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NOTE ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

JAROSLAV WERBOWSKI

We consider the asymptotic behavior of the solutions of the following differential equation

(1) $$x^{(n)}(t)+f(t,x(g_0(t)),x'(g_1(t)),\dots,x^{(n-1)}(g_{n-1}(t)))=0,\quad n\geqq 2,$$

where the functions g_k: $\langle 0,\infty)\to R$, $\lim_{t\to\infty} g_k(t)=\infty$ $(k=0, 1, \dots, n-1)$ and f: $\langle 0,\infty)\times R^n\to R$ are continuous and such that they guarantee the existence of solutions of (1) which are indefinitely extendable to the right. In the following we shall always suppose that the function f satisfies the conditions:

(2) $$x_0 f(t, x_0, \dots, x_{n-1})>0 \quad \text{for} \quad x_0\neq 0,$$

(3) $$|f(t, x_0, \dots, x_{n-1})|\leqq|f(t, y_0, \dots, y_{n-1})| \quad \text{for} \quad |x_k|\leqq|y_k|$$
$$(k=0, 1, \dots, n-1),\quad x_0y_0>0.$$

In this note we present a necessary and sufficient condition for the existence of a solution $x(t)$ of equation (1) with the property

(M) $$\lim_{t\to\infty} t^{k-m}x^{(k)}(t)=L_k\neq 0 \quad (k=0, 1, \dots, m),$$
$$\lim_{t\to\infty} x^{(k)}(t)=0 \quad (k=m+1, \dots, n-1),$$

where $m\in\{0, 1, \dots, n-1\}$ and L_k a constants. This problem for differential equations with a retarded argument in the cases $m=0$ and $m=n-1$ was investigated in [3—9]. The proofs of theorems of this note are based on combining the arguments of Bobrowski [1] and Kiguradze [2] with those of Marušiak [5].

Theorem 1. *If equation* (1) *has a solution* $x(t)$ *with the property* (M), *then*

$$\int^{\infty} t^{n-1-m}|f(t, C_0 g_0^m, C_1 g_1^{m-1}, \ldots, C_{m-1} g_{m-1}, C_m, 0, \ldots, 0)|\,dt < \infty,$$

$$0 \neq C_k = constant \quad (k=0, 1, \ldots, m).$$

Proof. Let the solution $x(t)$ of (1) have the property (M) and assume that $L_k > 0$ $(k=0, 1, \ldots, m)$ (a similar argument holds if $L_k < 0$). Then there exists a point $t_0 \geqslant 0$ such that

$$x^{(k)}(t) \geqslant C_k t^{m-k}, \quad C_k = \tfrac{1}{2} L_k, \quad (k=0, 1, \ldots, m),$$
$$(-1)^{n-1-k} x^{(k)}(t) \geqslant 0 \quad (k=m+1, \ldots, n-1)$$

for $t \geqslant t_0$. Choose a point $T \geqslant t_0$ such that $g_k(t) \geqslant t_0$ $(k=0, 1, \ldots, n-1)$ for $t \geqslant T$. Therefore for $t \geqslant T$ we have

$$\begin{aligned} &x^{(k)}(g_k) \geqslant C_k g_k^{m-k} \quad (k=0, 1, \ldots, m), \\ &(-1)^{n-1-k} x^{(k)}(g_k) \geqslant 0 \quad (k=m+1, \ldots, n-1). \end{aligned} \tag{4}$$

Multiplying both sides of equation (1) by t^{n-1-m} and integrating from T to t we obtain

$$(-1)^{n-m-1}[x^{(m)}(t) - x^{(m)}(T)] + P_m(T) = P_m(t) +$$
$$+ \int_T^t s^{n-1-m} f(s, x(g_0), \ldots, x^{(n-1)}(g_{n-1}))\,ds,$$

where

$$P_m(t) = \sum_{k=m+1}^{n-1} (-1)^{n-1-k} \frac{(n-1-m)!}{(k-m)!} t^{k-m} x^{(k)}(t)$$
$$\text{for} \quad 0 \leqslant m \leqslant n-2, \quad P_{n-1}(t) = 0.$$

