Štefan Černák

Cantor extension of a mixed product of directed groups

Mathematica Slovaca, Vol. 26 (1976), No. 2, 103--114

Persistent URL: http://dml.cz/dmlcz/136113

Math. Slov., 26, 1976, No. 2, 103—114

CANTOR EXTENSION OF A MIXED PRODUCT OF DIRECTED GROUPS

ŠTEFAN ČERNÁK

C. J. Everett [2] has defined the Cantor extension (C-extension) $C(H)$ of an Abelian l-group H. Let $H = {}^{l}\Pi A_{\lambda}(\lambda \in \Lambda)$ be an Abelian l-group, which is the lexicographic product of l-groups A_{λ}. In paper [1] the relation between the l-group $C(H)$ and the l-groups $C(A_{\lambda})$ was established.

The concept of the C-extension can be applied to Abelian directed groups. Let $G = \Omega A_{\lambda}(\lambda \in \Lambda)$ be an Abelian directed group which is the mixed product of directed groups A_{λ}, where the index λ runs over an arbitrary partially ordered set Λ. In this paper we describe the relation between $C(G)$ and the C-extensions of factors of the given mixed product. Let M be the set of all maximal elements in Λ. It will be shown that the directed group $C(G)$ is isomorphic with the mixed product $\Omega B_{\lambda}(\lambda \in \Lambda)$, where $B_{\lambda} = C(A_{\lambda})$ if $\lambda \in M$ and $B_{\lambda} = A_{\lambda}$ if $\lambda \in \Lambda \setminus M$.

Let S be a partially ordered set and N the set of all positive integers. We shall say that a sequence (x_n) is in S if $x_n \in S$ for each $n \in N$. The sequence (x_n) in S is called increasing if $x_n \leqslant x_{n+1}$ $(n \in N)$. Analogously we define a descending sequence. We say that the sequence (x_n) o-converges to $a \in S$ (or a is the o-limit of (x_n)) and we write $x_n \to a$ if there exist sequences (t_n) and (v_n) such that the sequence (t_n) is descending and the sequence (v_n) is increasing such that there exist $\bigwedge t_n, \bigvee v_n$ with properties

(i) $v_n \leqslant x_n \leqslant t_n$ $(n \in N)$,

(ii) $\bigwedge t_n = \bigvee v_n = a$.

It is easy to verify that if the sequence (x_n) is descending (increasing), then $x_n \to a$ if and only if $\bigwedge x_n = a$ ($\bigvee x_n = a$). In this case we shall write $x_n \downarrow a$ ($x_n \uparrow a$) instead of $x_n \to a$.

Now let S be a directed set. The set of all upper (lower) bounds of elements $x_1, x_2, \ldots, x_n \in S$ will be denoted by $U(x_1, x_2, \ldots, x_n)$ ($L(x_1, x_2, \ldots, x_n)$). Choose a fixed $n_0 \in N$ and form the sequences $h(n_0, x_n)$ and $d(n_0, x_n)$ as follows:

$$h(n_0, x_n) = d(n_0, x_n) = x_n (n \in N, n \geqslant n_0),$$

$h(n_0, x_n) = u$, where u is a fixed element of
$U(x_1, x_2, \ldots, x_{n_0})$,
$d(n_0, x_n) = l$, where l is a fixed element of
$L(x_1, x_2, \ldots, x_{n_0})$ $(n \in N,\ n < n_0)$.

We see that $d(n_0, x_n) \leqslant x_n \leqslant h(n_0, x_n)$ $(n \in N)$. It is evident that $h(n_0, x_n) \downarrow a (d(n_0, x_n) \uparrow a)$ if and only if $x_n \geqslant x_{n+1} (x_n \leqslant x_{n+1})$ $(n \in N,\ n \geqslant n_0)$ and $\bigwedge x_n (n \geqslant n_0) = a (\bigvee x_n (n \geqslant n_0) = a)$.

1. *$x_n \to a$ if and only if there exist sequences (t_n), (v_n) and $n_0 \in N$ such that* (i) *holds true for each $n \in N$, $n \geqslant n_0$ and $h(n_0, t_n) \downarrow a$, $d(n_0, v_n) \uparrow a$.*

Proof. If $x_n \to a$, the assertion is implied by the definition. Conversely, let there exist sequences (t_n) and (v_n) satisfying (i) for each $n \geqslant n_0$ and let $h(n_0, t_n) \downarrow a$, $d(n_0, v_n) \uparrow a$. We have to show that there exist sequences (t'_n), (v'_n) satisfying (i) and (ii) such that (t'_n) is descending and (v'_n) is increasing. Sequences (t'_n) and (v'_n) can be constructed by putting

$t'_n = t_n$ if $n \geqslant n_0$; $t'_n = u$, where u is a fixed element of $U(x_1, x_2, \ldots, x_{n_0-1}, t_{n_0})$ if $n < n_0$,

$v'_n = v_n$ if $n \geqslant n_0$; $v'_n = l$, where l is a fixed element of $L(x_1, x_2, \ldots, x_{n_0-1}, v_{n_0})$ if $n < n_0$.

Assume that G is a partially ordered Abelian group. A sequence (x_n) in G is said to be fundamental if there is a sequence (t_n) such that $t_n \downarrow 0$ and

$$-t_n \leqslant x_n - x_m \leqslant t_n \tag{1}$$

holds for each $n \in N$ and each $m \in N$, $m \geqslant n$.

2. *If $x_n \downarrow a$, $y_n \downarrow b$, then $x_n + y_n \downarrow a + b$.*

Proof. Obviously, $(x_n + y_n)$ is a descending sequence. By [3] (p. 47, the property (d)) we have $x_n + y_n \to a + b$.

By a zero sequence we understand a sequence which o-converges to 0, where 0 is the zero element of the group G. The set of all fundamental (zero) sequences in G denote by $H(E)$. Define the operation $+$ in H in a natural way by putting $(x_n) + (y_n) = (x_n + y_n)$.

