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NONLINEAR STATE PREDICTION BY SEPARATION APPROACH FOR CONTINUOUS–DISCRETE STOCHASTIC SYSTEMS

Jaroslav Švácha and Miroslav Šimandl

The paper deals with a filter design for nonlinear continuous stochastic systems with discrete-time measurements. The general recursive solution is given by the Fokker–Planck equation (FPE) and by the Bayesian rule. The stress is laid on the computation of the predictive conditional probability density function from the FPE. The solution of the FPE and its integration into the estimation algorithm is the cornerstone for the whole recursive computation. A new usable numerical scheme for the FPE is designed. In the scheme, the separation technique based on the upwind volume method and the finite difference method for hyperbolic and parabolic part of the FPE is used. It is supposed that separation of the FPE and choice of a suitable numerical method for each part can achieve better estimation quality comparing to application of a single numerical method to the unseparated FPE. The approach is illustrated in some numerical examples.

Keywords: stochastic systems, state estimation, nonlinear filters, Fokker–Planck equation, numerical solutions, finite volume method, finite difference method

AMS Subject Classification: 93E11

1. INTRODUCTION

The problem of state estimation of nonlinear continuous stochastic systems with discrete-time measurements is of special interest when dealing with real continuous processes and the digital devices used for processing measurements. The aim of the state estimation problem is to determine the filtering probability density function (pdf) at the measurement time instants and the predictive pdf on the intervals between measurements. A general recursive solution of the problem is given by the Fokker–Planck equation (FPE) and by the Bayesian rule (BR) [1, 5]. The FPE [14] is a partial differential equation (PDE) that governs the evolution of the predictive pdf between the measurement time instants, and the BR represents a correction of the previous predictive pdf at the measurement times. Exact solution is available for a few special cases only [1, 5]. These cases include for example the linear Gaussian system where the solution of the BR and the FPE is represented by the Kalman–Bucy filter [5, 6]. In other cases it is necessary to apply some ap-

proximative methods. These methods can be divided into two groups with respect to validity of the acquired estimates. The first group of the methods provides results with validity within some neighborhood of a point estimate only and thus can be called local methods. The second group of the methods provides results valid almost in the whole state space and thus they can be called global methods.

The local methods often approximate the pdf representing state estimate by Gaussian pdf. This means that the complete description of the estimated pdf can be given by the first two moments, i. e. mean value and covariance matrix. The main disadvantage of these methods is local validity of the state estimate and consequently impossibility to ensure convergence of the state estimate. The analytical approach to the local estimation uses approximation of a nonlinear function by the first few terms of the Taylor expansion [5, 15].

On the other hand, global methods, based on the approximation of the conditional pdf produce results with global validity. The disadvantage of these methods lies in the growth of theoretical and computational demands. There are three main approaches to the solution providing global estimates: analytical approach based on system approximation and Gaussian sum approximation of the pdf [1, 16, 19], numerical approach using numerical solution of the FPE [23] and simulation approach taking advantage of the Monte Carlo (MC) approximation [4].

The solution of the FPE and its integration into estimation algorithm can be viewed as a cornerstone for the whole recursive computation. Extensive numerical simulations of the FPE have been performed using finite element methods (FEM's) [11] or the Monte Carlo simulation [17]. Monte Carlo simulation methods are advantageous in high dimensional cases as their computational demands increase with dimension linearly only, but on the other hand they do not generally provide high estimation quality. Further, the finite difference methods (FDM's) [13, 25] belong to standard numerical approaches to the solution of PDE's and thus can also be applied to the approximation of the FPE. Nevertheless, most of these approaches are focused on specific physical processes and do not correspond to a direct manipulation in estimation algorithm based on the FPE.

The goal of the paper is to present a new usable and alternative numerical solution of the FPE in state estimation problem based on the separation of the FPE into two parts. The aim is to solve the first hyperbolic part by upwind finite volume methods (FVM's) [9] and the second parabolic part by the standard FDM's [13]. It is supposed that separation of the FPE and choice of a suitable numerical method for each part should achieve better estimation quality comparing to the application of a single numerical method to the unseparated FPE.

The paper is organized as follows: The problem formulation and general solution of the considered estimation problem is presented in Section 2. Section 3 is focused on the new numerical solution of the FPE. Afterwards, in Section 4 the grid cells design is discussed. The results of the paper are illustrated in some numerical examples in Section 5.

