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ON THE TANGENTIAL VELOCITY ARISING IN A CRYSTALLINE APPROXIMATION OF EVOLVING PLANE CURVES

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In a crystalline algorithm, a tangential velocity is used implicitly. In this short note, it is specified for the case of evolving plane curves, and is characterized by using the intrinsic heat equation.

Keywords: tangential velocity, intrinsic heat equation, crystalline algorithm, admissible polygonal curve

AMS Subject Classification: 34A26, 34A34, 35K65, 53A04, 53C80, 53C44, 65L20, 65M12, 65N12

1. INTRODUCTION

It is well known that the tangential velocity of evolution of plane curves causes stability from a numerical point of view. In this short note, the tangential velocity in a crystalline algorithm is specified, and it is compared with the one in other schemes by using the intrinsic heat equation. Our goal is simple. In the crystalline algorithm, the tangential velocity equals the quantity which is described as a negative of partial derivatives of the normal velocity with respect to the arc-length and its division by the curvature.

Following Mikula and Ševčovič [12], our target curves are defined as follows. We consider an embedded, orientable and closed plane curve Γ which is parameterized by a smooth function $\boldsymbol{x}: \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ such that $\Gamma = \mathrm{Image}(\boldsymbol{x}) = \{\boldsymbol{x}(u);\ u \in \mathbb{R}/\mathbb{Z}\}$ and $|\partial_u \boldsymbol{x}| > 0$. Here and hereafter, we denote $\partial_\xi(\cdot) = \partial(\cdot)/\partial\xi$. The unit tangent vector can be defined as $\boldsymbol{T} = \partial_u\boldsymbol{x}/|\partial_u\boldsymbol{x}| = \partial_s\boldsymbol{x}$, where s is the arc-length parameter and $ds = |\partial_u\boldsymbol{x}|du$, and the unit inward normal vector is defined by $\boldsymbol{N} = \boldsymbol{T}^\perp$, where $(a,b)^\perp = (-b,a)$. The curvature in the direction $\boldsymbol{N}$ is denoted by k, and the Frenet's formulae are $\partial_s\boldsymbol{T} = k\boldsymbol{N}$ and $\partial_s\boldsymbol{N} = -k\boldsymbol{T}$. Let ν be the angle of $\boldsymbol{T}$, i.e., $\boldsymbol{T} = (\cos\nu, \sin\nu)$ and $\boldsymbol{N} = (-\sin\nu, \cos\nu)$. Our problem is as follows: For a given initial curve $\Gamma^0 = \mathrm{Image}(\boldsymbol{x}^0) = \Gamma$, find a family of curve $\{\Gamma^t\}_{t\geq 0}$, $\Gamma^t = \{\boldsymbol{x}(u,t);\ u \in \mathbb{R}/\mathbb{Z}\}$ which starts from $\boldsymbol{x}(u,0) = \boldsymbol{x}^0(u)$ for $u \in \mathbb{R}/\mathbb{Z}$ and evolves according to the normal velocity

$$v = \beta(k, \nu). \tag{1}$$

Information of β determines movement of curves, and in many physical phenomena β includes k and ν. In general, β may also be a function of $\boldsymbol{x}$ and L^t, etc., other than k and ν, where $L^t = \int_{\Gamma^t} ds = \int_0^1 |\partial_u \boldsymbol{x}|\, du$ is the total length of the curve Γ^t. The normal velocity v is the normal component of the following evolution equation of a solution $\boldsymbol{x}$:

$$\partial_t \boldsymbol{x} = \beta \boldsymbol{N} + \alpha \boldsymbol{T}, \quad \boldsymbol{x}(\cdot, 0) = \boldsymbol{x}^0(\cdot). \tag{2}$$

Our main result is that, in a crystalline algorithm, the tangential velocity α is used essentially and implicitly such as:

$$\alpha = -\frac{\partial_s \beta}{k}. \tag{3}$$

This equation follows from the equality $\partial_t \nu = 0$, where $\partial_t \nu$ is described as

$$\partial_t \nu = \partial_s \beta + \alpha k, \quad \nu(\cdot, 0) = \nu^0(\cdot).$$

See [12, (3.8)].