3. *H is a group.*

Proof. Suppose that $(x_n), (y_n) \in H$. Then there are $u_n \downarrow 0$, $v_n \downarrow 0$ satisfying the following inequalities:

$$-u_n \leqslant x_n - x_m \leqslant u_n,$$

$$-v_n \leqslant y_n - y_m \leqslant v_n$$

for each $n \in N$ and each $m \geqslant n$. Then $-(u_n + v_n) \leqslant (x_n + y_n) - (x_m + y_m) \leqslant u_n + v_n$. In view of 2, we get $u_n + v_n \downarrow 0$. Indeed, if $(x_n) \in H$, then $(-x_n) \in H$ as well.

If for each $(x_n), (y_n) \in H$ the relation $(x_n) \leqslant (y_n)$ means that $x_n \leqslant y_n$ $(n \in N)$, H is a partially ordered group.

4. *Every sequence* $(x_n) \in H$ *is bounded.*

Proof. By the definition there is a sequence (t_n) with the properties $t_n \downarrow 0$ and $-t_n \leqslant x_n - x_m \leqslant t_n$ $(n \in N,\ m \geqslant n)$. Then $x_n - t_n \leqslant x_m \leqslant x_n + t_n$. If we put $n = 1$, then $x_1 - t_1$ is a lower bound and $x_1 + t_1$ is an upper bound of the sequence (x_n). In all that follows suppose that G is an Abelian directed group. Then 5 and 6 hold true.

5. *H is a directed group.*

Proof. Let $(x_n), (y_n) \in H$. In view of 4, there are $a, b, c, d \in G$ such that $a \leqslant x_n \leqslant b$, $c \leqslant y_n \leqslant d$ $(n \in N)$. Choose the elements $e \in L(a, c)$ and $f \in U(b, d)$ from G. Then $(e, e, \ldots) \leqslant (x_n), (y_n)$ and $(f, f, \ldots) \geqslant (x_n), (y_n)$ for each $n \in N$. Obviously, the constant sequences $(e, e, \ldots)$ and $(f, f, \ldots)$ belong to H.

6. *A sequence* (x_n) *is an element of H if and only if there exist* $n_0 \in N$ *and a sequence* (t_n) *such that* (1) *is satisfied for each* $n \in N$, $n \geqslant n_0$, *each* $m \in N$, $m \geqslant n$ *and* $h(n_0, t_n) \downarrow 0$.

Proof. If $(x_n) \in H$, the statement immediately follows from the definition. Conversely, let n_0 and (t_n) exist with the properties $h(n_0, t_n) \downarrow 0$, and let (1) hold for each $n \geqslant n_0$, $m \geqslant n$. Form a sequence (t'_n) in the following way:

$$t'_n = t_n, \text{ if } n \geqslant n_0$$

$$t'_n = u + t_{n_0}, \textit{ if } n < n_0, \text{ where } u \in U[\pm (x_1 - x_2), \ldots$$
$$\ldots, \pm (x_1 - x_{n_0}), \pm (x_2 - x_3), \ldots, \pm (x_2 - x_{n_0}), \ldots$$
$$\ldots, \pm (x_{n_0-1} - x_{n_0}), t_{n_0}].$$

Evidently, $t'_n \downarrow 0$ and (1) holds for each $n < n_0$ and each m such that $n \leqslant m \leqslant n_0$. Again, let $n < n_0$, but $m > n_0$. Then

$$-t'_n = -(u + t_{n_0}) = -u - t_{n_0} \leqslant (x_n - x_{n_0}) + (x_{n_0} - x_m) =$$
$$= x_n - x_m \leqslant u + t_{n_0} = t'_n.$$

The assumption implies that $-t'_n \leqslant x_n - x_m \leqslant t'_n$ $(n \in N,\ m \geqslant n)$.

One can easily verify that E is an o-ideal, i. e. a normal convex directed subgroup in H. Then we can form $H/E = C(G)$. The coset of $C(G)$ containing a sequence $(x_n) \in H$ will be denoted by $(x_n)^*$. The group $C(G)$ can be made into a partially ordered group by defining the order relation between the cosets by the rule $(x_n)^* \leqslant (y_n)^*$ if and only if $(x_n') \leqslant (y_n')$ for some $(x_n') \in (x_n)^*$ and some $(y_n') \in (y_n)^*$. Then (see [2]) for each $(x_n') \in (x_n)^*$ there exists $(y_n') \in \in (y_n)^*$ such that $(x_n') \leqslant (y_n')$. By virtue of 5 $C(G)$ is a directed Abelian group which is called the Cantor extension of G.

The inequality $(x_n)^* \leqslant (y_n)^*$ is valid exactly if $(x_n - y_n)^* \leqslant E$, that is, if we can find a sequence $(u_n) \in E$ such that $(x_n - y_n) \leqslant (u_n)$. The sequence (u_n) belongs to E if and only if there is a sequence (t_n) such that $t_n \downarrow 0$ and $-t_n \leqslant u_n \leqslant t_n$ $(n \in N)$; thus we conclude that $(x_n)^* \leqslant (y_n)^*$ if and only if there is a sequence (t_n) with the properties $t_n \downarrow 0$ and $(x_n) \leqslant (y_n) + (t_n)$.

For $(x_n) \in H$ denote $X_n = (x_n, x_n, \ldots)^*$.

