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THE LATTICE OF ALL SUBTREES OF A TREE

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The present paper will be concerned with trees. A tree is a connected undirected graph without circuits. It may be finite or infinite. In this paper the null graph K_0 (a graph whose vertex set and edge set are empty) and a graph consisting of one vertex and no edge will be considered also as trees. The null graph is a subgraph of every graph. The convenience of using the concept of the null graph is rather debatable, as shown in [2]. But in the present paper this concept is naturally needed.

If T_1 and T_2 are subtrees of a tree T, we put $T_1 \leqq T_2$ if and only if T_1 is a subtree of T_2. The relation $\leqq$ so defined is a partial ordering on the set of all subtrees of a given tree T. This set with the relation $\leqq$ is evidently a lattice; we denote it by $\mathfrak{L}(T)$. The lattice operations of join and meet will be denoted by $\vee$ and $\wedge$, respectively.

If $T_1 \in \mathfrak{L}(T)$, $T_2 \in \mathfrak{L}(T)$, then $T_1 \wedge T_2$ is the intersection of T_1 and T_2, i.e. the graph whose vertex set is the intersection of vertex sets of T_1 and T_2 and whose edge set is the intersection of edge sets of T_1 and T_2. It is evidently a tree and each common subtree of T_1 and T_2 is its subtree. If $T_1 \wedge T_2 \neq K_0$, then $T_1 \vee T_2$ is the union of T_1 and T_2, i.e. the graph whose vertex set is the union of vertex sets of T_1 and T_2 and whose edge set is the union of edge sets of T_1 and T_2. It is evidently a tree and is contained in each subtree of T which contains both T_1 and T_2 as subtrees. But if $T_1 \wedge T_2 = K_0$, the union of T_1 and T_2 is not a tree, because it is disconnected. To obtain the tree $T_1 \vee T_2$ from it, it is necessary to add a path of T connecting the pair of vertices u_1, u_2, where u_1 belongs to T_1, u_2 belongs to T_2 and the distance between u_1 and u_2 is the least of the distances of all such pairs of vertices (evidently this path is uniquely determined).

We shall prove some theorems on the structure of $\mathfrak{L}(T)$. In all theorems we shall tacitly suppose that T has at least three vertices.

Theorem 1. *The lattice $\mathfrak{L}(T)$ has the greatest element and the least one and is* atomic.

Proof. Evidently the least element of $\mathfrak{L}(T)$ is the null graph K_0 and the greatest element of $\mathfrak{L}(T)$ is the whole tree T. The atoms of $\mathfrak{L}(T)$ are all subtrees which consist only of one vertex. Any non-null subtree of T contains at least one vertex, therefore there exists an atom of $\mathfrak{L}(T)$ which is less than or equal to it.

Theorem 2. *The lattice $\mathfrak{L}(T)$ is dually atomic, if and only if there does not exist a proper subtree of T containing all the terminal vertices of T.*

Proof. Let T' be a dual atom of $\mathfrak{L}(T)$. As T' is a proper subtree of T, the set S of vertices belonging to T and not belonging to T' is non-empty. As T is connected, there exists at least one vertex v of S which is adjacent to some vertex w of T'. If we add the vertex v and the edge vw to T', we obtain a subtree T'' of T. We have $T' < T''$ and, as T' is a dual atom of $\mathfrak{L}(T)$, the equality $T'' = T$ holds. But then $S = \{v\}$. By deleting v from T we obtain a tree T', therefore v must be a terminal vertex of T. We have proved that each dual atom of $\mathfrak{L}(T)$ is obtained from T by deleting one terminal vertex. Let there exist a proper subtree T_0 of T containing all the terminal vertices of T. Then T_0 is not contained in a dual atom of $\mathfrak{L}(T)$, because to each dual atom of $\mathfrak{L}(T)$ there exists a terminal vertex of T not contained in it. On the other hand, if such a subtree does not exist, then to each proper subtree T_1 of T there exists a terminal vertex of T not contained in it. By deleting this vertex from T we obtain a dual atom of $\mathfrak{L}(T)$ containing T_1.