2. PROBLEM FORMULATION

Consider the problem of state estimation where the state $\boldsymbol{x}(t)$ evolves in continuous time according to the Itô stochastic differential equation (SDE)

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), t)\,\mathrm{d}t + \boldsymbol{G}(t)\,\mathrm{d}\boldsymbol{w}(t) \tag{1}$$

and the measurement $\boldsymbol{z}_k$ is given as

$$\boldsymbol{z}_k = \boldsymbol{h}(\boldsymbol{x}_k, t_k) + \boldsymbol{v}_k, \tag{2}$$

where t is time, t_k are time instants for $k = 0, 1, 2, \ldots$, $\boldsymbol{x}(t)$ is state vector with $\dim \boldsymbol{x}(t) = n$ (the short notation $\boldsymbol{x}_k = \boldsymbol{x}(t_k)$ is used), $\boldsymbol{z}_k$ represents measurement vector at time t_k with $\dim \boldsymbol{z}_k = m$, $\boldsymbol{f}(\boldsymbol{x}(t), t)$ and $\boldsymbol{h}(\boldsymbol{x}_k, t_k)$ are known vector functions, and $\boldsymbol{G}(t)$ is a known $n \times n$ matrix. The process noise, $\boldsymbol{w}(t)$, is an $\mathbb{R}^n$-valued Brownian motion with $\mathsf{E}(\mathrm{d}\boldsymbol{w}, \mathrm{d}\boldsymbol{w}^T) = \boldsymbol{I}\,\mathrm{d}t$. The measurement noise $\boldsymbol{v}_k$ is white and Gaussian with $\dim(\boldsymbol{v}_k) = m$, $\mathsf{E}(\boldsymbol{v}_k) = \boldsymbol{0}$ and $\mathrm{cov}(\boldsymbol{v}_k) = \boldsymbol{R}_k$, thus $p(\boldsymbol{v}_k) = \mathcal{N}(\boldsymbol{v}_k : \boldsymbol{0}, \boldsymbol{R}_k)$. The noises $\boldsymbol{w}(t)$, $\boldsymbol{v}_k$ and the random variable $\boldsymbol{x}(t_0)$ are mutually independent.

The aim is to determine the conditional filtering pdf $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ and predictive pdf $p(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ for $t \in I_{k,k+1} \triangleq (t_k, t_{k+1}\rangle$ (i. e. for the measurement times $t_k < t \leq t_{k+1}$), where $\boldsymbol{z}^k \triangleq [\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_k]^T$.

The general recursive solution of the filtering problem can be given by the Bayesian approach. The filtering pdf $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ at the measurement times represents a correction (update) of the previous predictive pdf $p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1})$ and has the following form

$$p(\boldsymbol{x}_k|\boldsymbol{z}^k) = \frac{p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1})p(\boldsymbol{z}_k|\boldsymbol{x}_k)}{\int p(\boldsymbol{x}_k|\boldsymbol{z}^{k-1})p(\boldsymbol{z}_k|\boldsymbol{x}_k)\,\mathrm{d}\boldsymbol{x}_k} \tag{3}$$

where $p(\boldsymbol{x}_0|\boldsymbol{z}^{-1})$ is the prior pdf of the initial state $\boldsymbol{x}_0$.

The predictive pdf $p(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ for $t \in I_{k,k+1}$ is given by the FPE

$$\begin{aligned} \frac{\partial p(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial t} = & - \frac{\partial p(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}(t)} \boldsymbol{f}(\boldsymbol{x}(t), t) - p(\boldsymbol{x}(t)|\boldsymbol{z}^k)\,\mathrm{tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}(t), t)}{\partial \boldsymbol{x}(t)}\right) \\ & + \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{Q}(t)\frac{\partial^2 p(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}^2(t)}\right) \end{aligned} \tag{4}$$

with the initial condition $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$, where $\frac{\partial p(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}(t)}$ is the gradient of $p(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ with respect to $\boldsymbol{x}(t)$, $\frac{\partial \boldsymbol{f}(\boldsymbol{x}(t),t)}{\partial \boldsymbol{x}(t)}$ is the Jacobian of $\boldsymbol{f}(\boldsymbol{x}(t), t)$ with respect to $\boldsymbol{x}(t)$, tr denotes "trace", $\frac{\partial^2 p(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}^2(t)}$ is the Jacobian of the transpose of the gradient $\frac{\partial p(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}(t)}$ and $\boldsymbol{Q}(t) = \boldsymbol{G}(t)\boldsymbol{G}(t)^T$.

The key idea of most numerical approaches for generating the conditional pdf's of the state is to substitute a nonnegligible part of the state space by a grid of cells. The values of the pdf are computed at the grid points only and thus the solution of (3) and (4) is performed numerically over the grid instead of the continuous support. The nonnegligible support is a region in the state space where the actual state is

probable to lie and hence the values of the pdf are nonnegligible there. Increasing the number of cells and extending the significant region causes approaching of the approximate posterior pdf to the true conditional pdf of the state. Obviously as the number of grid cells increases, the computational demands of the method increase as well. It is also clear that the computational demands rise with increasing state dimension dramatically.
The basic numerical scheme can be summarized in the following recursive algorithm:

Algorithm.

Initialization: Set $k = 0$ and suppose t_0.

Define a grid G_0 in $\mathbb{R}^n$ by grid points $\boldsymbol{x}[i_1, \ldots, i_n]$ for the prior pdf $p(\boldsymbol{x}_0|\boldsymbol{z}^{-1})$:

$$G_0 = \{\boldsymbol{x}[i_1, \ldots, i_n]\}, \tag{5}$$

where $i_l = 1, 2, \ldots, N_l$ and $l = 1, 2, \ldots, n$.