Since $\lim_{t\to\infty} x^{(m)}(t) = L_m < \infty$ and $P_m(t)$ is a nonnegative function, then the integral on the right side converges as $t \to \infty$. Thus in view of (2), (3) and (4) we obtain for all $t \geqslant T$

$$\int_T^t s^{n-1-m} f(s, C_0 g_0^m, C_1 g_1^{m-1}, \ldots, C_{m-1} g_{m-1}, C_m, 0, \ldots, 0)\,ds \leqslant$$
$$\leqslant \int_T^t s^{n-1-m} f(s, x(g_0), \ldots, x^{(n-1)}(g_{n-1}))\,ds < \infty.$$

Theorem 2. *Let* $m \in \{0, 1, \ldots, n-1\}$, *and let for any constants* $C_k \neq 0$ $(k = 0, 1, \ldots, n-1)$

(5) $$|f(t, C_0 g_0^m, C_1 g_1^{m-1}, \dots, C_{m-1}, C_m, \dots, C_{n-1})| \quad \textit{is bounded,}$$

(6) $$\int^{\infty} t^{n-1-m} |f(t, C_0 g_0^m, C_1 g_1^{m-1}, \dots, C_{m-1} g_{m-1}, C_m, \dots, C_{n-1})| \, dt < \infty .$$

Then equation (1) *has a solution $x(t)$ with the property* (M).

Proof. The existence of a solution $x(t)$ of (1), having the property (M), will be proved by the method of successive approximations. We denote

$$Q_l(t) = \int_t^{\infty} \frac{(s-t)^{n-1-l}}{(n-1-l)!} f\left(s, C_0 g_0^m, C_1 g_1^{m-1}, \dots, C_{m-1}, C_m, \frac{1}{2}, \dots, \frac{1}{2}\right) ds ,$$

$$C_k = \frac{1}{(m-k)!} \quad (k = 0, 1, \dots, m).$$

From the condition (6) it follows that $\lim_{t \to \infty} Q_l(t) = 0$ $(1 = m, \dots, n-1)$. Let a point $T > 0$ be chosen such that $Q_l(t) \leq \frac{1}{2}$ $(l = m, \dots, n-1)$ for all $t \geq T$. Consider the sequence of functions $\{y_{k,i}(t)\}_{i=0}^{\infty}$ $(k = 0, 1, \dots, n-1)$ defined as follows:

(7) $$y_{k,0}(t) = \begin{cases} C_k & (k = 0, 1, \dots, m) \\ 0 & (k = m+1, \dots, n-1) \end{cases} \quad \text{for} \quad t < T \text{ and } t \geq T,$$

(8) $$y_{k,i}(t) = \begin{cases} C_k & (k = 0, 1, \dots, m) \\ 0 & (k = m+1, \dots, n-1) \end{cases} \quad \text{for} \quad t < T,$$

and for $t \geq T$

(9) $$\begin{cases} y_{k,i}(t) = C_k + (-1)^{n-1-m} t^{k-m} \displaystyle\int_T^t \frac{(t-s)^{m-1-k}}{(m-1-k)!} R_{m,i-1}(s) \, ds , \\ \qquad (k = 0, 1, \dots, m-1), \\ y_{m,i}(t) = C_m + (-1)^{n-1-m} R_{m,i-1}(t), \\ y_{k,i}(t) = (-1)^{n-1-k} R_{k,i-1}(t), \quad (k = m+1, \dots, n-1), \end{cases}$$

where

$$R_{l,i}(t) = \int_t^{\infty} \frac{(s-t)^{n-1-l}}{(n-1-l)!} f(s, g_0^m y_{0,i}, g_1^{m-1} y_{1,i}, \dots, g_{m-1} y_{m-1}, y_{m,i}, \dots, y_{n-1,i}) \, ds ,$$

$$g_k = g_k(s), \quad y_{k,i} = y_{k,i}(g_k(s)), \quad (k = 0, 1, \dots, n-1).$$

From (7)—(9) it is obvious that the functions $\{y_{k,i}(t)\}_{i=0}^{\infty}$ $(k = 0, 1, \dots, n-1)$ are continuous for $t \geq T$.