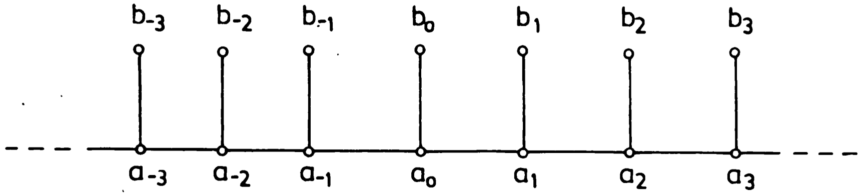

Fig. 1

Among the trees satisfying the condition from Theorem 2 there are all the trees without infinite paths, in particular all the finite trees. We shall show an example of a tree with infinite paths which satisfies it, too. The vertices of this tree are a_n and b_n and the edges are a_nb_n, a_na_{n+1} for all the integers n. This tree is in Fig. 1. An example of a tree which does not satisfy this condition is a tree consisting of one (one-way or two-way) infinite path. (If a tree T has no terminal vertices, then we consider it as a tree, any of whose subtrees contains all the terminal vertices of T.)

Theorem 3. *The lattice $\mathfrak{L}(T)$ is non-modular.*

Proof. As mentioned above, we suppose that T has at least three vertices. Let v_0 be a vertex of T of a degree at least two, let v_1, v_2 be two distinct vertices adjacent to v_0. By T_1 (or T_2) we denote the subtree of T consisting only of the vertex v_1 (or v_2 respectively). By T_3 we denote the subtree of T consisting of the vertices v_0 and v_1 and the edge joining them, by T_4 we denote the subtree of T consisting of the vertices v_0, v_1, v_2 and the edges v_0v_1, v_0v_2. We have $T_1 < T_3$. The modularity of $\mathfrak{L}(T)$ would imply $T_1 \vee (T_2 \wedge T_3) = (T_1 \vee T_2) \wedge T_3$. But $T_1 \vee (T_2 \wedge T_3) = T_1$, $(T_1 \vee T_2) \wedge T_3 = T_3$ and therefore $\mathfrak{L}(T)$ is not modular.

Theorem 4. *Each proper filter of the lattice $\mathfrak{L}(T)$ is a distributive lattice.*

Proof. Let $\mathfrak{F}$ be a proper filter of $\mathfrak{F}(T)$. Let $T_1 \in \mathfrak{F}$, $T_2 \in \mathfrak{F}$. As $\mathfrak{F}$ is a filter, $T_1 \wedge T_2 \in \mathfrak{F}$. As $\mathfrak{F}$ is a proper filter, $K_0 \notin \mathfrak{F}$ and thus $T_1 \wedge T_2 \neq K_0$. But then $T_1 \vee T_2$ is the union of T_1 and T_2. This holds for any T_1 and T_2 from $\mathfrak{F}$. As $T_1 \wedge T_2$ is always the intersection of T_1 and T_2, the filter $\mathfrak{F}$ is a sublattice of the lattice of all subsets of the union of the vertex set and the edge set of T. This lattice is distributive, therefore also $\mathfrak{F}$ is distributive.

Theorem 5. *The lattice $\mathfrak{F}(T)$ is complete.*

Proof is evident.

Theorem 6. *The lattice $\mathfrak{L}(T)$ is generated by its set of atoms.*

Proof. The assertion is evident when we know that the atoms of $\mathfrak{L}(T)$ are all one-vertex subtrees of T.

On the other hand, $\mathfrak{L}(T)$ is not generated by its dual atoms, even if it is dually atomic.

Now we shall define an important congruence on $\mathfrak{L}(T)$.

Theorem 7. *Let σ be a binary relation on $\mathfrak{L}(T)$ defined so that for two elements T_1, T_2 of $\mathfrak{L}(T)$ we have $(T_1, T_2) \in \delta$ if and only if the symmetric difference between the vertex sets of T_1 and T_2 is finite. Then δ is a congruence on $\mathfrak{L}(T)$.*