For each axis l it holds that $\boldsymbol{x}[i_1, \ldots, i_l, \ldots, i_n] < \boldsymbol{x}[i_1, \ldots, i_l+1, \ldots, i_n]$. The grid G_0 is orthogonal and equidistant and defines $N = N_1 \times N_2 \ldots \times N_n$ cells. The sizes $\Delta x_1, \ldots, \Delta x_n$ of the cells are given by the distances of two arbitrary neighbouring points on each axis, e. g. $\Delta x_1 = \boldsymbol{x}[2, i_2, \ldots, i_n] - \boldsymbol{x}[1, i_2, \ldots, i_n]$. E.g. for one-dimensional case the N intervals (grid cells) are given by end points $x[i - 1/2]$ and $x[i + 1/2]$ where $x[i - 1/2] = x[i] - \frac{\Delta x}{2}$ and $x[i + 1/2] = x[i] + \frac{\Delta x}{2}$ (the index 1 is dropped).

Compute the value

$$P'_0[i_1, \ldots, i_n] = \hat{p}_{\boldsymbol{x}_0|\boldsymbol{z}^{-1}}\left(\boldsymbol{x}[i_1, \ldots, i_n]|\boldsymbol{z}^{-1}\right), \tag{6}$$

where $P'_0[i_1, \ldots, i_n]$ represents the approximate value of the pdf at $\boldsymbol{x}[i_1, \ldots, i_n]$.

Step 1: At time t_k compute the values of the approximate filtering pdf $\hat{p}(\boldsymbol{x}_k|\boldsymbol{z}^k)$ at the grid points using

$$\begin{aligned} P_k[i_1, \ldots, i_n] &= \hat{p}_{\boldsymbol{x}_k|\boldsymbol{z}^k}\left(\boldsymbol{x}[i_1, \ldots, i_n]|\boldsymbol{z}^k\right) \\ &= c_k'^{-1} P'_k\left[i_1, \ldots, i_n\right] p_{\boldsymbol{v}_k}\left(\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}[i_1, \ldots, i_n])\right), \end{aligned} \tag{7}$$

where

$$c'_k = \sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} \Delta\boldsymbol{x} P'_k[i_1, \ldots, i_n] \cdot p_{\boldsymbol{v}_k}\left(\boldsymbol{z}_k - \boldsymbol{h}_k(\boldsymbol{x}[i_1, \ldots, i_n])\right) \tag{8}$$

and $\Delta\boldsymbol{x} = \Delta x_1 \Delta x_2 \ldots \Delta x_n$.

Consider $t_k^0 = t_k$. The time instant t_k^0 is used as the initial time for prediction between t_k and t_{k+1}.

Step 2: Define a new suitable grid G_k in $\mathbb{R}^n$ similarly to the initialization step for the predictive pdf $p(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ for $t_k^j = t_k + j \cdot \Delta t$, where $t_k^j \in I_{k,k+1}$ and $j = 0, \ldots, M$ (i. e. at the time instants $t_k, t_k + \Delta t, t_k + 2\Delta t, \ldots, t_k + (M - 1)\Delta t, t_{k+1}$), Δt is the time interval for the prediction. Compute the values $P'_j[i_1, \ldots, i_n]$ for $j = 0$ as follows

$$P'_j[i_1, \ldots, i_n] = \hat{p}_{\boldsymbol{x}_k|\boldsymbol{z}^k}\left(\boldsymbol{x}[i_1, \ldots, i_n]|\boldsymbol{z}^k\right). \tag{9}$$

Step 3: Compute values $P'_j[i_1, \ldots, i_n]$ of the approximate predictive pdf $\hat{p}(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ for $j = 1, \ldots M$ using a suitable numerical method for the FPE (4).

Let $k \longleftarrow k+1$ and continue with *Step 1.*

The given algorithm provides a basic frame only. The solution of the FPE (Step 2 and Step 3) is the cornerstone for the whole recursive computation. The next section is focused on these two steps of the recursive estimation algorithm within a new usable solution of the FPE.

3. NEW NUMERICAL SOLUTION OF THE FOKKER–PLANCK EQUATION

Numerical approaches to the FPE are discussed in many publications: classical FDM method [25], finite elements methods [11, 17], Monte Carlo simulations [18] or other possible techniques [24]. Overview and comparison of suitable implicit difference methods for the FPE can be found in [12, 24], where the well-known Chang–Cooper method [3] is also introduced. Unfortunately, most of these approaches are focused directly on specific physical FPE forms following certain processes.

