ and $X_1 \not\cong X_2$ there follows that X_1, X_2 themselves are permutation graphs. The only fact one can deduce from this situation seems to be that X_1 and X_2 must contain every graph without simple cut-set as a subgraph with the same multiplicity. If X_1 or X_2 does not possess simple cut-sets, then clearly $(X_1, p_1) \cong (X_2, p_2)$ implies $X_1 \cong X_2$.

We now consider the problem of when two permutation graphs (X, p_1) and (X, p_2) are isomorphic. The general problem here seems to be rather difficult and therefore we restrict ourselves to natural isomorphisms which correspond to the natural automorphisms of (X, p).

Definition 3. *Let* (X, p_1), (X, p_2) *be two permutation graphs belonging to the graph* $X=(V, E)$. *An isomorphism* φ φ: $(X, p_1)\to(X, p_2)$ *is called a natural isomorphism if* $\varphi V = V$ *or* $\varphi V = V'$.

From the definition it is clear that a natural isomorphism φ: $(X, p_1)\to(X, p_2)$ induces two automorphisms φ_1 and φ_2 *of* X. If $\varphi V = V$, $\varphi V' = V'$, then we have $\varphi x = \varphi_1 x$, $\varphi x' = (\varphi_2 x)'$; if $\varphi V = V'$, $\varphi V' = V$, then we have $\varphi x = (\varphi_1 x)'$ and $\varphi x' = \varphi_2 x$ for all $x \in V$.

Theorem 4. *If X is connected and does not possess a simple cut-set, then every isomorphism between two permutation graphs (X, p_1) and (X, p_2) is natural.*

Proof. One proceeds as in the proof of Theorem 3.

Theorem 5. *For every permutation p of V there exists a natural isomorphism φ from (X, p) onto (X, p^{-1}) with $\varphi V = V'$.*

Proof. Define φ by $\varphi x = x'$, $\varphi x' = x$ for all $x \in V$. Then we only have to consider the edges in E_p. Let therefore $[x, (px)'] \in E_p$; then $[\varphi x, \varphi(px)'] = [x', px] = [px, (p^{-1}(px))'] \in E_{p^{-1}}$, which shows φ to be an isomorphism.

Theorem 6. *If there exists a natural isomorphism φ from (X, p_1) onto (X, p_2) with $\varphi V = V$, then for the two automorphisms φ_1, φ_2 induced by φ there holds $p_2 = \varphi_2 p_1 \varphi_1^{-1}$. If conversely there exist two automorphisms φ_1, φ_2 with $p_2 = \varphi_2 p_1 \varphi_1^{-1}$, then a natural isomorphism φ from (X, p_1) onto (X, p_2) with $\varphi V = V$ is defined by $\varphi x = \varphi_1 x$, $\varphi x' = (\varphi_2 x)'$ for all $x \in V$.*

Proof. Let φ be given as in the Theorem and $[x, (p_1 x)'] \in E_p$. Then $[\varphi x, \varphi(p_1 x)'] = [\varphi_1 x, (\varphi_2 p_1 x)']$ belongs to E_p such that $(p_2 \varphi_1 x)' = (\varphi_2 p_1 x)'$ for all $x \in V$, i.e. $p_2 \varphi_1 = \varphi_2 p_1$. Let conversely φ be defined as in the Theorem. We only have to consider edges in E_p. Let $[x, (p_1 x)'] \in E_p$. Then $[\varphi x, \varphi(p_1 x)'] = [\varphi_1 x, (\varphi_2 p_1 x)'] \in E_p$, because $p_2 \varphi_1 x = \varphi_2 p_1 x$ by assumption.