7. *If $t_n \downarrow 0$, then $T_n \downarrow E$.*

Proof. From $t_n \geqslant 0$ we obtain $T_n \geqslant E$ $(n \in N)$. Assume that $(x_n)^* \in C(G)$, $(x_n)^* \leqslant T_m$ $(m \in N)$. According to the definition of the partial order in $C(G)$ for each fixed $m \in N$ there is a sequence (t_n^m) such that $t_n^m \downarrow 0$ and $(x_n) \leqslant \leqslant (t_m, t_m, \ldots) + (t_n^m)$. Since $(x_n) \in H$, there exists a sequence (v_s) with the properties $v_s \downarrow 0$ an $x_s - x_n \leqslant v_s$ $(s \in N,\ n \geqslant s)$. Then $x_s \leqslant x_n + v_s \leqslant t_m + + t_n^m + v_s$. Hence $x_s - v_s - t_m \leqslant t_n^m$ $(n \in N,\ n \geqslant s)$ and so $x_s - v_s - t_m \leqslant 0$ The inequality $x_s - v_s \leqslant t_m$ $(m \in N)$ implies $x_s - v_s \leqslant 0$ $(s \in N)$. Hence $(x_s)^* \leqslant \leqslant (v_s)^* = E$.

Let $\varphi : G \to C(G)$ be a mapping defined by the rule

$$\varphi(x) = (x, x, \ldots)^*$$

for every $x \in G$. Let $(x_n) \in H$. Denote $(x_n)^* = X$.

8. *If $(x_n) \in H$, then $X_n \to X$.*

Proof. We have to prove that $X_n - X \to E$. For an arbitrary fixed n_0 N we have

$$X_{n_0} - X = (x_{n_0}, x_{n_0}, \ldots)^* - (x_n)^* = (x_{n_0} - x_1, x_{n_0} - x_2, \ldots, x_n$$

$$- x_{n_0-1}, 0, x_{n_0} - x_{n_0+1}, x_{n_0} - x_{n_0+2}, \ldots)^*$$

$$= (0, x_{n_0} - x_{n_0+1}, x_{n_0} - x_{n_0+2}, \ldots)^* = (x_{n_0} - x_m)^* \ (m \geqslant n_0).$$

Since $(x_n) \in H$, we can find $t_n \downarrow 0$ such that

$$-t_n \leqslant x_n - x_m \leqslant t_n \ (n \in N,\ m \geqslant n).$$

Let $n \in N$ be fixed. Then

$$-T_n \leqslant (x_n - x_m)^* - X_n - X \leqslant T_n.$$

By 7 we get $T_n \downarrow E$ and the proof is complete. Moreover, we have proved

9. *For each coset* $X \in C(G)$ *there exists a sequence in* $\varphi(G)$ *which o-converges to* X.

We identify G and $\varphi(G)$ in the following theorem:

Theorem. *The Cantor extension* $C(G)$ *of an Abelian directed group* G *is an Abelian directed group. The mapping* $\varphi: x \to (x, x, \ldots)^*$ *from* G *into* $C(G)$ *is an o-isomorphism which preserves infinite joins and intersections. Every fundamental sequence in* G *has an o-limit in* $C(G)$ *and every element from* $C(G)$ *is an o-limit of some sequence from* G.

Proof. It is readily seen that the mapping preserves the group operation. With respect to 8 and 9 it remains to prove only that φ preserves infinite intersections. The idea of this proof is the same as in Everett, [2], where it was used in the case of the lattice ordered groups. Assume that $a_\gamma (\gamma \in \Gamma)$ and that there exists $\bigwedge a_\gamma = a$ in G. We intend to show that there is $\bigwedge \varphi(a_\gamma)$ in $C(G)$ and $\varphi(a) = \bigwedge \varphi(a_\gamma)$, i. e., $(a, a, \ldots)^* = \bigwedge (a_\gamma, a_\gamma, \ldots)^*$ holds. From $a \leqslant a_\gamma (\gamma \in \Gamma)$ we obtain $(a, a, \ldots)^* \leqslant (a_\gamma, a_\gamma, \ldots)^* \; (\gamma \in \Gamma)$. Assume that $(x_n)^* \in C(G)$, $(x_n)^* \leqslant (a_\gamma, a_\gamma, \ldots)^* \; (\gamma \in \Gamma)$. Then for each fixed $\gamma \in \Gamma$ there is a sequence (t_n^γ) such that $t_n^\gamma \downarrow 0$ and $x_n \leqslant a_\gamma + t_n^\gamma \; (n \in N)$. Because of $(x_n) \in H$, there exists a sequence (t_m), $t_m \downarrow 0$ and $x_m - x_n \leqslant t_m \; (m \in N, \; n \geqslant m)$. Then $x_m \leqslant x_n + t_m \leqslant a_\gamma + t_n^\gamma + t_m$, $x_m - a_\gamma - t_m \leqslant t_n^\gamma$. Since m and γ are fixed, we get $x_m - a_\gamma - t_m \leqslant 0$, $x_m - t_m \leqslant a_\gamma$, $x_m - t_m \leqslant a$, $x_m \leqslant a + t_m$. Thus $(x_n)^* \leqslant (a, a, \ldots)^*$.

Let us recall the definition of the mixed product of partially ordered groups. This concept is a common generalization of the concepts of the complete direct product and the lexicographic product (see Fuchs, [3]).

Let Λ be a partially ordered set and $A_\lambda (\lambda \in \Lambda)$ groups with nontrivial partial order. Let us form the complete direct product $C^g = \Pi A_\lambda (\lambda \in \Lambda)$ of the groups A_λ. For $x, y \in C^g$ we denote

$$\sigma(x, y) = \{\lambda \in \Lambda : x(\lambda) \neq y(\lambda)\}$$

and by $\min \sigma(x, y)$ the set of all minimal elements in $\sigma(x, y)$. We shall write $\sigma(x)$ instead of $\sigma(x, 0)$. Let G be the set of all $x \in C^g$ such that $\sigma(x)$ satisfies the descending chain condition. Indeed, G is a subgroup of C. If we put $x > 0$ if and only if $x(\lambda) > 0$ for each $\lambda \in \min \sigma(x)$, then G is a partially ordered group which is called the mixed product of partially ordered groups $A_\lambda (\lambda \in \Lambda$ and denoted by $G = \Omega A_\lambda (\lambda \in \Lambda)$.