Proof. First we shall consider an arbitrary non-empty set M and the Boolean algebra $\mathfrak{B}(M)$ of all subsets of M. The finite subsets of M form an ideal $\mathfrak{J}$ of $\mathfrak{B}(M)$. As $\mathfrak{B}(M)$ is a Boolean algebra, the ideal $\mathfrak{J}$ is the kernel of some congruence δ_0 on $\mathfrak{B}(M)$. If $A \in \mathfrak{B}(M)$, $B \in \mathfrak{B}(M)$, $(A, B) \in \delta_0$, then there exists $C \in \mathfrak{B}(M)$, $A_0 \in \mathfrak{J}$, $B_0 \in \mathfrak{J}$ such that $A = A_0 \cup C$, $B = B_0 \cup C$. Then the symmetric difference between A and B is contained in $A_0 \cup B_0$, which is a finite set, thus it is also finite. On the other hand, let $D \in \mathfrak{B}(M)$, $E \in \mathfrak{B}(M)$ and let the symmetric difference between D and E be finite. We have $D = (D \cap E) \cup (D - E)$, $E = (D \cap E) \cup (E - D)$. The sets $D - E$ and $E - D$ are subsets of the symmetric difference between D and E, therefore they are finite. Thus $D - E \in \mathfrak{J}$, $E - D \in \mathfrak{J}$ and we have $(D, E) \in \delta_0$. Now let M be the set of all vertices of T and consider the congruence δ_0 on $\mathfrak{B}(M)$. If T_1, T_2 are subtrees of T, let $V(T_1)$, $V(T_2)$, $V(T_1 \vee T_2)$ be vertex sets of T_1, T_2, $T_1 \vee T_2$ respectively. The tree $T_1 \vee T_2$ is either equal to the union of T_1 and T_2, or is obtained from this union by adding some finite path. In both cases the symmetric difference of the sets $V(T_1 \vee T_2)$ and $V(T_1) \cup V(T_2)$ is finite and thus $(V(T_1 \vee T_2), V(T_1) \cup V(T_2)) \in \delta_0$. Thus two subtrees of T are in the relation δ if and only if their vertex sets are in δ_0. Thus δ is a congruence on $\mathfrak{L}(T)$.

We shall prove some theorems concerning the factorlattice $\mathfrak{L}(T)/\delta$.

Theorem 8. *The factor-lattice $\mathfrak{L}(T)/\delta$ is distributive.*

Proof. From the proof of Theorem 7 it follows that each congruence class of δ consists of trees whose vertex sets are in one congruence class of δ_0. If T_1, T_2 are in $\mathfrak{L}(T)$, then the vertex set of $T_1 \vee T_2$ (or $T_1 \wedge T_2$) lies in the same congruence class of δ_0 as $V(T_1) \cup V(T_2)$ (or $V(T_1) \cap V(T_2)$, respectively). Thus $\mathfrak{L}(T)/\delta$ is isomorphic to a sublattice of $\mathfrak{B}(M)/\delta_0$. The lattice $\mathfrak{B}(M)/\delta_0$ is a Boolean algebra, therefore $\mathfrak{L}(T)/\delta$ must be distributive.

Before proving a further theorem, we shall say something about the concept of the end of a locally finite graph, defined by R. HALIN [1]. The rest of a one-way infinite path is a part of this path which is also a one-way infinite path. Two one-way infinite paths W_1, W_2 of a locally finite graph G are called equivalent, if and only if there exists a one-way infinite path W (not necessarily distinct from W_1 and W_2) in G such that each rest of W has common vertices with both W_1 and W_2. This relation is an equivalence on the set of all one-way infinite paths in G and its equivalence classes are called ends of G.

Now suppose that G is a locally finite tree. If two paths of a tree have two common vertices, then they have also the whole path connecting these two vertices in common. The path W from the definition of the end has infinitely many common vertices with W_1, therefore it has a common rest with W_1. Analogously it has a common rest with W_2 and therefore W_1 and W_2 have a common rest. On the other hand, if W_1 and W_2 have a common rest, we many put W to be this common rest. Thus two one-way infinite paths W_1, W_2 of a locally finite tree T belong to the same end of T, if and only if they have a common rest.

In the case of trees we can extend the concept of the end to all the trees, not only locally finite ones. Here we define the equivalence of two one-way infinite paths so that two one-way infinite paths are equivalent, if and only if they have a common rest. This is evidently an equivalence. The classes of this equivalence will be called the ends of the tree.

