Vladimír Kučera

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Kybernetika, Vol. 44 (2008), No. 3, 328--335

Persistent URL: http://dml.cz/dmlcz/135854

KYBERNETIKA — VOLUME *44* (2008), NUMBER 3, PAGES 328 – 335

SIXTY YEARS OF CYBERNETICS
A Comparison of Approaches to Solving the H_2 Control Problem

Vladimír Kučera

The H_2 control problem consists of stabilizing a control system while minimizing the H_2 norm of its transfer function. Several solutions to this problem are available. For systems in *state space* form, an optimal regulator can be obtained by solving two algebraic Riccati equations. For systems described by *transfer functions*, either Wiener–Hopf optimization or projection results can be applied. The optimal regulator is then obtained using operations with proper stable rational matrices: inner-outer factorizations and stable projections.

The aim of this paper is to compare the two approaches. It is well understood that the inner-outer factorization is equivalent to solving an algebraic Riccati equation. However, why are the stable projections not needed in the state-space approach?

The difference between the two approaches derives from a different construction of doubly coprime, proper stable matrix fractions used to represent the plant. The transfer-function approach takes any *fixed* doubly coprime fractions, while the state-space approach parameterizes *all* such representations and those selected then obviate the need for stable projections.

Keywords: linear systems, feedback control, stability, norm minimization

AMS Subject Classification: 93C05, 93D15, 49N10

1. INTRODUCTION

The H_2 control problem consists of stabilizing a control system while minimizing the H_2 norm of its transfer function. Several solutions to this problem are available. For systems in *state space* form, and under the standard regularity assumptions, Doyle et al. [2] obtained an optimal regulator in observer form by solving two algebraic Riccati equations. In the absence of the standard regularity assumptions, the H_2 control problem for systems in state space form was studied by Stoorvogel [10], who established a condition for an H_2 optimal controller to exist. Chen and Saberi [1] showed when such a controller is unique. Saberi et al. [9] then parameterized all H_2 optimal controllers and identified the fixed modes of the optimal control system.

For systems described by *transfer functions*, Park and Bongiorno [8] employed Wiener–Hopf optimization to obtain an optimal regulator transfer function via spectral factorizations and stable projections of rational matrices. Kwakernaak [5] derived an alternative solution in which operations with polynomial matrices replace those with rational matrices. Under the standard assumptions, Meinsma [6] applied projection results rather than Wiener–Hopf optimization to obtain a solution using operations with proper stable rational matrices: inner-outer factorizations and stable projections. Kučera [4] relaxed the standard assumptions and derived a general transfer-function solution in the sense that no assumptions on the system are made other than those securing the existence of outer factors.

The aim of this paper is to compare the state-space and the transfer-function approaches. It is well understood that the inner-outer (or spectral) factorization is equivalent to solving an algebraic Riccati equation. However, why are the stable projections not needed in the state-space approach?

The answer is complicated by the fact that the above approaches are not equivalent. Due to different mathematical tools applied, the H_2 control problem is solved at different levels of generality under different assumptions. Therefore same assumptions (namely the standard regularity assumptions) and same mathematical tool (namely the projection approach) are adopted first. Then the interpretation of the state-space solution presented in [2] in terms of the transfer-function solution obtained in [4] provides the answer.

2. PRELIMINARIES

The set of all real-rational matrix functions F of the complex variable s that are strictly proper and analytic on the imaginary axis is denoted by RL_2. The symbol RH_2 will be used to denote the set of strictly proper rational matrices that are analytic in the closed *right-half* complex plane, while $\mathrm{RH}_2^\perp$ will denote the set of strictly proper rational matrices that are analytic in the closed *left-half* complex plane. Then RH_2 is a subspace of RL_2 and $\mathrm{RH}_2^\perp$ is the orthogonal complement of RH_2 in RL_2.

The *H_2 norm* of a function F from RL_2 is defined as

$$\|F\| := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace } F^{\mathrm{T}}(-j\omega) F(j\omega) \, \mathrm{d}\omega \right)^{\frac{1}{2}}$$

where F^{T} denotes the transpose of F. In the sequel, we shall use the shorthand notation

$$F^*(s) := F^{\mathrm{T}}(-s)$$

for any rational matrix F.