Observe that $G = \Omega A_\lambda (\lambda \in \Lambda)$ is a directed group if A_λ is a directed group for each $\lambda \in \Lambda$. In fact, if $x, y \in G$, then there exists $z \in G$ such that $z(\lambda) > x(\lambda), y(\lambda)$ for each $\lambda \in \sigma(x) \cup \sigma(y)$ and $z(\lambda) = 0$ otherwise and there is fulfilled $z \geqslant x, y$.

If $\Omega A_\lambda (\lambda \in \Lambda)$ is a directed group, then A_λ need not be directed for each $\lambda \in \Lambda$.

Let Λ be a root system, i. e., a partially ordered set such that no pair of incomparable elements of Λ have a common lower bound and let $A_\lambda(\lambda \in \Lambda)$ be partially ordered groups. Now we state a necessary and sufficient condition for the mixed product of partially ordered groups to be a directed group provided the set Λ has the property mentioned above. The set of all minimal elements of Λ is denoted by Λ_0.

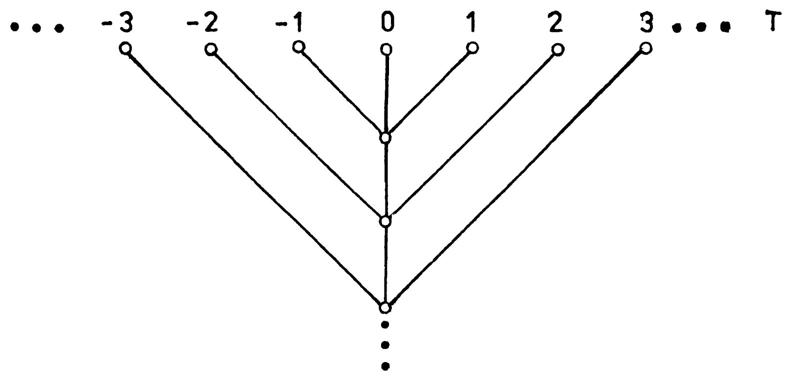
10. *Let Λ be a root system. Let A_λ be nontrivially ordered for each $\lambda \in \Lambda$. Then $G = \Omega A_\lambda(\lambda \in \Lambda)$ is a directed group if and only if A_λ is a directed group for each $\lambda \in \Lambda_0$.*

Proof. Let G be a directed group and $\Lambda_0 \neq \emptyset$. Pick out arbitrary $\lambda_0 \in \Lambda_0$ and $a, b \in A_{\lambda_0}$. We have to find an element $c \in A_{\lambda_0}$, $c \geqslant a, b$. Construct the elements $x, y \in G$ such that $x(\lambda_0) = a$, $y(\lambda_0) = b$, $x(\lambda) = y(\lambda) = 0$ for each $\lambda \in \Lambda$, $\lambda \neq \lambda_0$. If a and b are comparable, the assertion is obvious. Let $a \parallel b$ (i. e., a and b are incomparable). The assumption implies that there is $z \in G$, $z \geqslant x, y$. From $a \neq b$ it follows $z(\lambda_0) \neq a$ or $z(\lambda_0) \neq b$. If $z(\lambda_0) \neq a$, then $\lambda_0 \in \min \sigma(x, z)$, hence $z(\lambda_0) > a$. From $a \parallel b$ we get $z(\lambda_0) \neq b$. In a similar manner as above we obtain $z(\lambda_0) > b$. The proof is complete if we put $c = z(\lambda_0)$.

Conversely, let A_λ be a directed group for all $\lambda \in \Lambda_0$. If $x, y \in G$, denote $\Lambda_1 = \min \sigma(x)$, $\Lambda_2 = \min \sigma(y)$ and by $\Lambda_{1,2}$ we denote the set of all minimal elements of $\Lambda_1 \cup \Lambda_2$. Assume that $\lambda \in \Lambda_{1,2}$, $\lambda \notin \Lambda_0$ and pick $\mu_\lambda \in \Lambda \setminus \Lambda_{1,2}$ with $\mu_\lambda < \lambda$. From the fact that Λ is a root system we deduce that $\mu_{\lambda_1} \parallel \mu_{\lambda_2}$ whenever $\lambda_1, \lambda_2 \in \Lambda_{1,2}$, $\lambda_1, \lambda_2 \notin \Lambda_0$, $\lambda_1 \neq \lambda_2$. Consequently an element $z \in C^g$ such that $z(\mu_\lambda) > 0$ if $\lambda \in \Lambda_{1,2}$, $\lambda \notin \Lambda_0$, $z(\lambda) > x(\lambda), y(\lambda)$ if $\lambda \in \Lambda_{1,2} \cap \Lambda_0$ and $z(\lambda) = 0$ otherwise belongs to G and $z \geqslant x, y$ is valid.

If $\Lambda_0 = \emptyset$ and the set Λ is not a root system, then G fails to be a directed group in the general case.

Example. Let $A_\lambda = A(\lambda \in \Lambda)$ be an arbitrary partially ordered but not directed group and let Λ be a tree shown in the following figure:



There are elements $a, b \in A$ such that the set $U(a, b)$ in A is void. Elements $x, y \in G$ such that $x(\lambda) = a$, $y(\lambda) = b$ for each $\lambda \in T$ and $x(\lambda) = y(\lambda) = 0$ if $\lambda \in \Lambda \setminus T$ have no common upper bound in G.

In the following it will be assumed that $G = \Omega A_\lambda$ ($\lambda \in \Lambda$), where Λ is an arbitrary partially ordered set and A_λ a directed Abelian group for every $\lambda \in \Lambda$. Tnen G is again a directed Abelian group. In the sequel, we shall investigate the connection between $C(G)$ and $C(A_\lambda)$ ($\lambda \in \Lambda$). The set of all maximal elements of Λ is denoted by M.