In the crystalline algorithm, the class of polygonal curves is restricted to an admissible class and this restriction corresponds to the equality $\partial_t \nu = 0$. The admissibility is a powerful concept which may handle singular anisotropic energy such as crystalline energy (see references in the next section). We note that in the three dimensional crystalline algorithm for mean curvature flow has not been successfully constructed in the sense that the class of polyhedron is unclear. However, convergence of crystalline algorithm for the three dimensional Gauss curvature flow has been proved [16, 17].

In the next section, we will construct evolution of polygonal curves under the assumption $\partial_t \nu = 0$. In Section 3, the tangential velocity (3) will be compared with the one in other known schemes. In the last Section 4, some remarks on numerical scheme will be mentioned.

2. CRYSTALLINE ALGORITHM

A crystalline algorithm describes the motion of polygonal curves in an admissible class, and the admissible polygonal curves are the polygonal curves with each tangential angles belonging to a prescribed set. Let the prescribed set be $\mathcal{T} = \{\eta_1 < \eta_2 < \cdots < \eta_J < \eta_1 + 2\pi\}$, where J is a natural number $J \geq 3$ and $\eta_{j+1} - \eta_j < \pi$ is satisfied for all $j = 1, 2, \ldots, J$ ($\eta_{J+1} = \eta_1$). On a smooth curve $\Gamma = \{\boldsymbol{x}(u);\ u \in \mathbb{R}/\mathbb{Z}\}$, we distribute N points $\boldsymbol{x}_i = \boldsymbol{x}(u_i)$ ($u_1 < u_2 < \cdots < u_N < u_1 + 1$), which satisfy two conditions: (1) the tangential angle $\nu_i = \nu(u_i)$ of $\boldsymbol{x}_i$ belongs to $\mathcal{T}$, i.e., $(\cos \nu_i, \sin \nu_i) = \partial_s \boldsymbol{x}(\nu_i)$ and $\nu_i \in \mathcal{T}$ hold for $i = 1, 2, \ldots, N$; (2) For $i = 1, 2, \ldots, N$, there exists j such that $\{\nu_i, \nu_{i+1}\} = \{\eta_j, \eta_{j+1}\}$ holds ($\nu_{N+1} = \nu_1$). Let l_i be the straight line passing through $\boldsymbol{x}_i$ in the direction $\boldsymbol{T}_i = (\cos \nu_i, \sin \nu_i)$, and let $\boldsymbol{y}_i$ be the intersection point of l_{i-1} and l_i for all i. Thus a closed polygonal curve $\mathcal{P} = \bigcup_{i=1}^N [\boldsymbol{y}_i, \boldsymbol{y}_{i+1}]$ is constructed ($\boldsymbol{y}_{N+1} = \boldsymbol{y}_1$). This $\mathcal{P}$ is called $\mathcal{T}$-admissible polygonal curve. The length of the ith edge $d_i = |\boldsymbol{y}_{i+1} - \boldsymbol{y}_i|$ is given by

$$d_i = (\cot \xi_i + \cot \xi_{i+1}) h_i - h_{i+1} \operatorname{cosec} \xi_{i+1} - h_{i-1} \operatorname{cosec} \xi_i \tag{4}$$

for $i = 1, 2, \ldots, N$, where $\xi_i = \nu_i - \nu_{i-1}$ and $h_i = \boldsymbol{x}_i.\boldsymbol{N}_i$ ($\nu_0 = \nu_N$, $\boldsymbol{N}_i = \boldsymbol{T}_i^{\perp}$). And the ith vertex $\boldsymbol{y}_i$ is given by

$$\boldsymbol{y}_i = h_i \boldsymbol{N}_i + \frac{h_{i-1} - h_i \cos \xi_i}{\sin \xi_i} \boldsymbol{T}_i \tag{5}$$

for $i = 1, 2, \ldots, N$.

