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DISCONJUGACY CRITERIA FOR LINEAR VECTOR DIFFERENTIAL EQUATIONS

BLAŻEJ SZMANDA

1. We consider the n-th order linear vector differential equation

$$x^{(n)}=\sum_{i=1}^{n}A_i(t)x^{(n-i)}, \qquad x^{(0)}\equiv x,\ n\geq 2, \tag{1.1}$$

where the coefficients $A_i(t)$, $(i=1, \ldots, n)$ are real $m\times m$ matrix functions; x is the m-dimensional column-vector $x=(x_1, \ldots, x_m)$; $t\in[a, b]$.

By a solution of Eq. (1.1) we mean any function $x:[a, b]\to R^m$ with an absolutely continuous derivative of the $(n-1)$-th order, satisfying Eq. (1.1) almost everywhere in $[a, b]$.

If all components of a solution $x(t)$ are identically zero on $[a, b]$ (i.e. $x(t)\equiv\theta$, where $\theta=(0, \ldots, 0)$), then the solution is called trivial.

The differential equation (1.1) is said to be disconjugate on $[a, b]$ if no nontrivial solution of (1.1) has more than $n-1$ zeros, counting multiplicities, on $[a, b]$. The object of this paper is to derive disconjugacy criteria for (1.1) on a compact interval $[a, b]$, generally related to conditions of de la Vallée Poussin type. There is a large literature giving conditions of this type for (1.1) in case of $m=1$ (see for instance the references in [8]). Disconjugacy criteria for (1.1), $(m=1)$ of another type may be found for example in papers [2], [3].

The results obtained in this paper are generalizations of those obtained by de la Vallée Poussin [7], A. Levin (comp. e.g. A. Lasota, Z. Opial [4]) and M. Martelli [5].

2. Before establishing tests for disconjugacy, we prove first some lemmas. By $C^n(a, b)$ we denote the class of functions defined on $[a, b]$ with a continuous derivative of the order n.

Lemma 2.1. *If the function* $w:[a, b]\to R$, $w(t)\in C^n(a, b)$ *has at least n zeros in the interval* $[a, b]$ *(counting multiplicities), then*

$$(2.1)\qquad \left[\int_a^b |w(t)|^q \, dt\right]^{1/q} \leq \mu \frac{(b-a)^{n+1/q}}{n!(nq+1)^{1/q}}, \qquad q \geq 1,$$

where

$$(2.2)\qquad \mu = \max_{[a,b]} |w^{(n)}(t)| .$$

Proof. We denote the successive zeros of $w(t)$ by $t_1, \ldots, t_n$, (i.e. $w(t_i)=0$ for $i=1, \ldots, n$) where $a \leq t_1 \leq \ldots \leq t_n \leq b$.

If the function $w(t)$ satisfies the assumptions of lemma 2.1, then the following inequality holds

$$|w(t)| \leq \frac{\mu}{n!} |(t-t_1)(t-t_2)\ldots(t-t_n)| , \qquad t \in [a, b] ,$$

where μ is defined by the formula (2.2)*), (cf. [6] p. 156).

So we have

$$(2.3)\qquad \int_a^b |w(t)|^q \, dt \leq \left(\frac{\mu}{n!}\right)^q \int_a^b |(t-t_1)\ldots(t-t_n)|^q \, dt .$$

Denoting

$$\varphi(t) = |(t-t_2)\ldots(t-t_n)|^q ,$$

we get

$$I = \int_a^b |(t-t_1)\ldots(t-t_n)|^q \, dt = \int_a^b |t-t_1|^q \varphi(t) \, dt = \psi(t_1) ,$$

so

$$\psi(t_1) = \int_a^{t_1} (t_1 - t)^q \varphi(t) \, dt + \int_{t_1}^b (t-t_1)^q \varphi(t) \, dt .$$

First let us consider the case, where the first zero $t_1 \in [a, b]$ is floating while the other zeros are fixed. Then the function $\psi(t_1)$ is a continuous function of argument t_1 and has a derivative

$$\psi'(t_1) = q \int_a^{t_1} (t_1 - t)^{q-1} \varphi(t) \, dt - q \int_{t_1}^b (t-t_1)^{q-1} \varphi(t) \, dt , \qquad q \geq 1 .$$

When t_1 changes from a to b, then $\psi'(t_1)$ increases from negative to positive values. So the maximal value of I is reached either for $t_1 = a$ of for $t_1 = b$.