In the case when T is finite, the congruence δ is the universal relation on $\mathfrak{L}(T)$ and thus $\mathfrak{L}(T)/\delta$ is a trivial lattice consisting of one element. In the following we shall consider only infinite trees.

Theorem 9. *Let T be an infinite tree. The factor-lattice $\mathfrak{L}(T)/\delta$ is a Boolean algebra, if and only if the following conditions are satisfied:*

(a) *T has a finite number of vertices of a degree greater than two.*

(b) *T has a finite number of ends.*

(c) *The tree obtained from T by deleting all the terminal vertices has a finite number of the terminal vertices.*

Proof. Let (a), (b), (c) hold. Let T' be the least subtree of T containing all the vertices which have a degree greater than two in T. Let M be the set of these vertices. Let T_u be the subtree of T consisting only of the vertex u for any $u \in M$.

Then $T' = \bigvee_{u \in M} T_u$; it is the join of finitely many finite subtrees of T, therefore it is finite. The factor-lattice $\mathfrak{L}(T)/\delta$ is distributive; to prove that it is a Boolean algebra it is sufficient to prove its complementarity. Let T_0 be a subtree of T. The class of δ complementary to the class containing T_0 is the class of δ containing a tree $\bar{T}_0$ whose meet with T_0 is finite (belongs to the same class of δ as K_0) and whose join with T_0 is obtained from T by deleting finitely many vertices (belongs to the same class of δ as T). We shall construct such a tree $\bar{T}_0$. Let F be the forest obtained from T by deleting all vertices of T'. No connected component of F contains a vertex of a degree greater than two and each of them contains at least one vertex of a degree less than two (the vertex joined in T with a vertex of T'). Therefore each of them is either an isolated vertex, or a finite path, or a one-way infinite path. From (b) it follows that there are only finitely many infinite connected components of F. From (c) it follows that there are only finitely many finite connected components of F consisting of more than one vertex. Now the tree $\bar{T}_0$ is the join of T' and all connected components of F which are not subtrees of T_0. Consider the meet $T_0 \wedge \bar{T}_0$. If C is some connected component of F which is a one-way infinite path, then C is a subtree of $\bar{T}_0$ if and only if it is not contained in T_0. In this case T_0 contains only finitely many vertices of C; otherwise it would have to contain some rest of this path and its initial vertex, i.e. the whole C. Thus $T_0 \wedge \bar{T}_0$ contains only a finite number of vertices of any infinite connected component of F. It does not contain any vertex of a connected component of F consisting of one vertex; such a component is contained either in T_0, or in $\bar{T}_0$, but not in both of them. Thus the vertex set of $T_0 \wedge \bar{T}_0$ is the union of some subset of the vertex set of T' and of some finite subsets of vertex sets of connected components of F containing more than one vertex. As the number of such components is finite, also the vertex set of $T_0 \wedge \bar{T}_0$ is finite. Now from the construction it is clear that $T_0 \vee \bar{T}_0 = T$. We have proved that to the class of δ containing T_0 a complement in $\mathfrak{L}(T)/\delta$ exists. As T_0 was chosen arbitrarily, $\mathfrak{L}(T)/\delta$ is complementary and is a Boolean algebra.