Crystalline algorithm is formulated as the following problem: For a given initial $\mathcal{T}$-admissible N-sided polygonal curve $\mathcal{P}^0 = \bigcup_{i=1}^N [\boldsymbol{y}_i^0, \boldsymbol{y}_{i+1}^0] = \mathcal{P}$, find a family of $\mathcal{T}$-admissible polygonal curves $\{\mathcal{P}^t\}_{t \geq 0}$, $\mathcal{P}^t = \bigcup_{i=1}^N [\boldsymbol{y}_i(t), \boldsymbol{y}_{i+1}(t)]$ which starts from $\boldsymbol{y}_i(0) = \boldsymbol{y}_i^0$ for $i = 1, 2, \ldots, N$ and evolves according to the ith normal velocity

$$v_i = \beta(k_i, \nu_i), \quad i = 1, 2, \ldots, N, \tag{6}$$

where k_i is the ith discretized curvature, which is called crystalline curvature if $k_i = c_i/d_i$ for some geometric quantity c_i depending on $\mathcal{T}$. Since the problem is considered in the admissible class, the ith tangential angle ν_i does not depend on time t, i. e., $\partial_t \nu_i = 0$ holds for all i. Moreover, from (4), we have the evolution equations as a system of ordinary differential equations:

$$\dot{d}_i = (\cot \xi_i + \cot \xi_{i+1}) v_i - v_{i+1} \operatorname{cosec} \xi_{i+1} - v_{i-1} \operatorname{cosec} \xi_i \tag{7}$$

for $i = 1, 2, \ldots, N$, where $v_i = \dot{h}_i = \dot{\boldsymbol{x}}_i \cdot \boldsymbol{N}_i$. Here and hereafter, we denote $\dot{(\cdot)} = \mathrm{d}(\cdot)/\mathrm{d}t$. It is easy to check that (7) is equivalent to the evolution equations of vertices:

$$\dot{\boldsymbol{y}}_i = v_i \boldsymbol{N}_i + \frac{v_{i-1} - v_i \cos \xi_i}{\sin \xi_i} \boldsymbol{T}_i \tag{8}$$

for $i = 1, 2, \ldots, N$. Therefore, it is possible to interpret the crystalline approximation as follows. Let s_i be the arc-length defined as $s_i = s + \int_u^{u_i} |\partial_u \boldsymbol{x}| \, \mathrm{d}u$. By putting $\beta(s) = \beta(k(s), \nu(s))$ and $v_i = \beta(s_i)$, we have $v_{i-1} = \beta(s_i - a_i) = v_i - \partial_s \beta(s_i) a_i + o(a_i)$ ($a_i = s_i - s_{i-1}$). Hence it holds that

$$\frac{v_{i-1} - v_i \cos \xi_i}{\sin \xi_i} = \beta(s_i) \frac{1 - \cos \xi_i}{\sin \xi_i} - \frac{\partial_s \beta(s_i)}{\sin \xi_i} a_i + \frac{o(a_i)}{\sin \xi_i} \to -\frac{\partial_s \beta(s_i)}{k(s_i)}$$

as $s_{i-1} \to s_i$. Here we have used $\partial_s \nu = k$ and $\xi_i = \nu_i - \nu_{i-1} = \int_{s_{i-1}}^{s_i} k(s) \, \mathrm{d}s$. This corresponds to (3) at $s = s_i$.

This crystalline algorithm was introduced by J. E. Taylor [13, 14] and S. Angenent and M. E. Gurtin [1] at the end of 1980's. Since then crystalline curvature flow equation has been studied under various kinds of evolution law by several authors. We refer the reader, other than the above pioneer works, to the surveys by Taylor, Cahn and Handwerker [15] and the books by Gurtin [6] for a geometric and physical background. Besides this crystalline strategy, other strategies by subdifferential and level-set method have been extensively studied. See Giga [4] and references therein. Recently, the admissibility is extended as an essentially admissibility. See, e. g., Hontani, Giga, Giga and Deguchi [8] and Yazaki [19].