In this paper, the difference methods are preferred to be suitable within estimation algorithm for their simplicity. The basic idea of a new numerical solution of the FPE (4) is to see the FPE as a composition of a parabolic and a hyperbolic part, to consider them separately and subsequently to choose an efficient method for the solution of each part. The hyperbolic part

$$\frac{\partial p^*(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial t} = -\frac{\partial p^*(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}(t)} \boldsymbol{f}(\boldsymbol{x}(t), t) \tag{10}$$

with the initial condition pdf $\hat{p}(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ representing the estimate of the predictive pdf $p(\boldsymbol{x}(t)|\boldsymbol{z}^k)$, is solved by the upwind FVM's [9]. Upwind schemes based on FVM's represent a powerful class of numerical methods for the hyperbolic PDE's. The parabolic part

$$\frac{\partial \hat{p}(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial t} = -\hat{p}(\boldsymbol{x}(t)|\boldsymbol{z}^k) \operatorname{tr}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{x}(t), t)}{\partial \boldsymbol{x}(t)}\right) + \frac{1}{2} \operatorname{tr}\left(\boldsymbol{Q}(t) \frac{\partial^2 \hat{p}(\boldsymbol{x}(t)|\boldsymbol{z}^k)}{\partial \boldsymbol{x}^2(t)}\right) \tag{11}$$

with the initial condition pdf $p^*(\boldsymbol{x}(t)|\boldsymbol{z}^k)$ representing the solution of the hyperbolic part (10) of the FPE (4), is solved by the implicit scheme FDM's [13].

Now, Steps 2 and 3 from the basic algorithm considered in Section 2 will be designed.

3.1. Separation approach for one-dimensional system

Step 2: Divide the nonnegligible support of the filtering pdf $p(x_k|z^k)$ into N intervals (grid cells) by defining $x[i]$. The value $P_k[i]$ approximates the average of the pdf

$p(x_k|z^k)$ value over the ith interval at time t_k and also represents an approximate value of the pdf at $x[i]$:

$$P_k[i] = \hat{p}_{x_k|z^k}\left(x[i]|z^k\right) \simeq \frac{1}{\Delta x}\int_{x[i-1/2]}^{x[i+1/2]} p(x_k|z^k)\,\mathrm{d}x_k. \tag{12}$$

Step 3: The values $P_k[i]$ (12) represent the initial condition for numerical solution of the FPE (4) for $t_k^j \in I_{k,k+1}$, where $j = 0, \ldots, M$.

An explicit algorithm for the hyperbolic part FPE (10) can be developed [9]:

$$P'_{j+1}[i]^* = P'_j[i] - \frac{\Delta t}{\Delta x}\left(F^+(\Delta P'_j)[i-1/2] + F^-(\Delta P'_j)[i+1/2]\right). \tag{13}$$

The values $P'_{j+1}[i]^*$, where asterisk refers to the hyperbolic part (10), are modified at each time step t_k^j by (13) through the endpoints of the intervals. The specific variant of the FVM depends on numerical approximation of $F^+(\Delta P'_j)[i-1/2]$ and $F^-(\Delta P'_j)[i+1/2]$, e. g.:

$$F^+(\Delta P'_j)[i-1/2] = \max\{0, f\left(x_j[i], t_k^j\right)\}\left(P'_j[i] - P'_j[i-1]\right) \tag{14}$$

$$F^-(\Delta P'_j)[i+1/2] = \min\{0, f\left(x_j[i], t_k^j\right)\}\left(P'_j[i+1] - P'_j[i]\right). \tag{15}$$

Then the scheme (13) – (15) is an upwind FVM with first-order accuracy. For the explicit scheme to be stable, the condition

$$\left| f(x_j[i], t_k^j)\frac{\Delta t}{\Delta x}\right| \leq 1 \tag{16}$$

has to be satisfied for $i = 1, \ldots, N$ and $t_k^j \in I_{k,k+1}$. A more usable approximation of (13) can be found in [9]. The scheme (13) – (15) represents only one of several FVM variants.

Figure 1 illustrates time and state discretization (grid cells) for numerical solution of the FPE based on the FMV's.

Finally, the classical FDM can be used for the parabolic part (11) of the FPE. The discrete implicit scheme represents a matrix equation of Nth order and is unconditionally stable

$$\frac{P'_{j+1}[i] - P'_{j+1}[i]^*}{\Delta t} = -\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\bigg|_{x=x(t);t=t_{j+1}} P'_{j+1}[i] + \frac{1}{2}Q(t_{j+1})\frac{P'_{j+1}[i+1] - 2P'_{j+1}[i] + P'_{j+1}[i-1]}{\Delta x^2}. \tag{17}$$

3.2. Separation approach for n-dimensional system

Given scheme for the 1-dimensional system, the approach can be extended to a higher dimension:

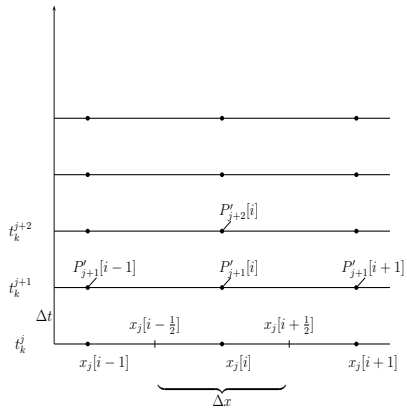


Fig. 1. Grid cells for the numerical solution of the FPE.

