Rastislav Potocký
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ON THE EXTENSION OF MEASURES

RASTISLAV POTOCKÝ

The purpose of this paper is to extend a measure defined on an algebra A and having values in a vector lattice to a measure on the smallest σ-algebra containing A. We present two classes of spaces in which the extension is possible. At the end of the paper we derive several results from the main theorem; some of them are known, the rest seem to be new.

We recall some notions and definitions which will be used throughout the paper. The vector lattice X is called

a) *Dedekind σ-complete if every non-empty at most countable subset of X which is bounded from above has a supremum.*

b) *σ-separable if every non-empty subset $Y \subset X$ possessing a supremum contains an at most countable subset possessing the same supremum as Y.*

We shall say that the sequence x_n in a Dedekind σ-complete vector lattice X is order convergent to an element x in X, if $\lim\sup x_n = \lim\inf x_n = x$. The above definitions as well as many interesting results on vector lattices can be found in [1], [2].

A set function m defined on an algebra A and having values in a Dedekind σ-complete vector lattice X is said to be a (vector) measure if

1) $m(\emptyset) = 0$;

2) $m(E) \geqq 0$ *for every E in A;*

3) $m(E) = \sum_{i=1}^{\infty} m(E_i)$ *for every disjoint sequence (E_n) of sets in A whose union is E.*

There is another definition of measure. *A set function m on an algebra A with values in a Dedekind σ-complete vector lattice X is a measure if*

1) $m(\emptyset) = 0$;

2) $m(E) \geqq 0$ *for every $E \in A$;*

3) $m(E) + m(F) = m(E \cup F) + m(E \cap F)$ *for every $E, F \in A$;*

4) $m(E) = \lim m(E_n)$ *(in o-sense) for every increasing sequence (E_n) of sets in A such that $E = \cup E_n \in A$.*

It is easy to prove that both definitions are, in fact, the same.

A linear functional on X is called

a) *positive (monotone) if* $Tx \geqq 0$ *for all* $x \geqq 0$;

b) *order continuous if for each sequence* (x_n) *in X with the order limit x,* Tx_n *converges to* Tx;

c) *o-bounded if it maps o-bounded sets into bounded sets.*

In what follows the set of all o-bounded linear functionals and the set of all linear functionals continuous with respect to a topology on X will be denoted by X^+ and X^*, respectively.

Theorem 1. *If m is a (vector) measure on an algebra A with values in a Dedekind σ-complete o-separable vector lattice such that the set of all o-continuous linear functionals on X separates points of X, then there is a unique (vector) measure* $\bar{m}$ *on the σ-algebr a* $S(A)$ *such that for E in A* $\bar{m}(E)=m(E)$.

Proof. The measure m is an operator on A with the following properties:

1) $E \subset F \Rightarrow m(E) \leqq m(F)$ for every E, $F \in A$;

2) $m(E)+m(F)=m(E \cup F)+m(E \cap F)$ for every E, $F \in A$;

3) $E \subset F \Rightarrow m(F)=m(E)+m(F \setminus E)$ for every E, $F \in A$;

4) $m(E \cup F) \leqq m(E)+m(F)$ for every E, $F \in A$;

5) $E_n \uparrow E$, $E_n, E \in A \Rightarrow m(E)=\lim m(E_n)$ for every sequence (E_n) of sets in A.

Let S denote the set of all subsets of the basic space Ω. Put $B=\{E \in S;\ \exists (E_n) \in A;\ E_n \uparrow E\}$ and define $m_1(E)=\lim m(E_n)$ for every E in B. The definition does not depend on the choice of the sequence (E_n).

Then define $m_2(E)=\inf\{m_1(F);\ E \subset F \in B\}$ for every set E in S. It follows that m_2 is a monotone operator on S with values in X such that $m_2(E \cup F) \leqq m_2(E)+m_2(F)$ for every E, $F \in S$. Moreover m_2 coincides with m on A.

For every monotone, o-continuous linear functional T on X define now an operator $*T$ from A into R (the field of real numbers) as follows: $*T(E)=Tm(E)$ for every $E \in A$. $*T$ has the following properties.

1) $E \subset F \Rightarrow *T(E) \leqq *T(F)$;

2) $*T(E)+*T(F)=*T(E \cup F)+*T(E \cap F)$;

3) $E \subset F \Rightarrow *T(F)=*T(E)+*T(E \setminus E)$;

4) $*T(E \cup F) \leqq *T(E)+*T(F)$;

5) $E_n \uparrow E$, $E_n, E \in A \Rightarrow *T(E)=\lim *T(E_n)$ for every sequence (E_n) of sets in A.

Then put $*T(E)=\sup *T(E_n)=\sup Tm(E_n)$ for every $E \in B$, $E_n \in A$, $E_n \uparrow E$. One can show that this is a correct definition. It follows that $T^*(E)=Tm_1(E)$.

Finally define $T^{**}(E)=\inf\{T^*(F);\ E \subset F \in B\}$ for every $E \in S$.

Since the field of real numbers is o-separable, we may suppose that there exists a decreasing sequence (F_n) of elements in B greater than E such that

$$T^{**}(E)=\inf\{T^*(F_n);\ E \subset F_n \in B\}.$$

On the other hand, since X is supposed to be o-separable, we have $m_2(E)=\inf\{m_1(G_n);\ E \subset G_n \in B\}$ and, consequently, $T^{**}(E)=\inf\{T^*(F_n);\ E \subset F_n \in B\}=$

$\inf\{Tm_1(F_n);\ E\subset F_n\in B\}=T\inf\{m_1(F_n);\ E\subset F_n\in B\}\geqq Tm_2(E)$ for every E in S. The reverse inequality is immediate.