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*) If two points e.g. t_1 and t_2 coincide, then instead of $(t-t_1)(t-t_2)$ we have $(t-t_1)^2$ and similarly in the case of coincidence of a larger number of points $t_1, \ldots, t_n$.

Arguing in the same way with respect to the points $t_2, \ldots, t_n$, we come to the conclusion that the maximal value of the integral I does not exceed the largest of the numbers

$$I_k = \int_a^b (t-a)^{qk}(b-t)^{(n-k)q}\,dt\,, \qquad (k=0, 1, \ldots, n)\,.$$

Substituing $t-a=(b-a)s$ we get

$$I_k=(b-a)^{nq+1}\frac{\Gamma(kq+1)\Gamma((n-k)q+1)}{\Gamma(nq+2)}\,,$$

where $\Gamma(\alpha)$ denotes the Gamma function.

In cases $k=0$ and $k=n$ it is obvious that

$$I_0=I_n=\frac{(b-a)^{nq+1}}{nq+1} \tag{2.4}$$

For other $k-s$ we have

$$2\leqslant nq+1-kq\leqslant nq+1-[kq]\,,$$

where $[kq]$ denotes the greatest integer contained in kq.

From the monotonicity of $\Gamma(\alpha)$ it follows

$$\Gamma((n-k)q+1)\leqslant\Gamma(nq-[kq]+1)\,. \tag{2.5}$$

From (2.5) and from the basic property of $\Gamma(\alpha)$ we get

$$I_k\leqslant(b-a)^{nq+1}\frac{kq(kq-1)\ldots\alpha_1\Gamma(\alpha_1)}{(nq+1)nq\ldots(nq-[kq]+1)}\,,$$

where $1\leqslant\alpha_1<2$. Since $0<k<n$ and $\Gamma(\alpha_1)\leqslant1$, the following inequality holds

$$I_k<(b-a)^{nq+1}\frac{1}{nq+1}\,, \qquad (k=1, \ldots, n-1)\,. \tag{2.6}$$

By virtue of (2.3), (2.6) and (2.4) one can write

$$\int_a^b |w(t)^q\,dt\leqslant\left(\frac{\mu}{n!}\right)^q\frac{(b-a)^{nq+1}}{nq+1}\,.$$

This completes the proof of lemma 2.1.

Remark 2.2. From the above considerations it follows that if at least two points from $t_1, \ldots, t_n$ are different and $w(t)\not\equiv0$, then the inequality (2.1) is sharp.

Remark. The inequality (2.1) is well known in the case of $q=1$ (comp. [6] p. 155). This inequality has also been proved recently by J. Brink [1]. In this paper an elementary proof of this inequality is given.

Corollary 2.3. *If the function $w(t)$ satisfies the assumptions of the lemma* 2.1, *then the following inequalities hold*

$$\left[\int_a^b |w^{(i)}(t)|^q \, dt\right]^{1/q} \leq \mu \frac{(b-a)^{n-i+1/q}}{(n-i)!((n-i)q+1)^{1/q}}, \qquad i=1, \ldots, n-1,$$

where μ is defined by (2.2).

In fact, as $w(t)$ has n zeros in $[a, b]$, then, by Rolle's Theorem, $v_i(t) = w^{(i)}(t)$ has at least $n-i$ zeros ($i = 1, \ldots, n-1$) in $[a, b]$. Putting $n-i$ instead of n in (2.1) and applying this inequality to the function $v_i(t)$, ($i=1, \ldots, n-1$), we get (2.7).

Let us denote by $\|x\| = \sum_{k=1}^m |x_k|$ the norm of an element $x \in R^m$ and similarly $\|A(t)\| = \max_j \sum_{k=1}^m |a_{kj}(t)|$ for an $m \times m$ matrix $A(t) = \{a_{kj}(t)\}$.