Step 2: Divide the nonnegligible support of the filtering pdf $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ into $N_1 \times N_2 \ldots \times N_n$ grid cells by defining $\boldsymbol{x}[i_1, \ldots, i_n]$. The value $P_k[i_1, \ldots, i_n]$ approximates the average of the pdf $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ over $[i_1, \ldots, i_n]$ cell at time t_k and also represents the approximate value of the pdf at $\boldsymbol{x}[i_1, \ldots, i_n]$:

$$\begin{aligned} P_k[i_1, \ldots, i_n] &= \hat{p}_{\boldsymbol{x}_k|\boldsymbol{z}^k}\left(\boldsymbol{x}[i_1, \ldots, i_n]|\boldsymbol{z}^k\right) \\ &\simeq \frac{1}{\Delta \boldsymbol{x}} \int_{x[i_1-1/2, i_2, \ldots, i_n]}^{x[i_1+1/2, i_2, \ldots, i_n]} \cdots \int_{x[i_1, i_2, \ldots, i_n-1/2]}^{x[i_1, i_2, \ldots, i_n+1/2]} p(\boldsymbol{x}_k|\boldsymbol{z}^k)\, \mathrm{d}\boldsymbol{x}_k. \end{aligned} \tag{18}$$

Step 3: The values $P_k[i_1, \ldots, i_n]$ (18) represent the initial condition for numerical solution of the FPE (4) for $t_k^j \in I_{k,k+1}$.

Figure 2 illustrates time discretization and state discretization (grid cells) for numerical solution of the FPE ($n = 2$).

An explicit algorithm for the hyperbolic part FPE (10) can be developed [9]:

$$\begin{aligned} &P'_{j+1}[i_1, \ldots, i_n]^* = P'_j[i_1, \ldots, i_n] \\ &- \sum_{l=1}^{n} \frac{\Delta t}{\Delta x_l} (F_l^+(\Delta P'_j)[i_1, \ldots, i_l - 1/2, \ldots, i_n] + F_l^-(\Delta P'_j)[i_1, \ldots, i_l + 1/2, \ldots, i_n]), \end{aligned} \tag{19}$$

where

$$\begin{aligned} &F_l^+(\Delta P'_j)[i_1, \ldots, i_l - 1/2, \ldots, i_n] \\ &= \max\{0, f_1\left(x_j[i_1, \ldots, i_l, \ldots, i_n], t_k^j\right)\} \left(P'_j[i_1, \ldots, i_n] - P'_j[i_1, \ldots, i_l - 1, \ldots, i_n]\right) \end{aligned} \tag{20}$$

$$\begin{aligned} &F_l^-(\Delta P'_j)[i_1, \ldots, i_l + 1/2, \ldots, i_n] \\ &= \min\{0, f_1\left(x_j[i_1, \ldots, i_l, \ldots, i_n], t_k^j\right)\} \left(P'_j[i_1, \ldots, i_l + 1, \ldots, i_n] - P'_j[i_1, \ldots, i_n]\right). \end{aligned} \tag{21}$$

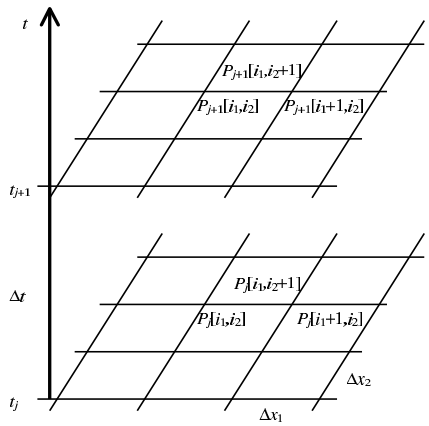


Fig. 2. Grid cells ($n = 2$) for numerical solution of the FPE.

The scheme (19) – (21) is an upwind FVM with first-order accuracy. For the explicit scheme to be stable, the condition

$$\sum_{l=1}^{n} |f_l(\boldsymbol{x}_j[i_1,\ldots,i_n], t_k^j)| \frac{\Delta t}{\Delta x_l} \leq 1 \tag{22}$$

has to be satisfied for $i_1 = 1,\ldots,N_1$, $i_2 = 1,\ldots,N_2 \ldots i_n = 1,\ldots,N_n$ and $t_k^j \in I_{k,k+1}$.