Let $n - 1 - m$ be odd. By mathematical induction we shall prove that

(10) $$\begin{matrix} \frac{1}{2} C_k \leq y_{k,i}(t) \leq C_k & (k = 0, 1, \dots, m) \\ 0 \leq (-1)^{n-1-k} y_{k,i}(t) \leq \frac{1}{2} & (k = m+1, \dots, n-1) \end{matrix} \qquad i = 1, 2, \dots$$

for $t \geq T$. From (7) it follows that for $t \geq T$ we have

$$y_{k,0}(g_k(t)) = \begin{cases} C_k & (k=0, 1, \ldots, m), \\ 0 & (k=m+1, \ldots, n-1). \end{cases}$$

Therefore, in view of (2) and (3) we obtain for $t \geq T$

$$R_{l,0}(t) \leq Q_l(t) \leq \tfrac{1}{2} \quad (l=m, \ldots, n-1).$$

Then from (9) we get for $t \geq T$

$$C_k \geq y_{k,1}(t) = C_k - t^{k-m} \int_T^t \frac{(t-s)^{m-1-k}}{(m-1-k)!} R_{m,0}(s)\,\mathrm{d}s \geq$$

$$\geq C_k - \tfrac{1}{2} t^{k-m} \int_T^t \frac{(t-s)^{m-1-k}}{(m-1-k)!}\,\mathrm{d}s \geq \tfrac{1}{2} C_k$$

$$(k=0, 1, \ldots, m-1),$$

$$C_m \geq y_{m,1}(t) = C_m - R_{m,0}(t) \geq \tfrac{1}{2} C_m,$$

$$0 \leq (-1)^{n-1-k} y_{k,1}(t) = R_{k,0}(t) \leq \tfrac{1}{2} \quad (k=m+1, \ldots, n-1).$$

Suppose that the condition (10) holds for some $i=j$. Then for $t \geq T$ we have

$$\tfrac{1}{2} C_k \leq y_{k,j}(g_k) \leq C_k \quad (k=0, 1, \ldots, m),$$

$$0 \leq (-1)^{n-1-k} y_{k,j}(g_k) \leq \tfrac{1}{2} \quad (k=m+1, \ldots, n-1),$$

which implies $R_{l,j}(t) \leq Q_l(t) \leq \frac{1}{2}$ $(1=m, \ldots, n-1)$. Then from (9) we obtain for $t \geq T$

$$C_k \geq y_{k,j+1}(t) = C_k - t^{k-m} \int_T^t \frac{(t-s)^{m-1-k}}{(m-1-k)!} R_{m,j}(s)\,\mathrm{d}s \geq \tfrac{1}{2} C_k$$

$$(k=0, 1, \ldots, m-1),$$

$$C_m \geq y_{m,j+1}(t) = C_m - R_{m,j}(t) \geq \tfrac{1}{2} C_m,$$

$$0 \leq (-1)^{n-1-k} y_{k,j+1}(t) = R_{k,j}(t) \leq \tfrac{1}{2} \quad (k=m+1, \ldots, n-1).$$

Therefore the condition (10) holds for $i=1, 2, \ldots$. From (5), (9) and (10) we get for $t \geq T$ and $i=1, 2, \ldots$

$$|y'_{k,i}(t)| = t^{-1} |y_{k+1,i}(t) - (m-k) y_{k,i}(t) \leq 2T^{-1} C_{k+1}$$

$$(k=0, 1, \ldots, m-1),$$

$$|y'_{k,i}(t)| = |y_{k+1,i}(t)| \leq \tfrac{1}{2} \quad (k=m, \ldots, n-2), \tag{11}$$

$$|y'_{n-1,i}(t)| = |R'_{n-1,i-1}(t)| \leq |f(t, C_0 g_0^m, C_1 g_1^{m-1}, \ldots, C_m, \tfrac{1}{2}, \ldots, \tfrac{1}{2})| \leq L,$$

where L is a positive constant. In view of (10) and (11) the family $\{y_{k,i}(t)\}_{i=0}^{\infty}$ $(k=0, 1, \ldots, n-1)$ is uniformly bounded and equicontinuous on $\langle T, A\rangle \subset \subset \langle T, \infty)$, ($A$ is arbitrary). We extract from $\{y_{k,i}(t)\}$ a uniformly convergent subsequence $\{y_{k,i_j}(t)\}$ on $\langle T, A\rangle$ and convergent on $\langle T, \infty)$, i.e. $\lim_{j\to\infty} y_{k,i_j}(t) = y_k(t)$ $(k=0, 1, \ldots, n-1)$ exist on $\langle T, \infty)$. Then $y_k(t)$ $(k=0, 1, \ldots, n-1)$ is a solution of the following system of integral equations