Lemma 2.4. *If the function $z: [a, b] \to R^m$, $z(t) \in C^n(a, b)$ has n zeros in $[a, b]$, (where the zeros are counted with their multiplicities), then*

$$(2.8) \qquad \left[\int_a^b \|z(t)\|^q \, dt\right]^{1/q} \leq \mu \frac{m^{1-1/q}(b-a)^{n+1/q}}{n!(nq+1)^{1/q}}, \qquad q \geq 1,$$

where

$$(2.9) \qquad \mu = \sum_{k=1}^m \mu_k \quad \mu_k = \max_{[a,b]} |z_k^{(n)}(t)|, \qquad (k=1, \ldots, m).$$

Proof. Applying Hölder's inequality for sums with the indices $q > 1$ and $q/(q-1)$, we have

$$(2.10) \left[\int_a^b \|z(t)\|^q \, dt\right]^{1/q} = \left[\int_a^b \left(\sum_{k=1}^m |z_k(t)|\right)^q dt\right]^{1/q} \leq \left[\int_a^b m^{q-1} \sum_{k=1}^m |z_k(t)|^q \, dt\right]^{1/q}.$$

Each of the functions $z_k(t)$ satisfies the assumptions of lemma 2.1. Thus from (2.1) it follows

$$(2.11) \qquad \int_a^b |z_k(t)|^q \, dt \leq \left(\frac{\mu_k}{n!}\right)^q \frac{(b-a)^{nq+1}}{nq+1}, \qquad (1 \leq k \leq m),$$

where μ_k is defined by (2.9).

By virtue of (2.10) and (2.11) we obtain

$$\left[\int_a^b \|z(t)\|^q \, dt\right] < \mu \frac{m^{1-1/q}(b-a)^{n+1/q}}{n!(nq+1)^{1/q}} \sum_{k=1}^m \mu_k,$$

that is (2.8).

In the case of $q=1$, the inequality (2.8) acquires the form

$$(2.12)\qquad \int_a^b \|z(t)\|\,dt \leqslant \mu\,\frac{(b-a)^{n+1}}{(n+1)!}.$$

In fact, by lemma 2.1, we have

$$\int_a^b \|z(t)\|\,dt = \sum_{k=1}^{m} \int_a^b |z_k(t)|\,dt \leqslant \frac{(b-a)^{n+1}}{(n+1)!} \sum_{k=1}^{m} \mu_k .$$

Remark 2.5. In view of remark 2.2 one may conclude that if at least two zeros of the function $z(t)$ are different and $z(t) \not\equiv \theta$, then the inequality (2.8) is sharp.

Lemma 2.6. *If the function* $z(t) = (z_1(t), \ldots, z_m(t))$ *satisfies the assumptions of lemma* 2.4, *then*

$$(2.13)\quad \left[\int_a^b \|z^{(i)}(t)\|^q\,dt\right]^{1/q} \leqslant \quad \leqslant \mu\,\frac{m^{1-1/q}(b-a)^{n-i+1/q}}{(n-i)!((n-i)q+1)^{1/q}},\ q \geqslant 1,\ i = 1, \ldots, n-1,$$

where μ *is defined in* (2.9).

Proof. Assume, $q > 1$. The inequality (2.10) takes the following form

$$\left[\int_a^b \|z^{(i)}(t)\|^q\,dt\right]^{1/q} \leqslant m^{1-1/q} \left[\int_a^b \sum_{k=1}^{m} |z_k^{(i)}(t)|^q\,dt\right]^{1/q}.$$

By virtue of (2.7), we have

$$\int_a^b |z_k^{(i)}(t)|^q\,dt \leqslant \left(\frac{\mu_k}{(n-i)!}\right)^q \frac{(b-a)^{(n-i)q+1}}{(n-i)q+1}, \qquad (k = 1, \ldots, m;\ i = 1, \ldots, n-1).$$

Thus we get

$$\left[\int_a^b \|z^{(i)}(t)\|^q\,dt\right]^{1/q} \leqslant \frac{m^{1-1/q}(b-a)^{n-i+1/q}}{(n-i)!((n-i)q+1)^{1/q}} \left(\sum_{k=1}^{m} \mu_k^q\right)^{1/q},$$
$$(i = 1, \ldots, n-1),$$

and this proves the inequalities (2.13).