The classical FDM is used for the parabolic part (11) of the FPE. The discrete implicit scheme represents a matrix equation of Nth order (where $N = N_1 \cdot N_2 \cdots N_n$) and is unconditionally stable

$$\begin{aligned} &\frac{P'_{j+1}[i_1,\ldots,i_n] - P'_{j+1}[i_1,\ldots,i_n]^*}{\Delta t} \\ &= -\sum_{l=1}^{n} \left.\frac{\partial f_l(\boldsymbol{x}(t),t)}{\partial x_l(t)}\right|_{\boldsymbol{x}=\boldsymbol{x}(t);t=t_{j+1}} P'_{j+1}[i_1,\ldots,i_n] + \frac{1}{2}\sum_{l=1}^{n} \boldsymbol{Q}_{l,l}(t_{j+1}) \\ &\cdot \frac{P'_{j+1}[i_1,\ldots,i_l+1,\ldots,i_n] - 2P'_{j+1}[i_1,\ldots,i_n] + P'_{j+1}[i_1,\ldots,i_l-1,\ldots,i_n]}{\Delta x_l^2}. \end{aligned} \tag{23}$$

From (19) and (23) it is noticeable that the computational complexity of the algorithm grows exponentially with increasing state dimension for a given accuracy. Also suitable design of grid cells and the condition of stability (22) are crucial for stable and time optimal computation at all time instants $t_k^j \in I_{k,k+1}$.

4. GRID CELLS DESIGN

The basic idea for an efficient grid cells design is similar to the point-mass (PM) approach [21, 22]. Utilizing the techniques proposed in the PM approach enables

to achieve the substantial reduction of numerical demands for the discrete problem formulation. Generally, it is necessary to cover a part (rectangle for $n = 2$) of the state space covering the nonnegligible part of the filtering pdf $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$ and all predictive pdf's $p(\boldsymbol{x}_j|\boldsymbol{z}^k)$ numerically computed in $t_k^j \in I_{k,k+1}$. Then the numerical solution of the FPE can run without any deformation on borders of the rectangle (border conditions). This easy trick is demonstrated in Figure 3. The large dashed rectangle covers individual small dashed rectangles for $t_k^j \in I_{k,k+1}$, $j = 0, 1, \ldots, M$. The question is how to determine this significant area for all predictive pdf's before the prediction step (before numerical computation of the FPE)? The suggestion is to set the rectangle from estimates of the first two moments of $\boldsymbol{x}_j$ given by the predictive pdf's $p(\boldsymbol{x}_j|\boldsymbol{z}^k)$, $j = 0, 1, \ldots, M$, i. e. the mean $\hat{\boldsymbol{x}}'_{j|k}$ and covariance matrix $\boldsymbol{P}'_{j|k}$. Two basic techniques are considered: (1) to find a difference stochastic state model relevant to the given SDE and apply the point mass approach [21], (2) to apply a conventional prediction approach – e. g. extended Kalman–Bucy filter (EKBF). The resulting rectangle is used for the computation of all predictive pdf's during the prediction step. Then the grid cells can be adapted after each filtering step before numerical computation of the FPE. Generally, the adaptation can proceed in the prediction step within one-step numerical solution of the FPE. This approach allows to decrease the nonnegligible part of the state space; on the other hand, it is very complex to realize this adaptation effectively. The last remaining questions in grid cells design are related to the size of the cells and the time step Δt for numerical scheme. These options are set by the designer and directly involve error/quality, computational demands and convergence of the used method (condition of stability (22)).

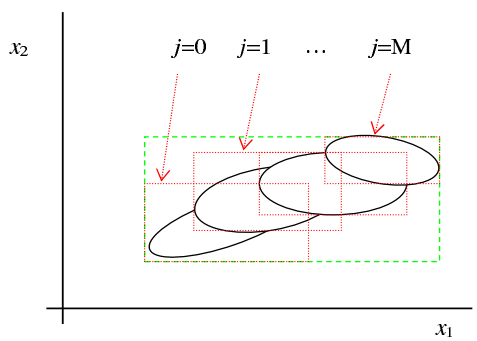


Fig. 3. Grid design ($n = 2$) for the FPE numerical solution.

Now, the algorithm for the grid cells design for n-dimensional systems, i. e. Step 2 in the basic numerical scheme, can be presented:

Step 1: Compute the estimates $\hat{\boldsymbol{x}}'_{j|k}$, $\boldsymbol{P}'_{j|k}$ for given Δt and $j = 1, \ldots, M$.

Note. The computation can be realized if a relevant discrete stochastic model is known for the SDE – e. g. a model acquired by the stochastic Euler scheme [2].

Other possibility is to apply a conventional prediction approach – e. g. the EKBF for the predictive estimates.

Step 2: Determine a nonnegligible part Ω_j of the state space for individual $p(\boldsymbol{x}_j|\boldsymbol{z}^k)$ pdf's for $j = 0, 1, \ldots, M$ based on $\hat{\boldsymbol{x}}'_{j|k}$, $\boldsymbol{P}'_{j|k}$ estimates. For $j = 0$ Ω_j is matched to the filtering nonnegligible part of $p(\boldsymbol{x}_k|\boldsymbol{z}^k)$.

Determine Ω_{k+1} covering all nonnegligible Ω_j for $j = 0, 1, \ldots, M$ – see Figure 3 where Ω_{k+1} represents the large dashed rectangle.