Now suppose that (a) does not hold. This means that the set M of vertices of degrees greater than two in T is infinite. First suppose that there exists a vertex a of T such that there are infinitely many branches of T outgoing from a which contain vertices of M. On each of these branches we choose one vertex of M; the set of those chosen vertices will be denoted by M_0. Let T_1 be the least subtree of T containing all vertices from M_0. Evidently each vertex of M_0 is a terminal vertex of T_1. For each $u \in M_0$ let $v_1(u)$, $v_2(u)$ be two distinct arbitrary vertices which are adjacent to u in T and do not belong to T_1; such vertices always exist, because u has a degree greater than two in T and the degree 1 in T_1. Let T_0 be the subtree of T whose vertex set consists of all the vertices of T_1 and of the vertices $v_1(u)$ for all $u \in M_0$. Suppose that there exists $\bar{T}_0$ with the above described properties. The join $T_0 \vee \bar{T}_0$ must be obtained from T by deleting finitely many vertices, thus $\bar{T}_0$ must contain an infinite number of vertices $v_2(u)$ for $u \in M_0$. Thus it contains an infinite subset M_0' of M_0 and the infinite subtree of T_1 which is the least subtree of T containing the set M_0'. But then this subtree is a subtree of $T_0 \wedge \bar{T}_0$, which is a contradiction with the finiteness of $T_0 \wedge \bar{T}_0$. Now suppose that for each vertex a only finitely many branches of T outgoing from a contain vertices of M. It is easy to prove that then there exists a one-way infinite path P in T which contains an infinite number of vertices of M. Denote these vertices by $a_1, a_2, a_3, \ldots$ in the ordering according to the distance from the initial vertex of P. For each a_n let b_n be a vertex adjacent to a_n and not belonging to P; as $a_n \in M$ for each n, the vertex b_n exists for each n. Evidently $b_m \neq b_n$ for $m \neq n$. Let the vertex set of T_0 consist of all the vertices b_n for n odd. Suppose that there exists $\bar{T}_0$ with the above described properties. As $T_0 \vee \bar{T}_0$ is obtained from T by deleting a finite number of vertices, $\bar{T}_0$ must contain infinitely many (all but a finite number) vertices b_n with n even, therefore also infinitely many vertices a_n with n even. This means that $\bar{T}_0$ contains a rest of P. This rest is contained in $T_0 \wedge \bar{T}_0$ and thus we have a contradiction with the finiteness of $T_0 \wedge \bar{T}_0$.

Now suppose that (a) holds and (b) does not hold. Consider the tree T_1 and the forest F defined above. The forest F has infinitely many connected components which are one-way infinite paths. Let the vertex set of T_0 consist of the vertex set of T_1 and of initial vertices of all of these paths. The tree $\bar{T}_0$ must contain all these paths, otherwise there would be infinitely many vertices belonging to T and not to $T_0 \vee T_0$. But then all of these initial vertices are in $T_0 \wedge \bar{T}_0$ and $T_0 \wedge \bar{T}_0$ is infinite, which is a contradiction.

Finally, suppose that (a) holds and (c) does not hold. We consider again the forest F. It has infinitely many connected components having more than one vertex. Consider a partition of the set of these components into two disjoint infinite subsets C_1, C_2. Let T_0 be the least subtree of T which contains T' as a subtree and contains one vertex from each of these components. If there exists $\bar{T}_0$ with the required properties, it must contain all but a finite number of these components; otherwise $T_0 \vee \bar{T}_0$ would not belong to the same class of δ as T. But then $T_0 \wedge \bar{T}_0$ contains a vertex from any of these components which belong to $\bar{T}_0$ and thus it is infinite, which is a contradiction.

Now we shall prove a lemma.

Lemma. *Let M be a countable set, let $\mathfrak{B}(M)$ be the Boolean algebra of all subsets of M, let δ_0 be the congruence on $\mathfrak{B}(M)$ defined so that two elements of $\mathfrak{B}(M)$ are in δ_0 if and only if their symmetric difference is finite. Then the cardinality of $\mathfrak{B}(M)/\delta_0$ is the power of the continuum.*

Proof. Let $\mathscr{C}$ be one equivalence class of δ_0, let $S \in \mathscr{C}$. Any set R from $\mathscr{C}$ can be expressed uniquely in the form $R = (S - T_R) \cup U_R$, where $T_R \subseteq S$, $U_R \subseteq M - S$ and T_R, U_R are finite. If we put $f(R) = [T_R, U_R]$ for each $R \in \mathscr{C}$, we obtain an injection

f of $\mathscr{C}$ into the family of ordered pairs of finite subsets of M. The family of all finite subsets of M is countable, therefore also the family of the ordered pairs of these subsets is countable. We see that each class $\mathscr{C}$ of is countable. As $\mathfrak{B}(M)$ is the union of all these classes $\mathscr{C}$ and is of the power of the continuum, $\mathfrak{B}(M)/\delta_0$ is also of the power of the continuum.

This means that in any infinite set there exists a family of its subsets which is of the power of the continuum and does not contain any two subsets which are in the relation δ.