In the case of $q = 1$ we argue as in the proof of lemma 2.4. We get then the inequalities

$$\int_a^b \|z^{(i)}(t)\|\,dt \leqslant \mu\,\frac{(b-a)^{n-i+1}}{(n-i+1)!}, \qquad (i = 1, \ldots, n-1),$$

which are identical with (2.13) for $q = 1$.

3. Now we shall prove the following:

Theorem 3.1. *Let the matrices* $A_i(t)$, $(i = 1, \ldots, n)$ *from Eq.* (1.1) *be measureable and bounded on* $[a, b]$*). *Then there exists* $\varepsilon > 0$ *such that the equation* (1.1) *is disconjugate on every subinterval* $[\alpha, \beta] \subset [a, b]$ *of a length less than* ε.

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*) This means that every element of the matrix has this property.

Proof. Let us denote

$$M=\max_{1\leq i\leq n}\sup_{t\in[a,b]}\|A_i(t)\| \tag{3.1}$$

and let

$$\varepsilon=\min\left[1,\frac{1}{nM}\right]. \tag{3.2}$$

Suppose, there exists a solution $x(t)\not\equiv\theta$ of Eq. (1.1) with n zeros (taking into account multiplicities) in some subinterval $[\alpha,\beta]\subset[a,b]$, where $\beta-\alpha<\varepsilon$. Then each component of the i-th derivative $x^{(i)}(t)=(x_1^{(i)}(t),\ldots,x_m^{(i)}(t))$ has $n-i$ zeros $[\alpha,\beta]$, where $i=1,\ldots,n-1$. By virtue of the mean value theorem, we get the following inequalities

$$|x_k^{(i)}(t)\leq\max_{[\alpha,\beta]}|x_k^{(i+1)i+1)}(\tau)|(\beta-\alpha),\qquad t\in[\alpha,\beta], \tag{3.3}$$

$$(k=1,\ldots,m;\ i=0,1,\ldots,n-2).$$

Let us denote

$$\mu_i=\sum_{k=1}^{m}\max_{[\alpha,\beta]}|x_k^{(i)}(t)|,\qquad (i=0,1,\ldots,n-1). \tag{3.4}$$

Since $\beta-\alpha<\varepsilon$, $\mu_0>0$, then by virtue of (3.3),

$$\mu_0<\varepsilon\mu_1,\quad \mu_1<\varepsilon\mu_2,\ \ldots,\ \mu_{n-2}<\varepsilon\mu_{n-1},$$

and hence

$$0<\mu_i<\varepsilon^{n-1-i}\mu_{n-1},\qquad (i=0,1,\ldots,n-2). \tag{3.5}$$

As we have seen there are points $\tau_k\in[\alpha,\beta]$, $(k=1,\ldots,m)$ such that

$$x_k^{(n-1)}(\tau_k)=0,\qquad (k=1,\ldots,m).$$

Therefore

$$x_k^{(n-1)}(t)=\int_{\tau_k}^{t}x_k^{(n)}(s)\,ds,\qquad (k=1,\ldots,m),$$

and so we get

$$|x_k^{(n-1)}(t)|\leq\int_{\alpha}^{\beta}|x_k^{(n)}(s)|\,ds,\qquad (k=1,\ldots,m),\ t\in[\alpha,\beta],$$

and finally,

$$\mu_{n-1} \leqslant \int_\alpha^\beta \|x^{(n)}(t)\| \, dt .$$

From the difnerential equation (1.1) and from the last inequality we have

$$\mu_{n-1} \leqslant \int_\alpha^\beta \|x^{(n)}(t)\| \, dt \leqslant \sum_{i=1}^n \int_\alpha^\beta \|A_i(t)\| \, \|x^{(n-i)}(t)\| \, dt ,$$

and hence, taking into account (3.4), we get

$$\mu_{n-1} \leqslant (\beta - \alpha) \sum_{i=1}^n \mu_{n-i} \sup_{[\alpha, \beta]} \|A_i(t)\| .$$

Taking into consideration condition $\varepsilon \leqslant 1$ and (3.1), (3.5) in the inequality (3.6), we obtain

$$\mu_{n-1} < M(\varepsilon + \varepsilon^2 + \ldots + \varepsilon^n)\mu_{n-1} \leqslant mn\varepsilon\mu_{n-1} ,$$

and so

$$1 < Mn\varepsilon ;$$

this contradicts the definition of ε (cf. (3.2)).