11. *Let (t_n) be a sequence in G with $t_n \downarrow 0$. Then for each $\lambda \in \Lambda$ there exists $n_0(\lambda) \in N$ satisfying*

(i) $h(n_0(\lambda), t_n(\lambda)) \downarrow 0$,

(ii) $t_n(\mu) = 0$ *for each* $n \in N$, $n \geqslant n_0(\lambda)$ *and each* $\mu \in \Lambda$, $\mu < \lambda$.

Proof. Assume that $t_n \downarrow 0$. The assertion is obvious if $t_n = 0$ holds true for some $n \in N$. Let $t_n > 0$ for every $n \in N$. First let us prove that for each $\lambda \in \Lambda$ there exists $n_0(\lambda)$ such that $t_{n_0(\lambda)}(\mu) = 0$ whenever $\mu \in \Lambda$, $\mu < \lambda$. Suppose by way of contradiction that for some $\lambda \in \Lambda$ and every $n \in N$ there is $\mu(n) \in \Lambda$, $\mu(n) < \lambda$ such that $t_n(\mu(n)) \neq 0$. Then for each $n \in N$ there is $\mu_0(n) \in \Lambda$, $\mu_0(n) < \lambda$, $\mu_0(n) \in \min \sigma(t_n)$, hence $t_n(\mu_0(n)) > 0$. Choose an element $g \in G$ such that $g(\lambda) > 0$ and $g(\nu) = 0$ for every $\nu \in \Lambda$, $\nu \neq \lambda$. Consequently, for each $n \in N$ and $\mu_0(n) \in \min \sigma(t_n, g)$ we have $t_n(\mu_0(n)) > g(\mu_0(n)) = 0$. We infer $0 < g < t_n$ ($n \in N$), which is contrary to $\bigwedge t_n = 0$.

Further we show that $t_n(\mu) = 0$ for each $\mu \in \Lambda$, $\mu < \lambda$ and each $n \in N$, $n \geqslant n_0(\lambda)$. Assume by way of contradiction that for some $n_1 \in N$, $n_1 > n_0(\lambda)$ there exists $\mu(n_1) \in \Lambda$, $\mu(n_1) < \lambda$ such that $t_{n_1}(\mu(n_1)) \neq 0$. Then there exists $\mu_0(n_1) \in \Lambda$, $\mu_0(n_1) < \lambda$, $\mu_0(n_1) \in \min \sigma(t_{n_1})$ and so $t_{n_1}(\mu_0(n_1)) > 0$. Since $\mu_0(n_1) \in$ $\in \min \sigma(t_{n_1}, t_{n_0(\lambda)})$, we have $t_{n_1}(\mu_0(n_1)) > t_{n_0(\lambda)}(\mu_0(n_1)) = 0$. This is impossible because of $t_{n_1} \leqslant t_{n_0(\lambda)}$ and thus (ii) is valid.

Therefore, we have also proved (i) for each $\lambda \in \Lambda \setminus M$. Suppose that $\lambda \in M$. As we have already proved above there exists $n_0(\lambda)$ such that $t_n(\mu) = 0$ for each $n \in N$, $n \geqslant n_0(\lambda)$, each $\mu \in \Lambda$, $\mu < \lambda$. If $n_1 \geqslant n_2 \geqslant n_0(\lambda)$, then either $t_{n_1}(\lambda) = t_{n_2}(\lambda)$ or $\lambda \in \min \sigma(t_{n_1}, t_{n_2})$, whence $0 \leqslant t_{n_1}(\lambda) \leqslant t_{n_2}(\lambda)$. To complete the proof it suffices to show that $\bigwedge t_n(\lambda)$ $(n \geqslant n_0(\lambda)) = 0$. Assume that there exists $a_\lambda \in A_\lambda$ such that $a_\lambda \leqslant t_n(\lambda)$ ($n \in N$, $n \geqslant n_0(\lambda)$). If we choose an element $x \in G$ such that $x(\lambda) = a_\lambda$ and $x(\nu) = 0$ for each $\nu \in \Lambda$, $\nu \neq \lambda$, then $x \leqslant t_n$ ($n \geqslant n_0(\lambda)$). Then $x \leqslant t_n (n \in N)$ and so the hypothesis implies $x \leqslant 0$. Hence $a_\lambda \leqslant 0$, and (i) holds.

If a sequence (t_n) in G fulfils (i) and (ii), then in general the assertion $t_n \downarrow 0$ is false. The following two counterexamples show that this fails already to hold in the cases of G being the complete direct product of partially ordered groups ($G = \Pi^* A_\lambda$) or G is the lexicographic product of partially ordered groups ($G = {}^l\Pi A_\lambda$; the lexicographic order goes from the left). In the following examples let $A_n (n \in N)$ and A_ω be the additive groups of all integers with the natural order.

Example 1. Let $G = \Pi^* A_n (n \in N)$. Define a sequence (t_n) in G as follows: $t_n(m) = 1$ if $m = n$ and $t_n(m) = 0$ if $m \neq n$. Then the sequence (t_n) fails to be a descending one.

Example 2. Let $G = {}^l\Pi A_n (n \in N)$. Let us consider a sequence (t_n) in G formed by the rule $t_{2n-1}(m) = 1$, $t_{2n}(m) = 2$ if $m = n$ and $t_{2n\ 1}(m) = t_{2n}(m) \quad 0$ if $m \neq n$.

If (t_n) is a sequence in G such that $t_n(\lambda) \downarrow 0$ for each $\lambda \in \Lambda$ and if (t_n) does not fulfil (ii), then in general $t_n \downarrow 0$ need not hold.