Step 3: Determine a suitable choice of $\Delta x_1, \ldots, \Delta x_{n_x}$ (or $N_1, \ldots, N_{n_x}$) and Δt corresponding to a required quality and the condition of stability (22) for the used explicit numerical scheme.

The described grid design procedure represents a basic frame only, e. g. in case of using a discrete implicit scheme for the hyperbolic part of the FPE, the condition of stability could be eliminated. Also, the considered grid cells design does not respect a possible rotation in eigenvector direction of the predictive covariance matrix to ensure a more efficient computation. This approach is a part of floating grid in the point mass method [21] for discrete case, but for the considered numerical methods the static grid of cells in prediction step $t_k^j \in I_{k,k+1}$ is required.

5. NUMERICAL ILLUSTRATION

5.1. Example 1

To show different performance of the FDM [13] and the new separation approach (SA), the following linear non-gaussian system is considered

$$\mathrm{d}x(t) = 0.4x(t)\,\mathrm{d}t + \mathrm{d}w(t), \quad z_k = 2x_k + v_k$$

with t_k ($t_0 = 0$s, $t_1 = 0.1$s, $t_2 = 0.2$s, …, 1s), the prior pdf $p(x_0|z^{-1}) = \mathcal{N}(x_0 : -2, 1)$ and $p(v_k) = 0.3\mathcal{N}(v_k : 0.5, 0.1) + 0.7\mathcal{N}(v_k : 2, 0.2)$. The new SA and the classical implicit FDM filter with grid parameters ($\Delta x = 0.1$, $x \in \langle -10, 10\rangle$, $N = 200$) and $\Delta t = 0.01$ are designed. The aim is to compare quality of these filters with the exact filtering pdf produced by the Gaussian sum filter (GSF) [19, 20].

The comparison is time evaluation of the filtering pdf's

$$J_{\mathrm{filt}} = \sum_{k=0}^{k=10} \int_{-10}^{10} (\hat{p}(x_k|z^k) - p(x_k|z^k))^2\,\mathrm{d}x$$

and predictive pdf's

$$J_{\mathrm{pred}} = \int_0^1 \int_{-10}^{10} (\hat{p}(x(t)|z^k) - p(x(t)|z^k))^2\,\mathrm{d}x\mathrm{d}t$$

for the FDM and the SA realized for 50 independent experiments – see Figure 4. It can be seen that the estimate quality of the SA is better than that of the FDM in majority of cases.

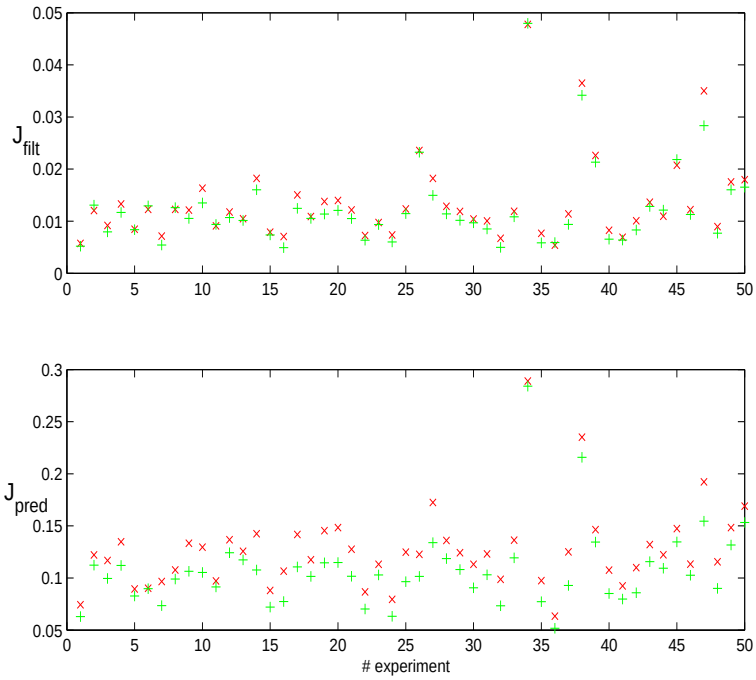


Fig. 4. Evolution of J_{filt} and J_{pred} for the FDM ($\times$ mark) and the SA ($+$ mark).

5.2. Example 2

To illustrate the estimate quality of the local EKBF and the global SA filter, the following nonlinear continuous stochastic process $x(t)$ observed at discrete time instants t_k ($t_0 = 0s$, $t_1 = 0.1s$, $t_2 = 0.2s, \ldots$)

$$\mathrm{d}x(t) = (x(t) - 0.4x^2(t))\,\mathrm{d}t + \mathrm{d}w(t), \quad z_k = x_k^2 + v_k$$

with the given prior pdf $p(x_0|z^{-1}) = p(x_0) = 0.5\mathcal{N}(x_0 : -2, 1) + 0.5\mathcal{N}(x_0 : 1, 1)$ and the pdf of the measurement noise $p(v_k) = \mathcal{N}(v_k : 0, 1)$ is considered.