Now we shall prove the theorem to be an extension of the well-known de la Vallée Poussin theorem [7].

Theorem 3.2. *If the coefficients $A_i(t)$, $(i = 1, \ldots, n)$ are measurable and bounded on $[a, b]$ $(\|A_i(t)\| \leqslant A_i$, $A_i = const.$, $i = 1, \ldots, n)$ and there holds the inequality*

$$\sum_{i=1}^n \frac{A_i(b-a)^i}{i!} \leqslant 1 , \tag{3.7}$$

then the differential equation (1.1) *is disconjugate on* $[a, b]$.

Proof. Let us suppose that the thesis of the theorem does not hold, and so there exists a solution $x(t) \not\equiv \theta$ with n zeros in $[a, b]$.

Of course at least two zeros are different, and so there exist points $\tau_k \in [a, b]$, $(k = 1, \ldots, m)$ at which

$$x_k^{(n-1)}(\tau_k) = 0 , \qquad (k = 1, \ldots, m) .$$

Let us denote

$$v = \sum_{k=1}^m \max_{[a, b]} |x_k^{(n-1)}(t)| . \tag{3.8}$$

Proceeding similarly as in the proof of theorem 3.1, we get

$$v \leqslant \sum_{i=1}^n \int_a^b \|A_i(t)\| \, \|x^{(n-i)}(t)\| \, dt , \tag{3.9}$$

and hence

$$(3.10) \qquad v \leqslant \sum_{i=1}^{n} A_i \int_a^b \|x^{(n-i)}(t)\| \, \mathrm{d}t \, .$$

In view of the inequality (2.12) and remark 2.5 we have

$$(3.11) \qquad \int_a^b \|x(t)\| \, \mathrm{d}t < v \frac{(b-a)^n}{n!} \, .$$

Taking into account the inequalities (2.14), we get

$$(3.12) \qquad \int_a^b \|x^{(n-i)}(t)\| \, \mathrm{d}t \leqslant v \frac{(b-a)^i}{i!} \, , \qquad i = 1, \ldots, n-1 \, .$$

Taking into consideration the estimates (3.11) and (3.12) in (3.10), we obtain

$$v < v \sum_{i=1}^{n} A_i \frac{(b-a)^i}{i!} \, .$$

Since $x(t) \not\equiv \theta$ in $[a, b]$, then $v > 0$; therefore dividing the above inequality by v we get an inequality which is contrary to the assumption (3.7). This completes the proof of theorem 3.2.

Let us denote by $\mathrm{L}^p(a, b)$ the class of functions summable with the p-th power $(p > 1)$ on $[a, b]$.

Now we consider the coefficients of a wider class, namely, of the class $\mathrm{L}^p(a, b)$. We shall prove.

Theorem 3.3. *If the functions* $\|A_i(t)\| \in \mathrm{L}^p(a, b)$, $(i = 1, \ldots, n)$ *and the* inequality

$$(3.13) \qquad m^{1/p} \sum_{i=1}^{n} \frac{\|A_i\|_p (b-a)^i}{(i-1)!((i-1)q+1)^{1/q}} \leqslant 1 \, , \quad 1/p + 1/q = 1 \, ,$$

hold, where

$$(3.14) \qquad \|A_i\|_p = \left[\frac{1}{b-a} \int_a^b \|A_i(t)\|^p \, \mathrm{d}t \right]^{1/p} , \qquad (i = 1, \ldots, n) \, ,$$

then the equation (1.1) *is disconjugate on* $[a, b]$.

Proof. The first part of this proof is the same as in the proof of theorem 3.2. Hence we start with the inequality (3.9). We have

$$(3.15) \qquad v \leqslant \sum_{i=1}^{n} \int_a^b \|A_i(t)\| \, \|x^{(n-i)}(t)\| \, \mathrm{d}t \, ,$$

where v is defined by (3.8).