$$(12)\quad \begin{cases} y_k(t) = C_k - t^{k-m} \displaystyle\int_T^t \frac{(t-s)^{m-1-k}}{(m-1-k)!} R_m(s)\,\mathrm{d}s, & (k=0, 1, \ldots, m-1), \\ y_m(t) = C_m - R_m(t), & \\ y_k(t) = (-1)^{n-1-k} R_k(t), & (k=m+1, \ldots, n-1). \end{cases}$$

for $t \geqslant T$, and

$$y_k(t) = \begin{cases} C_k & (k=0, 1, \ldots, m) \\ 0 & (k=m+1, \ldots, n-1) \end{cases} \quad \text{for} \quad t<T,$$

where $R_k(t) = \lim_{j\to\infty} R_{k,i_j}(t)$ $(k=m, \ldots, n-1)$. We prove that

$$(13)\qquad \lim_{t\to\infty} y_k(t) = \begin{cases} C_k & (k=0, 1, \ldots, m), \\ 0 & (k=m+1, \ldots, n-1). \end{cases}$$

From (8) and (10) it follows that for $t \geqslant T$ we have

$$\tfrac{1}{2}C_k \leqslant y_k(g_k) \leqslant C_k \quad (k=0, 1, \ldots, m),$$
$$0 \leqslant (-1)^{n-1-k} y_k(g_k) \leqslant \tfrac{1}{2} \quad (k=m+1, \ldots, n-1).$$

Thus, in view of (2) and (3), for $t \geqslant T$ we obtain $0 \leqslant R_l(t) \leqslant Q_l(t)$ $(l=m, \ldots, n-1)$. Since $\lim_{t\to\infty} Q_l(t) = 0$, then $\lim_{t\to\infty} R_l(t) = 0$ $(l=m, \ldots, n-1)$ and

$$\lim_{t\to\infty} t^{k-m} \int_T^t \frac{(t-s)^{m-1-k}}{(m-1-k)!} R_m(s)\,\mathrm{d}s = 0 \quad (k=0, 1, \ldots, m-1).$$

Therefore, from this and (12) we obtain (13). Now we put in (12)

$$y_k(t) = \begin{cases} t^{k-m} x_k(t) & (k=0, 1, \ldots, m), \\ x_k(t) & (k=m+1, \ldots, n-1). \end{cases}$$

Then, by easy calculation, it follows that $x_0(t)$ is the solution of (1) and it has the property (M).

In the case of $n-1-m$ being even the proof is similar.

Theorem 3. *Consider the equation*

$$(14) \qquad x^{(n)}(t)+f(t, x(g_0), x'(g_1), \ldots, x^{(n-2)}(g_{n-2}), K)=0,$$

where K is a constant and f satisfies the conditions (2) *and* (3). *If the condition* (6) *holds for* $C_{n-1}=K$, *then equation* (14) *has a solution* $x(t)$ *with the property* (M).

Proof. The proof of this theorem follows exactly the same procedure as the proof of Theorem 2.

Corollary 1. *Consider the equation*

$$(15) \qquad x^{(n)}(t)+p(t)F(x(g_0), x'(g_1), \ldots, x^{(n-2)}(g_{n-2}))=0,$$

where $p(t)>0$ *and the function* $p(t)F(x_0, \ldots, x_{n-2})$ *satisfies* (2) *and* (3). *From Theorems 1 and 3 it follows that*

$$\int^{\infty} t^{n-1}p(t)\,\mathrm{d}t<\infty$$

is a necessary and suficient condition for the existence of solution $x(t)$ *of* (15) *with the property*

$$\lim_{t\to\infty} x(t)=L_0\neq 0, \quad L_0=\text{constant},$$

$$\lim_{t\to\infty} x^{(k)}(t)=0 \quad (k=1, 2, \ldots, n-1).$$

Remark 1. A similar result as in Corollary 1 has been obtained by Marušiak [5] in the case $g_k(t)\leq t$.

Corollary 2. *Consider the equation*

$$(16) \qquad x^{(n)}(t)+p(t)[x(g_0(t))]^{\beta}=0,$$

where $p(t)>0$ *and* $\beta>0$ *is the ratio of odd integers. Then from Theorems 1 and 3 it follows that*

$$\int^{\infty} t^{n-1-m}[g_0(t)]^{\beta m}p(t)\,\mathrm{d}t<\infty$$

is a necessary and suficient condition for the existence of the solution of equation (16) *having the property* (M).

