Ladislav Mišík

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ADDITIVITY OF THE GAUGE

LADISLAV MIŠÍK

In his book Lectures on Analysis, Volume I, p. 359 G. Choquet presents Problem 19.5 under the title Additivity of the Gauge. In this problem he asserts the following: *Let X be a convex cone in a Hausdorff topological vector space E and $f\colon X \to \langle 0, \infty\rangle$ a positive homogeneous map. Let $X_f = \{x \in X\colon f(x) \leqq 1\}$. Then X_f is a closed convex set in X with convex complement iff f is lower semi-continuous and additive.*

The following example shows that the assertion in Problem 19.5 is wrong. Let E be the real euclidean space R^2 and $X = \{(x, y) \in R^2\colon x \geqq 0,\ y \geqq 0\}$. Let $f\colon X \to \langle 0, \infty\rangle$ be a function for which $f((x, y)) = 0$ for all $(x, y) \in X$ satisfying the inequality $0 \leqq y \leqq x$ and $f((x, y)) = \infty$ for all $(x, y) \in X$ satisfying the inequality $x < y$. The function f is a positive homogeneous map. The sets $X_f = \{(x, y) \in X\colon 0 \leqq y \leqq x\}$ and $X - X_f = \{(x, y) \in X\colon x < y\}$ are convex and the set X_f is closed in X. The function f is not additive as $f((1,1)) = 0 \neq \infty = f((0,1)) + f((1,0))$.

The exact formulation of Problem 19.5 should be as follows: *Let X be a convex cone in a Hausdorff topological vector space E and $f\colon X \to \langle 0, \infty\rangle$ a positive homogeneous map. Let $X_f = \{x \in X\colon f(x) \leqq 1\}$. Then X_f is a closed convex set in X with convex complement iff f is lower semi-continuous, subadditive and $f(x+y) = f(x) + f(y)$ for all $x, y \in X$ for which $x + y \in X$ and $f(x) > 0$ and $f(y) > 0$.*

Proof. First we prove the necessity. The lower semi-continuity of f is a consequence of the equations $X = \{x \in X\colon f(x) > \alpha\}$ for all $\alpha < 0$, $\{x \in X\colon f(x) > 0\} = \bigcup_{n=1}^{\infty} \left\{x \in X\colon f(x) > \frac{1}{n}\right\}$ and $\{x \in X\colon f(x) > \alpha\} = X - \alpha X_f$ for all $\alpha > 0$ and of closeness of X_f in X.

Let $x, y, x + y \in X$. The inequality $f(x+y) \leqq f(x) + f(y)$ is evident if $f(x) + f(y) = \infty$. *Let* $0 < \min(f(x), f(y)) \leqq \max(f(x), f(y)) < \infty$. Then $\frac{x}{f(x)}$, $\frac{y}{f(y)} \in X_f$. From the convexity of X_f we have $\frac{x+y}{f(x)+f(y)} = \frac{f(x)}{f(x)+f(y)}\frac{x}{f(x)} + \frac{f(y)}{f(x)+f(y)}\frac{y}{f(y)} \in X_f$. Therefore $f(x+y) \leqq f(x) + f(y)$. Let $f(x) = 0$ and $0 < f(y) < \infty$. Then αx, $\frac{y}{f(y)} \in X_f$ for all $\alpha > 0$. From the convexity of X_f we have

$$\frac{x+\left(1-\frac{1}{n}\right)y}{f(y)}=\frac{1}{n}\frac{nx}{f(y)}+\left(1-\frac{1}{n}\right)\frac{y}{f(y)}\in X_f$$ for $n=1, 2, 3, \ldots$. But X_f is closed in X. Therefore $\frac{x+y}{f(y)}\in X_f$ and $f(x+y)\leqq f(x)+f(y)$. The case $0<f(x)<\infty$, $f(y)=0$ *is now obvious. Let* $f(x)=0$ and $f(y)=0$. Then $2nx$, $2ny\in X_f$ for $n=1, 2, 3, \ldots$. Therefore $n(x+y)=\frac{1}{2}(2nx+2ny)\in X_f$ and $f(x+y)\leqq\frac{1}{n}$ for $n=1, 2, 3, \ldots$. This shows that $f(x+y)=0=f(x)+f(y)$. The subadditivity of f is proved.

Let now $x, y, x+y\in X$ and $f(x)>0$ and $f(y)>0$. To prove the equation $f(x+y)=f(x)+f(y)$ it suffices to prove $f(x+y)\geqq f(x)+f(y)$. But this holds as $f(x)+f(y)=\sup\{\alpha: 0<\alpha<f(x)\}+\sup\{\beta: 0<\beta<f(y)\}=$ $=\sup\{\alpha+\beta: 0<\alpha<f(x), 0<\beta<f(y)\}\leqq f(x+y)$. The last inequality follows from the convexity of $X-X_f$ and from the relations $\frac{x+y}{\alpha+\beta}=\frac{\alpha}{\alpha+\beta}\frac{x}{\alpha}+\frac{\beta}{\alpha+\beta}\frac{y}{\beta}$, $\frac{x}{\alpha}$, $\frac{y}{\beta}\in X-X_f$, which hold for all $0<\alpha<f(x)$ and $0<\beta<f(y)$.

Let now f be lower semi-continuous, subadditive and $f(x+y)=f(x)+f(y)$ for all x, y for which $x, y, x+y\in X$ and $f(x)>0$, $f(y)>0$. The closeness of X_f in X follows from the lower semi-continuity of f. The convexity of X_f is a consequence of the subadditivity and positive homogeneity of f. We get the convexity of $X-X_f$ as follows: Let x, $y\in X-X_f$, $\alpha>0$, $\beta>0$ and $\alpha+\beta=1$. Then $f(x)>1$ and $f(y)>1$, and so we have $f(\alpha x)>0$ and $f(\beta y)>0$. Therefore $f(\alpha x+\beta y)=$ $=f(\alpha x)+f(\beta y)=\alpha f(x)+\beta f(y)>1$. Then $\alpha x+\beta y\in X-X_f$.

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Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

АДДИТИВНОСТЬ МАСШТАБА

Ладислав Мишик

Резюме

В этой работе дана верная формулировка вопроса 19.5 из книги G. Choquet, Lectures on Analysis, Volume I.