Theorem 10. *Let T be an infinite tree. The lattice $\mathfrak{L}(T)/\delta$ is finite, if and only if the conditions* (a), (b), (c) *from Theorem 9 hold and T is locally finite. In the opposite case the cardinality of $\mathfrak{L}(T)/\delta$ is at least the power of the continuum.*

Proof. Let T be locally finite and let (a), (b), (c) hold. Consider again the tree T' and the forest F from the proof of Theorem 9. As T is locally finite, the number of the connected components of F is finite. Let T'' be the least subtree of T which contains T' and all finite connected components of F; the tree T'' is evidently finite. Let T_1, T_2 be two subtrees of T. Suppose that each infinite connected component of F having an infinite intersection with T_1 has an infinite intersection with T_2 and vice versa. Then the symmetric difference between the vertex sets of T_1 and T_2 can contain only the vertices of T'' and finitely many vertices from each connected component of F. Thus this symmetric difference is finite and $(T_1, T_2) \in \delta$. If T_1 has an infinite intersection with an infinite connected component of F (i.e. the rest of some one-way infinite path) with which T_2 has not an infinite intersection, then the symmetric difference between the vertex sets of T_1 and T_2 contains infinitely many vertices of this component and $(T_1, T_2) \notin \delta$. Analogously if T_2 has an infinite intersection with an infinite connected component of F with which T_1 has not. Thus to each subfamily of the family of all infinite connected components of F a class of δ corresponds uniquely so that a subtree of T belongs to this class if and only if it contains infinitely many vertices of each connected component of this subfamily and of no other component. Thus $\mathfrak{L}(T)/\delta$ is a finite Boolean algebra whose number of generators is equal to the number of infinite connected components of F, i.e. to the number of the ends of T.

Suppose that T is not locally finite. Let v be a vertex of T of an infinite degree, let M be the set of vertices of T adjacent to v. We choose a family of subsets of M of the power of the continuum with the property that no two elements of this family have a finite symmetric difference. To any of these sets we assign the least subtree of T which contains all the elements of this set. We obtain a family of subtrees of T of the power of the continuum with the property that no two of its elements are in δ.

Suppose that T is locally finite and (a) does not hold. Then there exists an infinite path P in T containing infinitely many vertices of a degree greater than two. We consider the vertices $a_1, a_2, a_3, \ldots$ and the vertices $b_1, b_2, b_3, \ldots$ as in the proof of Theorem 9. With the set $\{b_1, b_2, b_3, \ldots\}$ we do the same consideration as with M above.

Finally, suppose that T is locally finite, (a) holds and either (b) or (c) does not hold. In both these cases F has an infinite number of connected components. From each of them take a vertex which is adjacent to a vertex of T'. The set of these vertices is approached in the same way as M in the first case.

Theorem 11. *If the lattice $\mathfrak{L}(T)$ is given, then the tree T is determined uniquely up to isomorphism.*

Proof. The atoms of $\mathfrak{L}(T)$ are all one-vertex subtrees of T. Thus there is a one-to-one correspondence between the atoms of $\mathfrak{L}(T)$ and the vertices of T and the vertex set of T is determined by $\mathfrak{L}(T)$. Two vertices of a tree are adjacent, if and only if there exists a subtree of this tree which contains these two vertices and no others. Let u, v be two vertices of T corresponding to the atoms T_u, T_v of $\mathfrak{L}(T)$. Then u and v are adjacent in T, if and only if the join $T_U \vee T_v$ is incomparable with each atom of $\mathfrak{L}(T)$ distinct from T_u and T_v.

Now we shall give a characterization of lattices which are isomorphic to a maximal filter of the lattice of all subtrees of a tree. Any maximal filter of $\mathfrak{L}(T)$ consists of all subtrees of T which contain a given vertex of T. If T is a tree and u is a vertex of T, by $\mathfrak{F}(T, u)$ we denote the filter of $\mathfrak{L}(T)$ consisting of all subtrees of T which contain u. As any two of these subtrees have a non-empty intersection, $\mathfrak{F}(T, u)$ is isomorphic to a sublattice of the Boolean algebra of all the subsets of $V - \{u\}$, where V is the vertex set of T.