The symbol RH_∞ will be used to denote the set of proper rational matrices that are analytic in the closed right-half complex plane. A matrix $F \in \mathrm{RH}_\infty$ is said to be *inner* if $F^*F = I$. Left multiplication by an inner matrix preserves H_2 norms. A matrix $F \in \mathrm{RH}_\infty$ is said to be *outer* if $F(s)$ has full row rank for all s in the open right-half complex plane. An important result, see Vidyasagar [11], is that any RH_∞ matrix F of full rank can be factored as $F = F_i F_o$ where F_i is inner and F_o is outer. A matrix F is said to be *co-inner* if F^T is inner, and *co-outer* if F^T is outer.

A (linear, time-invariant, differential) system in state-space form is a quadruple of real matrices (A, B, C, D) such that

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where x is the state, u is the input, and y is the output. The transfer function of the system is $T(s) = C(sI - A)^{-1}B + D$, which is also denoted by

$$T := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

The pair (A, B) is said to be *stabilizable* if there exists a matrix L such that $A+BL$ has all eigenvalues with negative real parts and the pair (A, C) is said to be *detectable* if there exists a matrix K such that $A + KC$ has all eigenvalues with negative real parts.

3. PROBLEM FORMULATION

To fix ideas, the H_2 control problem in the "textbook" form [12] is considered. Given a state-space description of the system S, hereafter called the plant,

$$\dot{x} = Ax + B_1v + B_2u$$
$$z = C_1x + D_{11}v + D_{12}u$$
$$y = C_2x + D_{21}v + D_{22}u$$

find a system R, called the controller, that stabilizes S and minimizes the H_2 norm of the transfer function T from v to z in the standard control system configuration shown in Figure. In this figure, u is the control input, v is the external input, y is the measured output, and z is the controlled output. Stability means that the states of S and R go to zero from any initial state.

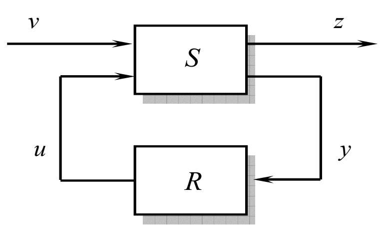

Fig. Standard control system.

It is assumed that the pair $(A,\ B_2)$ is stabilizable, the pair $(A,\ C_2)$ is detectable, the matrix

$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full column rank for all finite ω, the matrix

$$\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has full row rank for all finite ω, and

$$D_{11} = 0, \quad D_{12}^{\mathrm{T}} D_{12} = I, \quad D_{21} D_{21}^{\mathrm{T}} = I, \quad D_{22} = 0\,.$$

Under these conditions, a unique optimal controller exists.

4. TRANSFER FUNCTION SOLUTION

Firstly the transfer function of the plant, partitioned conformably with Figure,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} : = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{12} & 0 \end{array}\right],$$

is represented in terms of doubly (left and right) coprime matrix fractions over RH_∞

$$S = M^{-1} N = \bar{N} \bar{M}^{-1},$$

with the denominator matrices block triangular

$$M = \begin{bmatrix} I & M_{12} \\ 0 & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

and

$$\bar{N} = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} I & 0 \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}.$$

Then all controllers R_S that stabilize the plant S are parameterized as [3, 11, 12]

$$R_{\mathrm{S}}(W) := (X + W N_{22})^{-1} (Y + W M_{22}) = (\bar{Y} + \bar{M}_{22} W)(\bar{X} + \bar{N}_{22} W)^{-1}$$

where X, Y and $\bar{X}, \bar{Y}$ are RH_∞ matrices that satisfy the Bézout identity

$$\begin{bmatrix} X & -Y \\ -N_{22} & M_{22} \end{bmatrix} \begin{bmatrix} \bar{M}_{22} & \bar{Y} \\ \bar{N}_{22} & \bar{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and where W is a parameter that ranges over RH_∞.