The SA (with grid cell parameters $\Delta x = 0.05$, $x \in \langle -10, 10\rangle$, $N = 400$ and $\Delta t = 0.02$) and the EKBF are designed for this system to compare their filtering results. The prior pdf of the initial state $\boldsymbol{x}_0$ for the EKBF is derived from the equality of the mean and the variance as $p(x_0|z^{-1}) = p(x_0) = \mathcal{N}(x_0 : -0.5, 3.25)$. In Figure 5 the estimate development for the initial linearization point chosen to be negative is shown. It is obvious that the EKBF generates state estimates which diverge from the state. This fact is caused by the quadratic measurement function of the given system (25) and the given prior pdf. On the other hand, the SA provides correct estimates, which have global validity. Evolution of the filtering pdf's is shown in Figure 5.

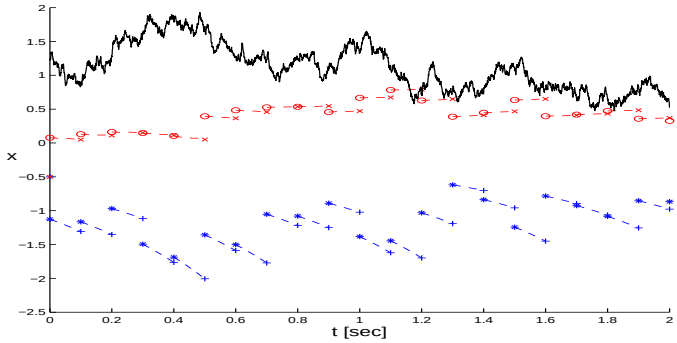
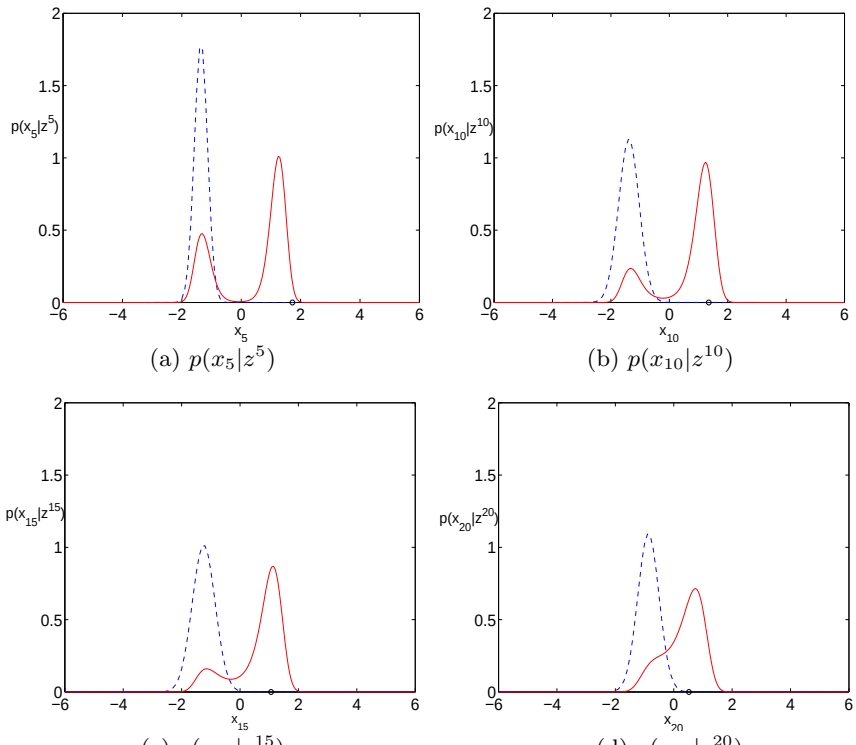


Fig. 5. Comparing the state estimates generated by the SA and the EKBF; the state $x(t)$ (solid line); SA – $\hat{x}_k$ (circle), $\hat{x}'_k$ ($\times$ mark), $\hat{x}'(t)$ (dashed line); EKBF – $\hat{x}_k$ (star), $\hat{x}'_k$ (plus), $\hat{x}'(t)$ (dashed line).



(a) $p(x_5|z^5)$ (b) $p(x_{10}|z^{10})$

(c) $p(x_{15}|z^{15})$ (d) $p(x_{20}|z^{20})$

Fig. 6. Evolution of the filtering pdf's $p(x_k|z^k)$ for $k = 5, 10, 15, 20$ generated from the SA (solid line) and the EKBF (dashed line), and the state x_k (circle).

6. CONCLUSION

A new separation approach for the numerical solution of the FPE was designed. The approach is based on the separation of the FPE into a hyperbolic and a parabolic part and the application of efficient numerical methods to each of them. The hyperbolic part is solved by an explicit FVM and the results are used in an implicit FDM for the parabolic part. The scheme has simple implementation and the extension to higher dimension is straightforward. In comparison with the classical implicit FDM, the SA approach can produce results with higher estimation quality.

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