Example 3. Let $\Lambda = N \cup \{\omega\}$ and $G = {}^l\Pi A_\lambda (\lambda \in \Lambda)$. Let us form a sequence (t_n) in G by putting $t_n(m) = 0$ if $m < n$, $t_n(m) = m - n + 1$ if $m \geqslant n$ and $t_n(\omega) = 0$. The condition (ii) is not fulfilled for $\lambda = \omega$. Let t be an arbitrary element from G satisfying $t \leqslant t_n$ $(n \in N)$. If we choose an element $g \in G$ with the components $g(\lambda) = t(\lambda)$ $(\lambda \in \Lambda,\ \lambda \neq \omega)$ and $g(\omega) = a$, where $a \in A_\omega$, $a > t(\omega)$, then $t < g \leqslant t_n (n \in N)$. Thus $\bigwedge t_n$ does not exist.

Remark. If (t_n) is a sequence in $G = \Pi^* A_\lambda (\lambda \in \Lambda)$ such that $t_n(\lambda) \downarrow 0$ $(\lambda \in \Lambda)$, then it is easy to verify that $t_n \downarrow 0$.

12. *Let (t_n) be a sequence in G. If for each $\lambda \in \Lambda$*

(i) $t_n(\lambda) \downarrow 0$,

(ii) *there exists $n_0(\lambda) \in N$ such that $t_n(\mu) = 0$ for each $n \in N$, $n \geqslant n_0(\lambda)$ and each $\mu \in \Lambda$, $\mu < \lambda$, then $t_n \downarrow 0$.*

Proof. It is clear that the sequence (t_n) is descending and $t_n \geqslant 0$. The statement is evident if $t_n = 0$ for some $n \in N$. Let $t_n > 0$ $(n \in N)$ and suppose that there exists $t \in G$ with property $t \leqslant t_n$ $(n \in N)$. Further, let $\lambda_0 \in \min \sigma(t)$. By (ii) there exists $n_0(\lambda_0) \in N$ such that $t_n(\mu) = 0$ for each $n \in N$, $n \geqslant n_0(\lambda_0)$ and each $\mu \in \Lambda$, $\mu < \lambda_0$. If $n \geqslant n_0(\lambda_0)$, then either $t(\lambda_0) = t_n(\lambda_0)$ or $\lambda_0 \in \min \sigma(t_n, t)$. Thus $t(\lambda_0) \leqslant t_n(\lambda_0)$ whenever $n \geqslant n_0(\lambda_0)$, hence by (i) $t(\lambda_0) < 0$ and so $t \leqslant 0$.

Let (x_n) be a sequence in G. We shall consider the following condition on (x_n):

(*) for every $\lambda \in \Lambda$ there exists $n_0(\lambda) \in N$ such that $x_n(\mu) = x_m(\mu)$ whenever $n, m \in N$, $n, m \geqslant n_0(\lambda)$, $\mu \in \Lambda$, $\mu < \lambda$.

Remark 1. If (x_n) fulfils (*), then for each $\mu \in \Lambda \setminus M$ there exists a uniquely determined element $x^\mu \in A_\mu$ such that for some $n_1(\mu) \in N$ we have $x_n(\mu) - x^\mu$ for each $n \geqslant n_1(\mu)$.

Remark 2. We see that (x_n) fulfils (*) if and only if $(g + x_n)$ fulfils (*), where g is an arbitrary element from G.

13. *If a sequence* (x_n) *satisfies* (*), *then the set* $S = \cup\, \sigma(x_n)$ *satisfies the descending chain condition.*

Proof. We have to show that an arbitrary descending chain

$$(1) \qquad \lambda_0 > \lambda_1 > \lambda_2 > \dots$$

in S is finite. The hypothesis implies that there exists $n_0(\lambda_0)$ such that $x_n(\lambda_p) = x^{\lambda_p}$ $(p = 1, 2, \dots;\ n \geqslant n_0(\lambda_0))$. Form the sets $A = \cup\, \sigma(x_n)$ $(n \leqslant n_0(\lambda_0))$ and $B = \cup\, \sigma(x_n)$ $(n > n_0(\lambda_0))$. Then $S = A \cup B$. For each λ_p $(p = 1, 2, \dots)$ from the chain (1) there is $n \leqslant n_0(\lambda_0)$ such that $x_n(\lambda_p) \neq 0$. Therefore, from λ_1 the chain (1) lies in A. The set A is a union of a finite number of sets satisfying the descending chain condition. Because of this fact, the set A fulfils this condition and thus the set S fulfils this condition as well.

Let us recall that by C^J we have denoted the complete direct product of the groups A_λ (without considering the partial orders on the groups A_λ).

Remark 3. If $g \in C^J$ such that $g(\mu) = 0$ for each $\mu \in \Lambda \setminus M$, then $g \in \Omega$ $A_\lambda(\lambda \in \Lambda)$. In fact, if $\nu \in \sigma(g)$, then $\nu \in M$ and thus $\nu \in \min \sigma(g)$.

Corollary 1. *Suppose that the sequence* (x_n) *in* G *fulfils* (*). *For each* $\mu \in \Lambda \setminus M$ *let* x^μ *be defined as in* Remark 1. *Then there exists* $x \in G$ *such that* $x(\mu) = x^\mu$ *for each* $\mu \in \Lambda \setminus M$.

Proof. First let us form the element $x' \in C^J$ such that $x'(\mu) = x^\mu$ for each $\mu \in \Lambda \setminus M$ and $x'(\mu) = 0$ if $\mu \in M$. Evidently, $\sigma(x') \subset S$ and consequently, in view of 13 the element x' belongs to G. By Remark 3, the element $g \in C^J$ such that $g(\mu) = 0$ if $\mu \in \Lambda \setminus M$ and $g(\mu) = x(\mu)$ if $\mu \in M$, belongs to G. Then $x' + g = x \in G$.

Since every constant sequence in G fulfils (*), we obtain the following assertion:

Corollary 2. *If* $z \in G$ *and* $z' \in C^J$ *such that* $z'(\mu) = z(\mu)$ *for each* $\mu \in \Lambda \setminus M$, *then* $z' \in G$.

Now we shall formulate a necessary and sufficient condition for a sequence (x_n) (expressed by means of components of the elements x_n) to be zero or fundamental. The set of all zero (fundamental) sequences in A_λ will be denoted by $E^\lambda(H^\lambda)$.