Finally, two dual projection results will be used:

(1) Let F and G be matrices with equally many rows, with F in RH_2 and G in RH_∞. Suppose that G is inner and $G^* F$ is in $\mathrm{RH}_2^\perp$. Then for any RH_2 matrix H,

$$\| F - GH \|^2 = \| F \|^2 + \| H \|^2 .$$

(2) Let F and G be matrices with equally many columns, with F in RH_2 and G in RH_∞. Suppose that G is co-inner and $FG^*$is in $\mathrm{RH}_2^\perp$. Then for any RH_2 matrix H,

$$\| F - HG \|^2 = \| F \|^2 + \| H \|^2 .$$

The strategy to find an optimal controller is to use doubly coprime matrix fractional representations for S and express the transfer function T of the *stable* closed-loop system as an affine function of the parameter W. This expression is then manipulated so that the two projection results may be applied to minimize the norm of T.

One obtains

$$T = S_{11} + S_{12}R_S(I - S_{22}R_S)S_{21} = N_{11} - VN_{21}$$

where

$$V: = M_{12}(\bar{X} + \bar{N}_{22}W) - N_{12}(\bar{Y} + \bar{M}_{22}W)$$

embodies the dependence of T on W. Write $N_{21} = U\tilde{N}_{21}$, where $\tilde{N}_{21}$ is co-inner and U is co-outer. Then

$$T\tilde{N}_{21}^* = N_{11}\tilde{N}_{21}^* - VU\,.$$

Let P denote the projection of $N_{11}\tilde{N}_{21}^*$ on RH_2. Then

$$T = T_1 - V_1\tilde{N}_{21}$$

with $T_1 := N_{11} - PN_{21}$ in RH_2 and $T_1\tilde{N}_{21}^*$ in $\mathrm{RH}_2^{\perp}$. Therefore, applying the dual projection result, one has

$$\| T \|^2 = \| T_1\|^2 + \| V_1\|^2$$

where only V_1 depends on W.

Now

$$V_1 = VU - P = \bar{N}_{11K} - \bar{N}_{12}W_2$$

where

$$\bar{N}_{11K}: = (M_{12}\bar{X} - N_{12}\bar{Y})U - P\ , \quad W_2 := WU.$$

Write $\bar{N}_{12} = \tilde{N}_{12}\bar{U}$, where $\tilde{N}_{12}$ is inner and $\bar{U}$ is outer. Then

$$\tilde{N}_{12}^*\,V_1 = \tilde{N}_{12}^*\,\bar{N}_{11K} - \bar{U}\,W_2\,.$$

Denote $\bar{P}$ the projection of $\tilde{N}_{12}^*\bar{N}_{11K}$ on RH_2. Then

$$V_1 = \bar{T}_2 - \tilde{N}_{12}\bar{V}$$

where $\bar{T}_2 := \bar{N}_{11K} - \tilde{N}_{12}\bar{P}$ is in RH_2 and $\tilde{N}_{12}^*\bar{T}_2$ is in $\mathrm{RH}_2^{\perp}$. Then the primal projection result can be applied and

$$\| V_1\|^2 = \left\| \bar{T}_2\right\|^2 + \left\| \bar{V}\right\|^2$$

where only $\bar{V}$ depends on W.

To summarize,

$$\| T \|^2 = \| T_1\|^2 + \left\| \bar{T}_2\right\|^2 + \left\| \bar{V}\right\|^2$$

provided $\bar{V} = \bar{U}WU - \bar{P}$ is strictly proper.

The optimal controller R_0 corresponds to $\bar{V} = 0$, hence

$$R_0 = R_S(\bar{U}^{-1}\bar{P}U^{-1}).$$

Indeed, the optimal controller depends on the outer factor U, on the co-outer factor $\bar{U}$, and on the projection $\bar{P}$ (which in turn depends on P).