By similar arguments or by the use of Theorem 5 one proves:

Theorem 7. *If there exists a natural isomorphism φ from (X, p_1) onto (X, p_2) with $\varphi V = V'$, then for the two automorphisms φ_1, φ_2 induced by φ there holds $p_2 = \varphi_2 p_1^{-1} \varphi_1^{-1}$. If conversely there exist two automorphisms φ_1, φ_2 with $p_2 = \varphi_2 p_1^{-1} \varphi_1^{-1}$, then a natural isomorphism φ from (X, p_1) onto (X, p_2) with $\varphi V = V'$ is defined by $\varphi x = (\varphi_1 x)'$, $\varphi x' = \varphi_2 x$ for all $x \in V$.*

Corollary 1. *There exists a natural isomorphism from (X, p_1) onto (X, p_2) with $\varphi V = V$ iff the intersection of the two cosets $p_2 G(X)$ and $G(X) p_1$ is not empty. Similarly a φ with $\varphi V = V'$ exists iff $p_2 G(X) \cap G(X) p_1^{-1} \neq \emptyset$ (or equivalently $p_1 G(X) \cap G(X) p_2^{-1} \neq \emptyset$).*

Corollary 2. *Let $X = (V, E)$ be connected without simple cut-sets. Then $(X, p_1) \cong (X, p_2)$ iff p_1 and p_2 or p_1 and p_2^{-1} belong to the same double coset $G(X) p G(X)$ of $G(X)$ in the symmetric group on V.*

Proof. From Theorem 4 we have that every isomorphism from (X, p_1) onto (X, p_2) is a natural isomorphism and because of Theorem 5 we can restrict ourselves to natural isomorphisms φ with $\varphi V = V$. But then the corollary follows immediately from Theorems 6 and 7.

Corollary 3. *Let $X = (V, E)$ be an arbitrary graph. Then the number of isomorphism classes of permutation graphs (X, p) is at most equal to the number of double cosets of $G(X)$ in the symmetric group on V.*

If we call a double coset $G(X)pG(X)$ an *inverse double coset* if $p^{-1} \in G(X)pG(X)$, then we can make a precise assertion about the number of isomorphism classes of permutation graphs (X, p).

Corollary 4. *Let $X = (V, E)$ be a connected graph without simple cut-sets. Then the number of isomorphism classes of permutation graphs is equal to $\frac{1}{2}N + M$, where N = number of noninverse double cosets and M = number of inverse double cosets of $G(X)$ in the symmetric group on V.*

In general this is not true because there may be nonnatural isomorphisms between permutation graphs belonging to different double cosets.

Example. The following two permutation graphs (Fig. 3) are isomorphic and a nonnatural isomorphism is given by mapping vertices with the same label onto one another. There does not exist a natural isomorphism.

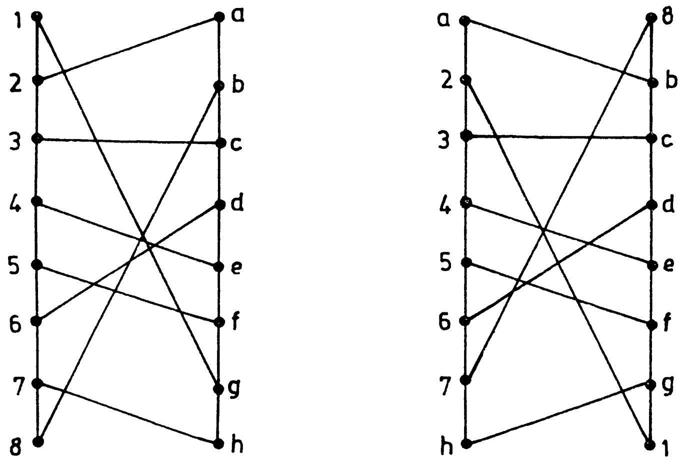

Fig. 3

Up to this point all results except the last two corollaries remain true also in the case of infinite graphs. In the next theorem we use for the first time the finiteness of the considered graphs.

Theorem 8. *Let $X = (V, E)$ be an arbitrary (finite) graph. Then $(X, \mathrm{id}) \cong$ $\cong (X, p)$ iff p is an automorphism of X.*