We apply Hölder's inequality with the indices p and $q = p/(p-1)$ to the right-hand side of (3.15), thus obtaining

$$(3.16) \qquad v \leq \sum_{i=1}^{n} \left[\int_a^b \|A_i(t)\|^p \, dt \right]^{1/p} \left[\int_a^b \|x^{(n-i)}(t)\|^q \, dt \right]^{1/q} .$$

By virtue of lemma 2.4 and remark 2.5 we get

$$(3.17) \qquad \left[\int_a^b \|x(t)\|^q \, dt \right]^{1/q} < v \frac{m^{1-1/q}(b-a)^{n-1+1/q}}{(n-1)!((n-1)q+1)^{1/q}} ,$$

while from the lemma 2.6 the following inequalities follow

$$\left[\int_a^b \|x^{(n-i)}(t)\|^q \, dt \right]^{1/q} \leq$$

$$(3.18) \qquad \leq v \frac{m^{1-1/q}(b-a)^{i-1+1/q}}{(i-1)!((i-1)q+1)^{1/q}} , \qquad (i = 1, \ldots, n-1) .$$

Taking into account (3.17), (3.18) in (3.16) we get

$$v < v \sum_{i=1}^{n} \frac{m^{1-1/q}(b-a)^{i-1+1/q}}{(1-1)!((i-1)q+1)^{1/q}} \left[\int_a^b \|A_i(t)\|^p \, dt \right]^{1/p} .$$

Dividing this inequality by $v > 0$ and applying (3.14), we obtain an inequality contradicting (3.13). Thus, theorem 3.3 is proved.

Remark. Taking in particular $m = 1$ in theorem 3.3, one may conclude Martelli's theorem given in [5] and Levin's theorem (cf. e.g. [4]).

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КРИТЕРИЯ ОТСУТСТВИЯ СОПРЯЖЕННЫХ ТОЧЕК ДЛЯ ЛИНЕЙНЫХ ВЕКТОРНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Блажей Шманда

Резюме

В настоящей работе рассматривается линейное дифференциальное уравнение

$$x^{(n)} = \sum_{i=1}^{n} A_i(t) x^{(n-i)}, \qquad n \geq 2, \tag{1.1}$$

где $A_i(t)$, $(i = 1, \ldots, n)$ являятся вещественными матрицами размера $m \times m$; x — векторная функция со значениями в R^m; $t \in [a, b]$.

В статье представлены достаточные условия для того, чтобы каждое нетривиальное решение $x(t)$ уеавнения (1.1) имело не более $n-1$ нулей в интервале $[a, b]$ с учётом кратности. Полученные результаты являются обобщением результатов Валле—Пуссена, А. Левина, М. Мартелля, касающихся уравнения (1.1) в случае $m = 1$.

BOOK REVIEWS

V. Cruceanu: ELEMENTE DI ALGEBRA LINIARA SI GEOMETRIE, Editura didactica si pedagogica Bucuresti 1973.

Kniha je vysokoškolskou účebnicou lineárnej algebry a geometrie a podáva obligátny základný kurz týchto spriaznených disciplín. Prvých 12 kapitol (190 strán) je venovaných lineárnej algebre. Hlavné pojmy: grupa, okruh, pole, vektorový priestor, matice a determinanty, systémy rovníc, lineárne transformácie, lineárne,bilineárne, kvadratické a polylineárne formy, skalárny súčin, ortogonálne transformácie, vonkajšie formy. Zvyšných 10 kapitol (150 strán) aplikuje a rozvíja vyložený algebraický aparát v oblasti geometrie. Hlavné pojmy: afinný priestor a grupa jeho transformácií, nadkvadriky a ich klasifikácia, izometrické transformácie, nadkvadriky v euklidovskom priestore, kužeľosečky v E^2 a kvadriky v E^3.

Do materiálu nie je pojatý pojem duálneho priestoru, projektívneho priestoru a konformnej grupy. Nad zvyčajný rozsah je tu pojednané o vonkajšom súčine.

Látka je didakticky dobre utriedená a proporcionálne vyvážená. Výklad je primeraný študentovi prvých ročníkov vysokej školy. Sympatické je delenie textu na kratšie celky. Dostatočný počet obrázkov uľahšuje názornú predstavu. Pomerne skromná je zásoba príkladov, na ktorých by bola teória vysvetlená ilustratívne. Škoda, že učebnica neobsahuje register pojmov.

Milan Hejný, Bratislava