5. STATE SPACE SOLUTION

The state-space approach is based on expressing the doubly coprime matrix fractions of S and R in terms of stabilizing state feedback and output injection gains. Let K and L be matrices such that $A + B_2 L$ and $A + KC_2$ have all eigenvalues with negative real parts; then

$$M := \left[\begin{array}{c|cc} A + KC_2 & 0 & K \\ \hline C_1 & I & 0 \\ C_2 & 0 & I \end{array}\right], \quad N := \left[\begin{array}{c|cc} A + KC_2 & B_1 + KD_{21} & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

$$\bar{N} := \left[\begin{array}{c|cc} A + B_2 L & B_1 & B_2 \\ \hline C_1 + D_{12} L & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \quad \bar{M} := \left[\begin{array}{c|cc} A + B_2 L & B_1 & B_2 \\ \hline 0 & I & 0 \\ L & 0 & I \end{array}\right]$$

and

$$X := \left[\begin{array}{c|c} A + KC_2 & -B_2 \\ \hline L & I \end{array}\right], \quad Y := \left[\begin{array}{c|c} A + KC_2 & -K \\ \hline L & 0 \end{array}\right]$$

$$\bar{Y} := \left[\begin{array}{c|c} A + B_2 L & -K \\ \hline L & 0 \end{array}\right], \quad \bar{X} := \left[\begin{array}{c|c} A + B_2 L & -K \\ \hline C_2 & I \end{array}\right].$$

The strategy is then to take specific gains K and L that will make the optimizing choice of W obvious. In particular, take

$$K = -(B_1 D_{21}^{\mathrm{T}} + Q_K C_2^{\mathrm{T}})$$

where Q_K is the largest symmetric non-negative definite solution of the algebraic Riccati equation

$$AQ_K + Q_K A^{\mathrm{T}} + B_1 B_1^{\mathrm{T}} = (B_1 D_{21}^{\mathrm{T}} + Q_K C_2^{\mathrm{T}})\,(B_1 D_{21}^{\mathrm{T}} + Q_K C_2^{\mathrm{T}})^{\mathrm{T}}.$$

Then N_{21} is co-inner and $N_{11} N_{21}^*$ belongs to $\mathrm{RH}_2^\perp$ so that the dual projection result is readily applicable. Further take

$$L = -(B_2^{\mathrm{T}} Q_L + D_{12}^{\mathrm{T}} C_1)$$

where Q_L is the largest symmetric non-negative definite solution of the algebraic Riccati equation

$$A^{\mathrm{T}} Q_L + Q_L A + C_1^{\mathrm{T}} C_1 = (B_2^{\mathrm{T}} Q_L + D_{12}^{\mathrm{T}} C_1)^{\mathrm{T}} (B_2^{\mathrm{T}} Q_L + D_{12}^{\mathrm{T}} C_1).$$

Then $\bar{N}_{12}$ is inner and $\bar{N}_{12}^* \bar{N}_{11}$ belongs to $\mathrm{RH}_2^\perp$. Since $\bar{N}_{11K}$ can be obtained from $\bar{N}_{11}$ by replacing B_1 with K, then $\bar{N}_{12}^* \bar{N}_{11K}$ belongs to $\mathrm{RH}_2^\perp$ as well and the primal projection result is readily applicable.

It follows that

$$\|\, T\,\|^2 = \|\, N_{11}\|^2 + \left\|\, \bar{N}_{11K}\right\|^2 + \|\, W\|^2$$

for any W in RH_2. The minimum of the norm is achieved for $W = 0$, and

$$R_0 = R_S(0) = X^{-1} Y = \bar{Y} \bar{X}^{-1} := \left[\begin{array}{c|c} A + B_2 L + KC_2 & -K \\ \hline L & 0 \end{array}\right].$$

Therefore, the optimal controller is in observer form and depends on the stabilizing gains K and L.

6. COMPARISON

The difference between the two approaches derives from a different construction and use of doubly coprime fractional representations. The transfer-function approach takes any *fixed* doubly coprime fractions of S, while the state-space approach parameterizes *all* such fractions in terms of K and L. This difference shows in full when the transfer function T is manipulated so that the projection results may be applied. While the transfer-function approach simply extracts the inner factor from $\bar{N}_{12}$ and the co-inner factor from N_{21}, the state-space approach shapes the two doubly coprime fractions so as to *make* them inner/co-inner by appropriately selecting K and L. This is achieved by solving two algebraic Riccati equations.

