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ADMISSIBLE OPERATIONS IN CATEGORIES OF ALGEBRAS

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Primitive classes of algebras have the following property: No further operations can be added to the basic operations and polynomials (in the sence of Grätzer, [2]) without changing the category of all algebras of that primitive class. In more general categories of algebras, however, there may occur further admissible operations (i.e. operations compatible with all the homomorphisms; a more precise definition will be given below). In the present paper some conditions for the existence of admissible operations and of proper and improper free algebras, which will be defined, are given.

1. Admissible operations.

Define a system of admissible operations in a category $\mathcal{K}$.

Definition. *Let $\mathcal{K}$ be a category of algebras. Let for every algebra $A \in \mathrm{Ob}\,\mathcal{K}$ $f^A: A^n \to A$ be a mapping where n is a non-negative integer and let for arbitrary B, $C \in \mathrm{Ob}\,\mathcal{K}$, $b_1, \ldots, b_n \in B$ and $\varphi \in \mathrm{Hom}\,(B, C)$*

$$f^B(b_1, \ldots, b_n)\varphi = f^C(b_1\varphi, \ldots, b_n\varphi)$$

hold. If for every polynimial symbol $\mathbf{g}$ there is an algebra $B \in \mathrm{Ob}\,\mathcal{K}$ with $f^B \neq g^B$ (where by g^B we denote the polynomial associated to $\mathbf{g}$ on the algebra B), then

$$f = \{f^A \mid f^A: A^n \to A, A \in \mathrm{Ob}\,\mathcal{K}\}$$

is called a system of n-ary admissible operations in the category $\mathcal{K}$.

Two admissible systems of n-ary operations f and g in the category $\mathcal{K}$ are different when there is an algebra $A \in \mathrm{Ob}\,\mathcal{K}$ with $f^A \neq g^A$.

Definition. *The mapping f^A from the system of admissible operations f will be called an admissible operation on the algebra A.*

Example 1. Let $\mathcal{M}$ be the category of all semigroups with the unit and having the property that all elements have an inverse element. The basic operation on the objects is multiplication and the polynomials are

$$f(x_1, \ldots, x_n) = x_{k_1} \ldots x_{k_j}$$

where n is a natural number and $k_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, j$. The nullary operation of the unit and the unary operation of the inverse element are admissible operations in the category $\mathcal{M}$ because they cannot be expressed as the multiplication of the above mentioned form and it can be shown that if a mapping between two semigroups from the category $\mathcal{M}$ preserves the operation of multiplication, then it preserves also the unit and the inverse element.

2. Proper and improper free algebras.

Free algebras in primitive classes of algebras can be defined in two ways:

Definition. *$\mathcal{F}(X)$ is a free algebra on a set X over the class K if*

(i) $\mathcal{F}(X) \in K$;

(ii) *$\mathcal{F}(X)$ is generated by X (thus every element of the algebra $\mathcal{F}(X)$ can be expressed in the form $f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in X$ and f is an n-ary* polynomial);

(iii) *every mapping $X \to A$, $A \in K$ can be extended to a homomorphism $\mathcal{F}(X) \to A$.*

Definition. *$\mathcal{F}(X)$ is a free algebra on a set X over the class K if*

(i) $\mathcal{F}(X) \in K$;

(ii) *every mapping $X \to A$, $A \in K$ can be uniquely extended to a homomorphism $\mathcal{F}(X) \to A$.*

In the case of a non-primitive class the last two definitions need not be equivalent. We shall use the latter one, therefore the free algebra on the set X over a non-primitive class of algebras need not be generated by the set X.

Definition. *A free algebra on the set X over a class K is called a proper free algebra if it is generated by X; otherwise it is called an improper free algebra.*

Example 2. The set of all integers with addition is the free algebra on the one-element set $\{1\}$ over the category $\mathcal{M}$ from Example 1 and the subalgebra generated by $\{1\}$ is only the set of all positive integers, thus the algebra is an improper free algebra on the set $\{1\}$ over $\mathcal{M}$.

3. Existence of admissible operations.

Theorems 1 and 2 show some relationship between the existence of admissible operations and the existence of proper and improper free algebras.

Theorem 1. *If there exists a proper free $\mathcal{K}$-algebra with the property* card $X \geq n$, *then no admissible n-ary operation in the category $\mathcal{K}$ exists.*

Proof. Let the assumption of the theorem be satisfied and an n-ary admissible operation f exist. If $x_1, \ldots, x_n \in X$, where $x_i \neq x_j$ for $i \neq j$, then

$$f^{\mathscr{F}(X)}(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$$

where $y_1, \ldots, y_m \in X$ and g is a polynomial, thus

$$f^A(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$$

holds in every algebra $A \in \mathrm{Ob}\, \mathscr{K}$ where $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are considered as variables. Hence f is equal to the polynomial g apart from the order of the arguments. It is clear that there exists a polynomial equal to f.