Now let M be a finite or countable set, let $\mathfrak{B}(M)$ be the Boolean algebra of all the subsets of M. By $\mathscr{A}(M)$ we denote the family of all the complete sublattices $\mathfrak{L}$ of $\mathfrak{B}(M)$ with the following properties:

(α) If x is an element of a finite height in $\mathfrak{B}(M)$ and $x \in \mathfrak{L}$, then there exists an element $y \in \mathfrak{L}$ such that x covers y in $\mathfrak{B}(M)$.

(β) If an element $x \in \mathfrak{L}$ of a finite height covers only one element $y \in \mathfrak{L}$ in $\mathfrak{B}(M)$, then y covers only one element of $\mathfrak{L}$ in $\mathfrak{B}(M)$ or is the least element of $\mathfrak{L}$.

(γ) Each element of $\mathfrak{L}$ of an infinite height in $\mathfrak{B}(M)$ is the join of some elements of $\mathfrak{L}$ of finite heights in $\mathfrak{B}(M)$.

(δ) $\mathfrak{L}$ contains the greatest and the least element of $\mathfrak{B}(M)$.

If $x \in \mathfrak{B}(M)$, then x is a subset of M and its height in $\mathfrak{B}(M)$ is evidently equal to the number of its elements. A complete sublattice is a sublattice closed under forming infinite joins and infinite meets.

Let $\mathfrak{L}_0$ be the subset of $\mathfrak{L}$ consisting of all the elements x of $\mathfrak{L}$ with the property that x has a finite height in $\mathfrak{B}(M)$ and covers only one element of $\mathfrak{L}$ in $\mathfrak{B}(M)$ (where $\mathfrak{L} \in \mathscr{A}(M)$). By induction we can prove that the principal ideal of $\mathfrak{L}$ determined by an element of $\mathfrak{L}_0$ is a chain.

Theorem 12. *Let $\mathfrak{L}$ be a lattice. Then the following two assertions are* equivalent:

(i) *$\mathfrak{L}$ is isomorphic to a lattice from $\mathscr{A}(M)$ for some finite or countable set M.*

(ii) *$\mathfrak{L}$ is isomorphic to $\mathfrak{F}(T, u)$ for some tree T and a vertex u of T.*