Remark 2. If $g_0(t)\leq t$ in (16), then from Corollary 2, in the cases $m=0$ and $m=n-1$, we obtain some results of Odarich [6], Odarich and Shevelo [7, 9]. If $g_0(t)=t$ in (16), then from Corollary 2 we obtain some result of Kiguradze [2].

Theorem 4. *Assume for equation* (1) *that*

$$\int^{\infty} t^{n-1}|f(t, C, 0, \ldots, 0)|\,dt = \infty, \quad 0 \neq C = constant. \tag{17}$$

Then

(*i*) *for n even every bounded solution of* (1) *is oscillatory,*

(*ii*) *for n odd every bounded solution of* (1) *is either oscillatory or tends monotonically to zero as* $t \to \infty$.

Proof. Assume that under the condition (17) there exists a nonoscillatory bounded solution $x(t)$ of equation (1) and let $x(t) > 0$ for $t \geq t_0 \geq 0$.

(*i*) Let n be even. Then $x(t)$ is nondecreasing and the limit is finite as $t \to \infty$. Hence the argument in the proof of Theorem 1 is applicable, which leads us to a contradiction.

(*ii*) Let n be odd. Then $x(t)$ is nonincreasing. We prove that $x(t) \to 0$ as $t \to \infty$. Suppose $x(t) \to L_0 > 0$ as $t \to \infty$. Then analogously as in the proof of (*i*) we obtain a contradiction to the condition (17).

REFERENCES

[1] BOBROWSKI, D.: Asymptotyczne własności rozwiązań pewnego równania różniczkowego nieliniowego n-tego rzędu. Fasciculi Math., 3, 1967, 67—79.

[2] KIGURADZE, I. T.: On the Oscillation of Solutions of the Equation $\frac{d^m u}{dt^m} + a(t)|u|^m \operatorname{sign} u = 0$. Matem. Sborn., 2, 1964, 172—187 (Russian).

[3] MARUŠIAK, P.: Note on the Ladas' Paper on Oscillation and Asymptotic Behavior of Solutions of Differential Equations with Retarded Argument. J. diff. Equations, 1, 1973, 150—156.

[4] MARUŠIAK, P.: The Differential Equation with Retarded Argument Asymptotically Equivalent to the Equation $y^{(n)} = 0$. Mat. Čas., 1, 1974, 45—54 (Russian).

[5] MARUŠIAK, P.: Oscillation of Solutions of Nonlinear Delay Differential Equations. Mat. Čas., 4, 1974, 371—380.

[6] ODARICH, O. N.: On Asymptotic Behavior of Solutions of Nonlinear Differential Second Order Delay Equation. Dopov. AN URSR 8, 1968, 712—715 (Ukrainian).

[7] ODARICH, O. N.—SHEVELO, V. N.: On Asymptotic Behavior of Monotonous Solutions of Nonlinear Differential Second Order Delay Equations. Dopov. AN URSR 12, 1971, 1072—1075 (Ukrainian).

[8] ODARICH, O. N.—SHEVELO, V. N.: Some Problems of Asymptotic Behavior of Solutions of Nonlinear Differential Equations with Retarded Argument. Diff. Uravn., 4, 1973, 637—646 (Russian).

[9] ODARICH, O. N.—SHEVELO, V. N.: Some Theorems on Existence of Nonoscillating Solutions for Nonlinear Differential Delay Equations. Dopov. AN URSR, 4, 1974, 307—310 (Ukrainian).

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POLSKO

ЗАМЕТКА ОБ АСИМПТОТИЧЕСКОМ СВОЙСТВЕ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

Ярослав Вербовски

Резюме

Для дифференциального уравнения с отклоняющимся аргументом

$$x^{(n)}(t)+f(t, xg_0(t)), \ldots, x^{(n-1)}(g_{n-1}(t)))=0$$

показано необходимое и достаточное условие чтобы существовало решение со свойствами

$$\lim_{t\to\infty} t^{k-m}x^{(k)}(t)=L_k\neq 0, \quad L_k - \text{константа}, \quad k=(0, 1, \ldots, m),$$

$$\lim_{t\to\infty} x^{(k)}(t)=0 \quad (k=M+1, \ldots, n-1)$$

для $m\in\{0, 1, 2, \ldots, n-1\}$.