Proof. Suppose first that $p \in G(X)$. Then the mapping φ: $(X, \mathrm{id}) \to (X, p)$ defined by

$$\varphi x = x,\quad \varphi x' = (px)' \text{ for all } x \in X$$

is an isomorphism. Clearly we only have to consider edges in E_{id}, i.e. the edges $[x, x']$. There holds $\varphi[x, x'] = [x, (px)'] \in E_p$, which proves the assertion. Let conversely $p \notin G(X)$. We count the number of quadrangles in (X, id) and in (X, p). If N is the number of quadrangles in X, then in (X, id) we have $2N + |E|$ quadrangles. The quadrangles of (X, p) are the quadrangles in X and X', resp., and quadrangles with vertices x_1, $(px_1)'$, x_2, where $[x_1, x_2] \in E$ and $[px_1, px_2] \in E$. Therefore the number of quadrangles in (X, p) is at most equal to $2N + |E|$ with equality iff to every edge in E there corresponds a quadrangle as above. Since $p \notin G(X)$, there exists an edge $[x_1, x_2] \in] \in E$ such that $[px_1, px_2] \notin E$ and evidently there cannot correspond to this edge a quadrangle of (X, p). This implies that (X, id) is not isomorphic with (X, p).

Remark. If $p \in G(X)$, then $(X, \text{id}) \cong (X, p)$ with an isomorphism which on V induces the identity. This holds also under a more general assumption.

Theorem 9. *If p_1, p_2 belong to the same right coset of $G(X)$ in the symmetric group on V, then $(X, p_1) \cong (X, p_2)$ with a natural isomorphism φ such that $\varphi V = V$ and φ induces the identity on V.*

Proof. It follows that $p_2 p_1^{-1} \in G(X)$. We define the mapping $\varphi:(X, p_1) \to (X, p_2)$ by

$$\varphi x = x \text{ and } \varphi x' = (p_2 p_1^{-1} x)' \text{ for all } x \in V.$$

To show then that φ is an isomorphism we only have to consider the edges in E_p. Let $[x, (p_1 x)'] \in E_p$. Then $[\varphi x, \varphi(p_1 x)'] = [x, (p_2 p_1^{-1} p_1 x)'] = [x, (p_2 x)'] \in E_p$, which proves the assertion.

Similarly one proves

Theorem 9'. *If p_1 and p_2 belong to the same left coset of $G(X)$ in the symmetric group on V, then $(X, p_1) \cong (X, p_2)$ with a natural isomorphism φ such that $\varphi V' = V'$ and φ induces the identity on V'.*

Definition 4. *Let $X = (V, E)$ be a graph, $X' = (V', E')$ a disjoint isomorphic copy of x and f: $V \to V$ an arbitrary mapping. The mapping graph (X, f) is defined as the union of X and X' with the additional edges $[x, (fx)']$ for all $x \in V$.*

The concepts of *natural automorphism* and *natural isomorphism* are defined as for permutation graphs. If f is not a permutation, then of course only those natural isomorphisms φ and automorphisms φ exist for which $\varphi V = V$. We restrict ourselves in the following to proper mapping graphs (X, f), where f is not a permutation. Similarly as Theorems 1 and 6 one can prove

Theorem 10. *Let (X, f) be a mapping graph of X. If $\varphi_1 \in G(X)$, then there* *exists a natural automorphism φ of (X, f) with $\varphi \mid V = \varphi_1$ iff there exists $\varphi_2 \in G(X)$ with $\varphi_2 f = f\varphi_1$. In this case $\varphi_2 = \varphi \mid V'$.*

Theorem 11. *If there exists a natural isomorphism φ from (X, f_1) onto (X, f_2), then for the two automorphisms φ_1, φ_2 induced by φ there holds $\varphi_2 f_1 = f_2 \varphi_1$.*

If conversely there exist two automorphisms $\varphi_1, \varphi_2 \in G(X)$ with $\varphi_2 f_1 = f_2 \varphi_1$, then by $\varphi x = \varphi_1 x$, $\varphi x' = (\varphi_2 x)'$ a natural isomorphism $\varphi: (X, f_1) \to (X, f_2)$ is defined.

Remark. The automorphism $\varphi_2 \in G(X)$ in Theorem 10 fulfils $\varphi_2(fV) = fV$, i.e. φ_2 belongs to the stabilizer of fV in $G(X)$. As an example consider the case that f is a constant mapping with $fx = x_0$ for all $x \in V(X)$ and that X is connected. Then $\varphi_2 f = f\varphi_1$ holds for all $\varphi_1 \in G(X)$ and φ_2 belonging to the stabilizer of x_0 in $G(X)$. Therefore the group of natural automorphisms $G_n(X, f)$ in this case is isomorphic with the direct product of $G(X)$ and the stabilizer of x_0 in $G(X)$.