14. $(x_n) \in E$ *if and only if for each* $\lambda \in \Lambda$ *the following conditions hold true:*

(i) $(x_n(\lambda)) \in E^\lambda$,

(ii) *there exists* $n_0(\lambda) \in N$ *such that* $x_n(\mu) = 0$ *for each* $n \in N$, $n \geqslant n_0(\lambda)$ *and each* $\mu \in \Lambda$, $\mu < \lambda$.

Proof. Suppose that $(x_n) \in E$. Then there exists a sequence (t_n) in G such that $t_n \downarrow 0$ and $-t_n \leqslant x_n \leqslant t_n$ $(n \in N)$. If λ and $n_0(\lambda)$ are as in 11, we get $t_n(\mu) = 0$, hence $x_n(\mu) = 0$ for each $n \geqslant n_0(\lambda)$ and each $\mu < \lambda$. Thus (ii) is proved. With respect to 1 we have also proved the assertion (i) for each $\lambda \in \Lambda \setminus M$. Let $\lambda \in M$ and let $n_0(\lambda)$ be as above. If $\lambda \in \sigma(x_n, t_n)$ $(n \geqslant n_0(\lambda))$, then by (ii) $\lambda \in \min \sigma(x_n, t_n)$, hence $-t_n(\lambda) \leqslant x_n(\lambda) \leqslant t_n(\lambda)$ $(n \geqslant n_0(\lambda))$. By (i) in 11, $h(n_0(\lambda), t_n(\lambda)) \downarrow 0$ and according to 1 we obtain $(x_n(\lambda)) \in E^\lambda$.

Conversely, suppose that (i) and (ii) are fulfilled and let $\lambda \in \Lambda$. From (i) it follows that there is a sequence (t_n^λ) in A_λ such that $t_n^\lambda \downarrow 0$ and $-t_n^\lambda \leqslant x_n(\lambda) \leqslant$ $\leqslant t_n^\lambda (n \in N)$. If there is $k(\lambda) \in N$ such that $x_n(\lambda) = 0$ for each $n \in N$, $n \geqslant k(\lambda)$, then by $p(\lambda)$ we denote the least element of N with this property. Let $(t_n'^\lambda)$ be a sequence in A_λ defined as follows: if there is $p(\lambda)$, then we put $t_n'^\lambda = 0$ $(n \in N$, $n \geqslant p(\lambda))$ and $t_n'^\lambda = t_n^\lambda (n \in N,\ n < p(\lambda))$. If $p(\lambda)$ does not exist, then we put $t_n'^\lambda = t_n^\lambda$ $(n \in N)$. For each $n \in N$ let us form the element $t_n' \in C^J$ such that $t_n'(\lambda) = t_n'^\lambda(\lambda \in \Lambda)$. Because of (ii), the sequence (x_n) fulfils (*). Let S be as in 13. Since $\sigma(t_n') \subset S(n \in N)$, according to 13, (t_n') is a sequence in G. As for $-t_n'(\lambda) \leqslant x_n(\lambda) \leqslant t_n'^\lambda(\lambda \in \Lambda)$, we have $-t_n' \leqslant x_n \leqslant t_n'$. From (ii) and from the fact that $n_0(\lambda) \geqslant p(\mu)$ $(\mu < \lambda)$ we infer that $t_n'(\mu) = 0$ $(\mu < \lambda,\ n \geqslant n_0(\lambda))$. Further, we see that $t_n'(\lambda) \downarrow 0$ $(\lambda \in \Lambda)$. Then by 12, $t_n' \downarrow 0$ and the proof is complete.

15. *If a sequence* (x_n) *satisfies* (*), *then the set* $S' = \cup\, \sigma(x_n - x_m)$ $(n \in N$ $m \geqslant n)$ *fulfils the descending chain condition.*

Proof. We have to prove that an arbitrary chain in S' of the form (1) is finite. If n is a fixed positive integer and $m \geqslant n$, then in view of Remark 2 after 12, the sequence $(x_n - x_m)$ has the property (*). Hence by 13 the set $A_n = \cup\, \sigma(x_n - x_m)$ $(m \geqslant n)$ fulfils the descending chain condition. Let $n_0(\lambda_0)$ be as in 13. Denote $A = \cup\, A_n(n \leqslant n_0(\lambda_0))$ and $B = \cup\, A_n(n > n_0(\lambda_0))$. Then $S' = A \cup B$. For λp $(p = 1, 2, \ldots)$ from the chain (1) we get $x_n(\lambda_p) = x_m(\lambda_p)$ $= x^{\lambda_p}$, i. e., $x_n(\lambda_p) - x_m(\lambda_p) = 0$ $(n \geqslant n_0(\lambda_0),\ m \geqslant n)$. Thus λ_p $(p = 1, 2, \ldots)$ belongs to A. The set A fulfils the descending chain condition and so the chain (1) is finite.

16. $(x_n) \in H$ *if and only if for each* $\lambda \in \Lambda$ *the following conditions hold true:*

(i) $(x_n(\lambda)) \in H^\lambda$,

(ii) *there exists* $n_0(\lambda) \in N$ *such that* $x_n(\mu) = x_m(\mu)$ *for each* $\mu \in \Lambda$, $\mu < \lambda$, $n \geqslant n_0(\lambda)$, $m \geqslant n$.

Proof. If $(x_n) \in H$, there exists a sequence (t_n) such that $t_n \downarrow 0$ and $-t_n \leqslant$ $\leqslant x_n - x_m \leqslant t_n$ $(n \in N,\ m \geqslant n)$. If $n_0(\lambda)$ is as in 11, then $(x_n - x_m)(\mu) = 0$, that is $x_n(\mu) = x_m(\mu)$ $(n \geqslant n_0(\lambda),\ m \geqslant n$ and $\mu \in \Lambda,\ \mu < \lambda)$ and (ii) is proved. We have also shown that (i) holds true for each $\lambda \in \Lambda \setminus M$. Now let $\lambda \in M$.

