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A NOTE ON MAPPINGS OF BAIRE SPACES

TIBOR NEUBRUNN

There are some assertions concerning Baire spaces in the proof of which the following false statement is used:

(1) *If f is a one-to-one, feebly continuous mapping from X onto Y, then f is almost continuous.* (See, e.g. [4] p. 217).

We shall deal with the following three assertions of the mentioned type.

(2) *If f is a one-to-one feebly continuous* and *feebly open mapping of X onto Y, then X is a Baire space if and only if Y is a Baire space.*

(3) *If f is a one-to-one feebly continuous and feebly open mapping of X onto Y, then X is totally inexhaustible if and only if Y is totally inexhaustible.*

(4) *If f is a one-to-one feebly continuous and feebly open mapping of a regular space X onto* a *totally inexhaustible space Y, then X is a Baire space.*

Concerning (2), see [3], Corollary p. 383. As to (3) and (4), see [4], Corollaries 3.6 and 3.7.

We shall prove that (2) is true while (3) and (4) are false. Note that all the results of [3] are correct. The only place where (1) was used is the assertion (2), which is also true. Among the corect results of [4], the false corollaries (3) and (4) appear. In their proofs (1) was used.

A topological space X is said to be a Baire space if the intersection of any sequence $\{G_n\}_{n=1}^{\infty}$ of dense open sets in X is dense in X. It is said to be inexhaustible if it is not of the first category (or, in the notations of [1], if it is not a meagre set) relative to itself.

Hence a space X is inexhaustible if and only if it is not a contable union of closed nowhere dense sets in X or, which is the same, if the intersection of any countable sequence $\{G_n\}_{n=1}^{\infty}$ of open dense sets is nonempty.

A topological space X is said to be totally inexhaustible, (totally non-meagre in the notations of [1]) if any closed subspace of X is inexhaustible.

A mapping from X onto Y is said to be feebly continuous (feebly open) if for any nonempty open set $V \subset Y$ ($U \subset X$), the set int $(f^{-1}(V))$ (int $(f(U))$) is nonempty.

A mapping from X onto Y is said to be almost continuous if $f^{-1}(G) \subset$ $\subset \overline{\operatorname{int} f^{-1}(G)}$ for any open set $G \subset Y$.

Note that the notion "almost continuous" as defined above is known in literature also under different names. We omit various equivalent definitions and different names of this notion.

Proposition 1. *Statement* (1) *is not true.*

For the proof it is sufficient to take $X = Y = (-\infty, \infty)$ with the topology of the real line and to put $f(x) = x$, if $x \neq 0$, $x \neq 1$, $f(0) = 1$, $f(1) = 0$. Clearly f is one-to-one feebly continuous moreover it is also feebly open. If $G = (-\frac{1}{2}, \frac{1}{2}) \subset Y$, then $f^{-1}(G) = (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \cup \{1\}$. Hence $f^{-1}(G) \not\subset \overline{\operatorname{int} f^{-1}(G)}$.
Thus f is not almost continuous.

Now we shall give the proof of (2).

Theorem. *If f is a one-to-one feebly continuous and feebly open mapping of X onto Y, then X is a Baire space if and only if Y is a Baire space.*

Proof. Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of open sets which are dense in Y. We shall prove that the sets $\operatorname{int} f^{-1}(G_n)$, $n = 1, 2, \ldots$ are dense in X. Let $x_0 \in X$, n arbitrarily fixed, and U any neighbourhood containing x_0. Since f is feebly open, we have $\operatorname{int} f(U) \neq 0$. Hence a nonempty open set V exists such that $V \subset f(U)$. The set $V \cap G_n$ is a nonempty open set. Since f is also feebly continuous, we have $\emptyset \neq W = \operatorname{int}(f^{-1}(V \cap G_n))$.

But

$$W \subset f^{-1}(V \cap G_n) \subset f^{-1}(f(U)) = U.$$

Since U was an arbitrary neighbourhood of x_0 and W is a nonempty open subset of $f^{-1}(G_n)$ contained in U, the statement $x_0 \in \overline{\operatorname{int}(f^{-1}(G_n))}$ is true. Thus $Z_n = \operatorname{int} f^{-1}(G_n)$ are nonempty open and dense subsets of X. Since X is a Baire space, the set $\bigcap_{n=1}^{\infty} Z_n$ is dense in X. This and the feeble continuity of f imply that $f(\bigcap_{n=1}^{\infty} Z_n)$ is dense in $f(X) = Y$.

Since $\bigcap_{n=1}^{\infty} G_n \supset \bigcap_{n=1}^{\infty} f(Z_n) \supset f(\bigcap_{n=1}^{\infty} Z_n)$, the set $\bigcap_{n=1}^{\infty} G_n$ is dense in Y.

The "only if" part follows from the fact that the inverse mapping f^{-1} is also feebly continuous and feebly open.

Note that the above theorem remains to be true if the words "X is a Baire space, Y is a Baire space" are substituted by words "X is of the second category, Y is of the second category". For this case the proof is quite analogical. This case is also included in [2], see Theorem 17, Corollary 17.1, where statement (1) was not used.

In Theorem 18 of [2], (1) was used only indirectly. In fact in 18 only the fact was used that (2) is true. As we have seen (2) is true, hence Theorem 18 of [2] and its proof are corect.

Proposition 2. *Statements* (3) *and* (4) *are not true.*

Proof. Denote successively by Q, R, Z, N the following subsets of E^2: the set of all $(r, 0)$, where r is rational; the set of all $(x, 0)$ where x is real; the set of all $(0, y)$, where y is real; the set of all $(0, n)$, where n is a positive integer. Put $X = E^2 - R$, $Y = X \cup Q$. Both X and Y will be considered with the topology given by the Euclidean metric.

The space X is totally inexhaustible. In fact we may consider X as a subspace of E^2. If F is a nonempty closed substet of X, then the closure $\bar{F}$ (in E^2) is of the form $\bar{F} = (\bar{F} \cap R) \cup F$. The set $\bar{F}$ is a nonempty and closed subset of E^2, hence it is of the second category in itself. The set $\bar{F} \cap R$ is nowhere dense in $\bar{F}$ as can be immediately verified. Hence F is of the second category in $\bar{F}$. The last implies that F is of the second category in itself. Thus we have that F is totaly inexhastible.

The space Y is not totally inexhaustible. It is sufficient to take the subspace $Q \subset Y$, which is closed in Y, but it is of the first category.

We shall define now a feebly continuous and feebly open mapping from X onto Y. First of all let φ be some one-to-one function frome N onto Q. Let ψ be some one-to-one function from $Z - N$ onto Z.

Define $f: X \to Y$ as:

$$f(t) = \begin{matrix} t & \text{if} & t \notin Z \\ \varphi(t) & \text{if} & t \in N \\ \psi(t) & \text{if} & t \in Z - N. \end{matrix}$$

The function f is a one-to-one, feebly continuous and feebly open mapping of X onto Y. Thus (3) is not true. The fact that (4) is not true follows immediately if we consider instead of f the inverse mapping f^{-1} from the regular space Y onto the totally inexhaustible space X.

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ЗАМЕЧАНИЕ ОБ ОТОБРАЖЕНИЯХ БЭРОВСКИХ ПРОСТРАНСТВ

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Резюме

Образ бэровского пространства при значительно обобщенном гомеоморфизме является также бэровским пространством. Но при том же гомеоморфизме образ пространства, у которого каждое замкнутое подпространство второй категории, необязательно того же типа. Этих два утверждения появляются в работе, пополняя тем самым некоторые известные результаты в этом направлении.