By the *kernel* of $f: V \to V$ one means the equivalence relation on V defined by $x_1 \sim x_2$ iff $fx_1 = fx_2$. We say that a *permutation φ on V fixes the kernel* of f if $\varphi A = A$ for every class of the kernel. Then there holds that the group of automorphisms φ_1 of X which fix the kernel of f is isomorphic with a subgroup of $G(X, f)$. This follows from the fact that $f\varphi_1 = f = f \circ \mathrm{id}$ for every φ_1 fixing the kernel of f.

The next theorem describes a class of mapping graphs for which the identity is the only natural automorphism.

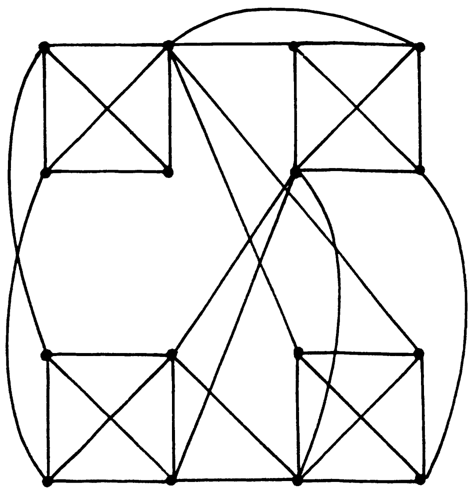

Fig. 4

Theorem 12. *Let X be a graph with an intransitive automorphism group $G(X)$ such that some nontrivial orbit of $G(X)$ contains a vertex x_0 which is not fixed by* *any $\varphi \in G(X)$, $\varphi \neq \mathrm{id}$. Then there exist f such that (X, f) has no nontrivial natural automorphisms.*

Proof. If $G(X)=\{\mathrm{id}\}$, then the assertion is trivial. Otherwise let A be an orbit of $G(X)$ with $|A| \geqq 2$, $B \neq A$ another orbit and $x_0 \in A$ be a vertex with $\varphi x_0 \neq x_0$ for every $\varphi \neq \mathrm{id}$, $\varphi \in G(X)$. Then define f by $fx = x$ for $x \in A$, $x \neq x_0$, $fx_0 = y \in B$, $fy = x_0$ and elsewhere arbitrarily. Now suppose there exist $\varphi_1, \varphi_2 \in G(X)$ with $\varphi_2 f = f\varphi_1$. This implies $\varphi_2 f x_0 = f\varphi_1 x_0$. If $\varphi_1 \neq \mathrm{id}$, then $f\varphi_1 x_0 \in A$ but $\varphi_2 f x_0 \in B$ and therefore $\varphi_1 = \mathrm{id}$, $\varphi_2 f = f$. But now $\varphi_2 f y = \varphi_2 x_0 \neq x_0 = fy$ if $\varphi_2 \neq \mathrm{id}$ and we conclude that $\varphi_2 = \mathrm{id}$, too.

By a partial generalization of Theorem 3 one can easily show that there exist mappings $f\colon V \to V$ which are not permutations and for which (X, f) has nonnatural automorphisms if X is not connected or has a simple cut-set. But the converse is not true as it is shown by the following example. The graph X has no simple cut-set but nevertheless the mapping graph (X, f) as given in the figure 4 has nonnatural automorphisms.

Remark. A part of this paper will appear in a shortened version without full proofs in the Proceedings of International Coll. Problèmes Combinatoires et Théorie des Graphes (Paris, July 9—13, 1976).

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Received January 17, 1977

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О ГРАФАХ ОТОБРАЖЕНИЙ И ГРАФАХ ПЕРЕСТАНОВОК

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Резюме

Пусть даны граф X и перестановка p его множества вершин. Граф перестановки (X, p) состоит из двух непересекающихся экземпляров графа X и всех ребер соединяющих вершину x первого экземпляра с вершиной $p(x)$ второго экземпляра. Если p — какое-нибудь отображение, то (X, p) — граф отображения. Доказываются теоремы об автоморфизмах графа (X, p) и особенно исследуется группа естественных автоморфизмов определенных таким свойством, что данные два экземпляра графа X образуют систему блоков этой группы.