By (ii) either $(x_n - x_m)(\lambda) = 0$ or $\lambda \in \min \sigma(x_n - x_m)$ $(n \geqslant n_0(\lambda),\ m \geqslant n)$. Then either $(x_n - x_m)(\lambda) = t_n(\lambda)$ or $\lambda \in \min \sigma(x_n - x_m, t_n)$. Hence $-t_n(\lambda) \leqslant (x_n - x_m)(\lambda) = x_n(\lambda) - x_m(\lambda) \leqslant t_n(\lambda)$. From 11 we get $h(n_0, t_n(\lambda)) \downarrow 0$. Then in view of 6 we obtain $(x_n(\lambda)) \in H^\lambda$.

Conversely, assume that (i) and (ii) are fulfilled and further that λ is an arbitrary element from Λ. With respect to (i) there is a sequence (t_n^λ) in A_λ such that $t_n^\lambda \downarrow 0$ and $-t_n^\lambda \leqslant x_n(\lambda) - x_m(\lambda) \leqslant t_n^\lambda$ $(n \in N,\ m \geqslant n)$. From (ii) it follows that $x_n(\mu) - x_m(\mu) = 0$ $(n \geqslant n_0(\lambda),\ m \geqslant n,\ \mu \in \Lambda,\ \mu < \lambda)$. If there exists $k(\lambda) \in N$ such that $x_n(\lambda) - x_m(\lambda) = 0 (n \geqslant k(\lambda),\ m \geqslant n)$ denote by $p(\lambda)$ the least positive integer with this property. Let us form sequences $(t_n'^\lambda)$ and (t_n') in the same way as in the proof of 14. From (ii) it follows that (x_n) satisfies (*). Since $\sigma(t_n') \subset S'(n \in N)$, therefore, by 15, (t_n') is a sequence in G. The proof can be finished in a similar way as in 14.

Theorem. *$C(G) \simeq \Omega B_\lambda(\lambda \in \Lambda)$, where $B_\lambda = A_\lambda$ if $\lambda \in \Lambda \setminus M$ and $B_\lambda = C(A_\lambda)$ if $\lambda \in M$.*

Proof. Let $(x_n) \in H$ and let $x \in G$ be as in the Corollary 1 of the assertion 13. We denote by b an element from the complete direct product of the groups $B_\lambda(\lambda \in \Lambda)$ such that $b(\lambda) = x(\lambda)$ if $\lambda \in \Lambda \setminus M$ and $b(\lambda) = (x_n(\lambda))^*$ if $\lambda \in M$. From 16 it follows $(x_n(\lambda)) \in H^\lambda$ and so $(x_n(\lambda))^* \in C(A_\lambda)$ $(\lambda \in \Lambda)$. Therefore, if we apply Corollary 2 to the complete direct product of groups $B_\lambda(\lambda \in \Lambda)$ and to $B = \Omega B_\lambda(\lambda \in \Lambda)$, we get $b \in B$.

Let $\varphi : C(G) \to B$ be a mapping defined by the rule

$$\varphi((x_n)^*) = b.$$

Let $(x_n), (y_n) \in H$, $\varphi((x_n)^*) = b_1$, $\varphi((y_n)^*) = b_2$. Assume that $(x_n)^* = (y_n)^*$. If $\lambda \in \Lambda \setminus M$ and $\lambda_0 \in \Lambda$, $\lambda_0 > \lambda$, by 16 there is $n_0(\lambda_0)$ such that $x_n(\lambda) = b_1(\lambda)$, $y_n(\lambda) = b_2(\lambda)$ $(n \geqslant n_0(\lambda_0))$. Since $(x_n - y_n) \in E$, by using 14 we get $x_n(\lambda) - y_n(\lambda) = b_1(\lambda) - b_2(\lambda) = 0 (n \geqslant n_0(\lambda_0))$, that is $b_1(\lambda) = b_2(\lambda)$. If $\lambda \in M$, again by 14 we obtain $((x_n - y_n)(\lambda)) \doteq (x_n(\lambda) - y_n(\lambda)) \in E^\lambda$, i. e., $(x_n(\lambda))^* = (y_n(\lambda))^*$, that is again $b_1(\lambda) = b_2(\lambda)$. We infer that $b_1 = b_2$. Conversely, if $b_1 = b_2$, then by 14 we obtain $(x_n)^* = (y_n)^*$. We conclude that the mapping φ is correctly defined and one-to-one.

It can be verified that φ is a mapping from $C(G)$ onto B. In fact, if $b \in B$, then $b(\lambda) \in A_\lambda$ for $\lambda \in \Lambda \setminus M$ and $b(\lambda) = (x_n^\lambda)^* \in C(A_\lambda)$ for $\lambda \in M$, where $(x_n^\lambda) \in H^\lambda$. For each $n \in N$ let us form an element $x_n \in C^\gamma$ such that $x_n(\lambda) = b(\lambda)$ if $\lambda \in \Lambda \setminus M$ and $x_n(\lambda) = x_n^\lambda$ if $\lambda \in M$. Corollary 2 implies that (x_n) is a sequence in G and by 16, $(x_n) \in H$. We conclude $(x_n)^* \in C(G)$ is the origin of b under the mapping φ.

We can easily verify that φ preserves the group operation and the partial order relation.

REFERENCES

[1] ČERNÁK, Š.: The Cantor extension of a lexicographic product of l-groups. Mat. Čas. 23, 1973, 97—102.
[2] EVERETT, C. J.: Sequence completion of lattice modules. Duke Math. J. 11, 1944, 109—119.
[3] ФУХС, Л.: Частично упорядоченные алгебраические системы. Москва 1965.

Received November 21, 1974

Katedra matematiky Strojníckej fakulty
Vysokej školy technickej
Švermova 5
040 01 Košice