Denote by L the set of all $E\in S$ such that

$$\sup\{m_2(C);E\supset C\in D\}=\inf\{m_2(F);E\subset F\in B\},$$

where D is the set of all $E\in S$ for which there exists a decreassing sequence (A_n) of elements of A such that $A_n\downarrow E$.

We define, similarly,

$$L^*=\{E\in S;\sup\{T^{**}(C);E\supset C\in D\}=\inf\{T^{**}(F);E\subset F\in B\}\}.$$

Since $\sup\{m_2(C_n);\ E\supset C_n\in D\}=\inf\{m_2(F_n);\ E\subset F_n\in B\}$ with an increasing sequence (C_n) and a decreasing sequence (F_n) implies that $\sup\{T^{**}(C_n);\ E\supset C_n\in D\}=\inf\{T^{**}(F_n);\ E\subset F_n\in B\}$, we have $L\subset L^*$.

The next problem is to prove that if (E_n) is a monotone sequence in L which converges to a set E in S, then E belongs to L. Since $L\subset L^*$, we obtain from the extension theorem for real valued measures that $E\in L^*$, i.e. $\sup\{T^{**}(C_n');\ E\supset C_n'\in D\}=\inf\{T^{**}(F_n');E\subset F_n'\in B\}$. It follows, since the set of all o-continuous linear functionals separates points of X, that $\sup\{m_2(C_n');\ E\supset C_n'\in D\}=\inf\{m_2(F_n');\ E\subset F_n'\in B\}$, i.e. that $E\in L$.

Since L contains A, we may suppose the existence of the smallest set N containing A with the following property: $F_n\in N,\ F_n\uparrow F\in S(F_n\downarrow F\in S)\Rightarrow F\in N$.

Since $N=S(A)$, we define $\bar{m}(E)=m_2(E)$ for $E\in N$.

It is evident that $\bar{m}(\emptyset)=0$ and $\bar{m}(E)\geqq 0$ for every $E\in S(A)$. In order to prove the continuity from below, consider arbitrary $E_n\in S(A)$, $E_n\uparrow E$. We have immediately that $\bar{m}(E)\geqq\lim\bar{m}(E_n)$ since $\bar{m}$ is monotone. The desired result follows then from the fact that $T^{**}(E)=\lim T^{**}(E_n)$, i.e. $Tm_2(E)=\lim Tm_2(E_n)$ for every linear functional under consideration and from the fact that the set of all o-continuous linear functionals separates points of X.

The equality $\bar{m}(E)+\bar{m}(F)=\bar{m}(E\cup F)+\bar{m}(E\cap F)$ and the uniqueness of $\bar{m}$ follow without difficulty.

Corollary 1. (cf. [3], th. 11) *If X is a regular Dedekind σ-complete vector lattice such that X^+ separates points of X, then the extension theorem holds.*

Proof. Every regular Dedekind σ-complete vector lattice is o-separable and every o-bounded linear functional on such a space is o-continuous.

Theorem 2. *Let X be a Dedekind σ-complete, o-separable locally convex space with an ordering given by a closed cone. Let $x_n\xrightarrow{o}x$ imply $T(x_n)\to T(x)$ for every $T\in X^*$. Then for every measure on an algebra A with values in X there exists a unique extension to $S(A)$.*

Proof. Analoguous to that of Theorem 1.

So far we have been concerned with a set function which was a measure in the o-sense. We can, however, extend our results to the case when we are primarily interested in the topology of X. Substituting in the above definition of measure a topological convergence for the o-convergence, we shall speak about a (vector) measure in the topological sense. The following results should be compared with [4], [5], [6].

Theorem 3. *Let X be a Dedekind σ-complete, o-separable locally convex space oredered by normal cone and let every continuous linear functional be o-continuous. Then every measure (in the topological sense) on an algebra A with values in X can be uniquely extended to $S(A)$.*

Proof. Since the cone is closed, the set function under consideration is a measure in the o-sense as well. If $\bar{m}$ means the extension to $S(A)$ mentioned in Theorem 2, we have that $\bar{m}(E_n) \xrightarrow{w} \bar{m}(E)$ whenever $E_n \uparrow E$, E_n, $E \in S(A)$. Since the cone is normal, the result follows.

Theorem 4. *Let X be a Dedekind σ-complete, o-separable complete metrizable locally convex space ordered by a closed cone and let every continuous linear functional be o-continuous. Then for every measure on an algebra A with values in X there is a unique extension to $S(A)$.*

Proof. The above assumptions imply normality of the cone.

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Katedra numerickej matematiky
a matematickej štatistiky PFUK
Mlynská dolina
816 31 Bratislava

О ПРОДОЛЖЕНИИ МЕР

Растислав Потоцки

Резюме

Пусть m – векторная мера определена на алгебре A с значениями в σ – полной o – сепарабельной векторной решетке X такой, что семейство всех о – непрерывных линейных форм разделяет ее точки. Тогда существует векторная мера $\bar{m}$ на σ-алгебре $S(A)$ порожденной алгеброй A, являющаяся продолжением m. Мера $\bar{m}$ определена однозначно.