Corollary. *If there is a proper free $\mathscr{K}$-algebra on an infinite set, then no admissible operation in the category $\mathscr{K}$ exists.*

Theorem 2. *If there exists an improper free $\mathscr{K}$-algebra on a finite set $X = \{x_1, \ldots, x_n\}$, then there are n-ary admissible operations in the category $\mathscr{K}$.*

Proof. Let $[X]$ be the set of all elements from $\mathscr{F}(X)$ generated by the set X. To every element $y \in \mathscr{F}(X) - [X]$ an n-ary admissible operation f_y defined by

$$f_y^A(a_1, \ldots, a_n) = y\varphi \text{ for } \quad a_1, \ldots, a_n \in A$$

can be associated, where $\varphi \colon \mathscr{F}(X) \to A$ is the homomorphism defined by

$$x_i\varphi = a_i \quad \text{for} \quad i = 1, \ldots, n\,.$$

If $\psi \colon A \to B$ is a homomorphism, then $f_y^A(a_1, \ldots, a_n)\, \psi = y\varphi\psi = f_y^B(a_1\psi, \ldots, a_n\psi)$. Clearly, every f_y is different from all polynomials in the category $\mathscr{K}$.

Corollary. *If there exists a proper free $\mathscr{K}$-algebra on X, then no improper free $\mathscr{K}$-algebra on a finite set Y with* card $Y \leq$ card X *exists.*

Remarks: 1. By omitting some homomorphisms in the category $\mathscr{K}$ some admissible operations may arise. Let the objects of the category $\mathscr{K}$ be the cyclic groups Z_3, Z_4 with a nullary operation 0 and a binary operation $+$ and let the morphisms be all group homomorphisms. By omitting all zero homomorphisms an admissible operation $0 \in Z_3$, $2 \in Z_4$ arises. For every remaining homomorphism $Z_3 \ni 0 \mapsto 0 \in Z_3$, $Z_4 \ni 2 \mapsto 2 \in Z_4$ hold and the element $2 \in Z_4$ cannot be obtained by using the operations 0 and $+$, thus it is not the value of any nullary polynomial.

2. By omitting some algebras in the category $\mathscr{K}$ some admissible operations may disappear. Let the objects of the category $\mathscr{K}$ be the cycklic groups Z_3, Z_4 with the same operations as above and let the morphisms be the following group homomorphisms: $Z_3 \ni 1 \mapsto 1 \in Z_3$, $Z_4 \ni 1 \mapsto 0 \in Z_4$, $Z_4 \ni 1 \mapsto 1 \in Z_4$, $Z_4 \ni 1 \mapsto 2 \in Z_4$, $Z_4 \ni 1 \mapsto 3 \in Z_4$. There is an admissible nullary operation $2 \in Z_3$, $0 \in Z_4$ in $\mathscr{K}$. By omitting the group Z_3 this admissible operation disappears.

3. By omitting some algebras in the category $\mathscr{K}$ we omit also the corresponding homomorphisms, thus some admissible operations may arise and some others may disappear. Let the objects of the category $\mathscr{K}$ be the cyclic groups Z_2, Z_4 with

a nullary operation 0 and a binary operation + and let the morphisms be the following group homomorphisms: $Z_2 \ni 1 \mapsto 1 \in Z_2$, $Z_2 \ni 1 \mapsto 0 \in Z_4$, $Z_4 \ni 1 \mapsto 1 \in Z_4$, $Z_4 \ni 1 \mapsto 3 \in Z_4$. There is an admissible nullary operation $1 \in Z_2$, $0 \in Z_4$ in $\mathcal{K}$. By omitting the group Z_2 this admissible operation disappears and a new admissible nullary operation $2 \in Z_4$ arises.

REFERENCES

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ДОПУСТИМЫЕ ОПЕРАЦИИ В КАТЕГОРИЯХ АЛГЕБР

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Резюме

Допустимая операция в категории алгебр не является ни основной операцией ни полиномом, но сохраняют ее все морфизмы в этой категории. В не примитивных классах алгебр могут существовать не собственные свободные алгебры, которые не порождены своими свободными порождающими элементами. В работе показана относительность между существованием допустимых операций и существованием собственных и не собственных свободных алгебр.