Proof. (i)$\Rightarrow$(ii). Let (i) hold; we may consider $\mathfrak{L}$ directly as a lattice from $\mathscr{A}(M)$. Let $\mathfrak{L}_0$ be the above defined subset of $\mathfrak{L}$. Let T be a graph with the vertex set $\mathfrak{L}_0 \cup \{o\}$, where o is the least element of $\mathfrak{L}$, such that each element of $\mathfrak{L}_0$ is joined by an edge with the element $y \in \mathfrak{L}_0 \cup \{o\}$ which is covered by it. The graph T is evidently a tree. Now let T' be a subtree of T containing o. Each vertex of T' different from o is some element of $\mathfrak{L}_0$; let $\alpha(T')$ be the join of all of them; if T' consists only of o, then $\alpha(T') = o$. Thus each subtree of T containing o is mapped by α onto some element of $\mathfrak{L}$. Now let T_1, T_2 be two different subtrees of T, both containing o. As they are different, at least one of them contains a vertex which is not contained in the other. Thus, without loss of generality, let T_2 contain some vertex v which is not in T_1. Suppose $\alpha(T_1) = \alpha(T_2)$. Then $v \vee \alpha(T_1) = \alpha(T_1)$ and $v \leqq \alpha(T_1)$ in $\mathfrak{L}$. If $V(T_1)$ is the vertex set of T_1, then $\alpha(T_1) = \bigvee_{z \in V(T_1)} z$ and thus $v \leqq \bigvee_{z \in V(T_1)} z$. As $\mathfrak{L}$ is a complete sublattice of $\mathfrak{B}(M)$, it is infinitely distributive, therefore $v = v \wedge \alpha(T_1) = v \wedge \bigvee_{z \in V(T_1)} z = \bigvee_{z \in V(T_1)} (v \wedge z)$. Suppose $v \wedge z \neq v$ for all $z \in V(T_1)$. Then $v \wedge z < v$ for each $z \in V(T_1)$. The element v covers only one element of $\mathfrak{L}$; let this element be w. This means $v \wedge z \leqq w$ for each $z \in V(T_1)$. But then $v = \bigvee_{z \in V(T_1)} (v \wedge z) \leqq w$, which is a contradiction. Thus $v \wedge z = v$ for some $z \in V(T_1)$; this means $z \geqq v$. But then v belongs to the principal ideal of $\mathfrak{L}$ determined by z; this ideal is a chain and all of its elements are in T_1. Thus v is in T_1, which is a contradiction. Thus for arbitrary two subtrees T_1, T_2 of T containg o the inequality $T_1 \neq T_2$ implies $\alpha(T_1) \neq \alpha(T_2)$. The mapping α is an injection. Now we shall prove that each $a \in \mathfrak{L}$ is equal to $\alpha(T')$ for some subtree T' of T containing o. For elements of a finite height we prove this by induction. Let a be such an element. If the height of a is 0, the element $a = o$ and $a = \alpha(T_0)$, where T_0 is the subtree of T consisting only of the element o. Now let the height of a be $h > 0$ and suppose that the assertion is true for all the elements of a height less than h. If $a \in \mathfrak{L}_0$, then $a = \alpha(T_a)$, where T_a is the subtree of T whose vertex set is the principal ideal determined by a (this ideal is a chain). If $a \notin \mathfrak{L}_0$, then it covers at least two elements y_1, y_2 of $\mathfrak{L}$ whose heights in $\mathfrak{B}(M)$ are equal to $h-1$. By the induction assumption the assertion holds for these elements y_1, y_2. Thus $y_1 = \alpha(T_1)$, $y_2 = \alpha(T_2)$ where T_1, T_2 are subtrees of T containing o. Then $a = y_1 \vee y_2 = \alpha(T_1) \vee \alpha(T_2)$ and this is evidently equal to $\alpha(T')$, where T' is the join of T_1 and T_2. Thus we have proved the assertion for all the elements of $\mathfrak{L}$ which have finite heights. Each element of $\mathfrak{L}$ having an infinite height is the join of some elements of finite heights. If $a = \bigvee_{\iota \in I} a_\iota$, where I is some subscript set, then $a = \alpha(T')$, where $T' = \bigvee_{\iota \in I} T_\iota$, the trees T_ι being such subtrees of T containing o for which $\alpha(T_\iota) = a_\iota$. We have proved that α is a surjection. As it is also an injection, it is a bijection. Evidently $\alpha(T_1 \vee T_2) = \alpha(T_1) \vee \alpha(T_2)$, $\alpha(T_1 \wedge T_2) = \alpha(T_1) \wedge \alpha(T_2)$. Therefore α is an isomorphic mapping of $F(T, o)$ onto $\mathfrak{L}$.

(ii)$\Rightarrow$(i). Consider the lattice $F(T, u)$. If $T_1 \in F(T, u)$ and is finite, then it contains at least one terminal vertex distinct from u. By deleting such a vertex we obtain a tree T_2 from T_1; in $F(T, u)$ evidently T_1 covers T_2 and (α) is satisfied. If T_1 covers T_2 and no other element of $F(T, u)$, then evidently T_1 has only one terminal vertex distinct from u. Then T_2 (described above) can have at most one terminal vertex distinct from u and thus it covers only one tree T_3; this tree is obtained from T_2 by deleting this vertex; (β) holds. If some subtree $T' \in F(T, u)$ is infinite, then it is evidently the join of some finite trees from $F(T, u)$, therefore (γ) holds. The validity of (δ) is evident.

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РЕШЕТКА ВСЕХ ПОДДЕРЕВЬЕВ ДЕРЕВА

Богдан Зелинка

Резюме

В статье изучается структура решетки $\mathfrak{L}(T)$ всех поддеревьев дерева T. Особое внимание уделяется фактор-решетке $\mathfrak{L}(T)/\delta$ решетки $\mathfrak{L}(T)$ по конгруэнции δ определенной так, что для двух поддеревьев T_1, T_2 дерева T мы имеем $(T_1, T_2) \in \delta$ тогда и только тогда, когда симметричная разность множеств вершин деревьев T_1 и T_2 конечна. Доказано, что дерево однозначно определено своей решеткой поддеревьев. Характеризованы максимальные фильтры решетки $\mathfrak{L}(T)$.