Now it is seen why no stable projection is needed in the state-space approach. The process of shaping $\bar{N}_{12}$ and N_{21} results in *trivial* outer and co-outer factors, $\bar{U} = I$ and $U = I$. Consequently, the inner $\bar{N}_{12}$ and the co-inner N_{21} cancel *all* the stable dynamics when forming $N_{11}N_{21}^*$ and $\bar{N}_{12}^*\bar{N}_{11K}$. That is why $P = 0$ and $\bar{P} = 0$.

7. CONCLUSION

The state-space model implies that the state vector x of the plant is available for feedback and output injection. Following Nett et al. [7], all doubly coprime fractional representations of the plant can be parameterized in terms of stabilizing state feedback gain K and stabilizing output injection gain L. The norm minimization procedure then makes use of the design parameters K and L so as to select the inner-outer factors that obviate the need for stable projections.

It is further noted that the design parameters K and L in the doubly coprime fractional representation of the plant directly define the optimal controller R_0. Consequently, the doubly coprime fractions need *not* be explicitly calculated.

This advantage is not available in the transfer-function approach, where one has no clue as to which doubly coprime fractional representation to take. Having no control over the shape of the resulting inner-outer factors, one has to apply proper stable projections.

In addition to the conceptual advantages, the state-space approach is also superior in computational terms. The critical part of the transfer-function algorithm is the final substitution of the optimal W into K_S to obtain K_0. This operation generically results in common factors that must be cancelled to obtain K_0 in reduced form. Another difficulty is related to degree control. When operations with proper stable rational matrices are implemented using polynomial matrix operations, polynomials may result whose leading coefficients are small and care must be taken when setting them to zero.

In general, the computational complexity of the state space synthesis depends largely on the size of the state vector x whereas the transfer-function algorithm depends critically on the number of the control inputs u and the measurement outputs y. That is why the latter algorithm is most useful in the single-input single-output case.

ACKNOWLEDGEMENT

Supported by the Ministry of Education, Youth and Sports of the Czech Republic, Research Program MSM 6840770038. A preliminary version of this paper was presented at the IEEE Mediterranean Conference on Control and Automation, Ancona 2006.

REFERENCES

[1] B. M. Chen and A. Saberi: Necessary and sufficient conditions under which an H_2 optimal control problem has a unique solution. Internat. J. Control *58* (1993), 337–348.

[2] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis: State space solutions to standard H_2 and H_∞ control problems. IEEE Automat. Control *34* (1989), 831–847.

[3] V. Kučera: Discrete Linear Control: The Polynomial Equation Approach. Wiley, Chichester 1979, pp. 115–118.

[4] V. Kučera: The H_2 control problem: a general transfer-function solution. Internat. J. Control *80* (2007), 800–815.

[5] H. Kwakernaak: H_2 optimization – Theory and applications to robust control design. In: Proc. 3rd IFAC Symposium on Robust Control Design, Prague 2000, pp. 437–448.

[6] G. Meinsma: On the standard H_2 problem. In: Proc. 3rd IFAC Symposium on Robust Control Design, Prague 2000, pp. 681–686.

[7] C. N. Nett, C. A. Jacobson, and N. J. Balas: A connection between state-space and doubly coprime fractional representations. IEEE Automat. Control *29* (1984), 831–832.

[8] K. Park and J. J. Bongiorno: A general theory for the Wiener–Hopf design of multivariable control systems. IEEE Automat. Control *34* (1989), 619–626.

[9] A. Saberi, P. Sannuti, and A. A. Stoorvogel: H_2 optimal controllers with measurement feedback for continuous-time systems – Flexibility in closed-loop pole placement. Automatica *32* (1996), 1201–1209.

[10] A. A. Stoorvogel: The singular H_2 control problem. Automatica *28* (1992), 627–631.

[11] M. Vidyasagar: Control System Synthesis: A Factorization Approach. MIT Press, Cambridge, Mass. 1985, pp. 108–116.

[12] K. Zhou: Essentials of Robust Control. Prentice Hall, Upper Saddle River 1998, pp. 261–265.

Vladimír Kučera, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Praha 6, and Institute of Information Theory and Information, Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.

e-mail: kucera@fel.cvut.cz