3. COMPARISON WITH OTHER SCHEMES

In this section, the tangential velocity (3) is compared with the one in other schemes by using the intrinsic heat equation for a solution $\boldsymbol{x}$:

$$\partial_t \boldsymbol{x} = \theta_1^{-1}\partial_s\left(\theta_2^{-1}\partial_s \boldsymbol{x}\right), \quad \boldsymbol{x}(\cdot,0) = \boldsymbol{x}^0(\cdot),$$

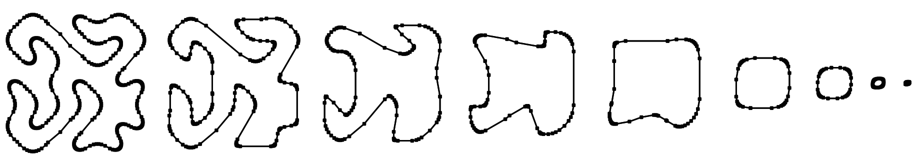
where $\theta_1\theta_2 = k/\beta$. Hence, the tangential velocity α is described as $\alpha = \theta_1^{-1}\partial_s\theta_2^{-1}$.

We will sketch the known results. Dziuk [3] studied a numerical scheme for the case $\beta(k) = k$ with $\alpha = 0$, i. e., $\theta_1 = \theta_2 = 1$. Kimura [10] proposed a uniform redistribution scheme in the case $\beta(k) = k$ by using a nontrivial tangential velocity α which satisfies $\theta_1\theta_2 = 1$ and discretizes the conditions $\int_0^1 \alpha\, \mathrm{d}u = 0$ and $|\partial_u \boldsymbol{x}| = L^t$. At this stage, the relation between these conditions and θ_1, θ_2 is not clear. Another scheme for the case $\beta(k) = k$ was proposed by Deckelnick [2], who used a nontrivial $\alpha = -\partial_u(|\partial_u \boldsymbol{x}|^{-1})$ which satisfies $\theta_1 = |\partial_u \boldsymbol{x}| = 1/\theta_2$. Mikula and Ševčovič [11] generalized these results for the case $\beta(k) = |k|^{m-1}k$ by putting $\alpha = \partial_s|k|^{m-1}/2$ and $\theta_1 = \theta_2 = \sqrt{k/\beta(k)}$. This result was improved in Mikula and Ševčovič [12] for the general case $v = \beta(k,\nu)$. In this, θ_1 and θ_2 satisfy $\alpha = \theta_1^{-1}\partial_s\theta_2^{-1}$ and $\partial_s\alpha = \overline{kv} - kv$, where $\overline{f} = \int_{\Gamma^t} f\, \mathrm{d}s/L^t$ is the average of f.

Applying the crystalline case (3) for $\beta(k) = |k|^{m-1}k$, we obtain $\alpha = -\bar{m}\partial_s|k|^{m-1}$, $\bar{m} = m/(m-1)$ if $m \neq 1$, and $\alpha = -\partial_s \log|k|$ if $m = 1$; and $\theta_1 = k/\beta^2$, $\theta_2 = \beta$.

4. REMARKS ON NUMERICAL SCHEMES

Discretizing (7) (or (8)) in time and using the adapted time step control, we obtain a sequence of $\mathcal{T}$-admissible polygonal curves $\mathcal{P}^0 = \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots$, where $\mathcal{P}_j$ approximates $\mathcal{P}^{t_j}$ $(0 = t_0 < t_1 < t_2 < \cdots)$. The following figure is a numerical scheme in the case $\beta(k) = |k|^{m-1}k$ with $m = 0.1$.



We can observe that vertices are concentrated at sharp corners. But scheme does not break down, since the elimination of some edges and renumbering are done. (In this figure, the last partition number is $N = 36$ starting with the initial partition number $N = 380$.) In other words, the solution is extended in several times while the solution curve keeps admissibility. However, in general, preservation of admissibility is open problems. Indeed, in the case where $\beta(k,\nu) = a(\nu)|k|^{m-1}k$, there exist $m \in (0,1)$, $a(\cdot)$ and $\mathcal{P}^0$ such that the admissibility collapses in finite time, i. e., we have the examples that admissible nonconvex polygonal curve becomes nonadmissible in finite time. See, e. g., Hirota, Ishiwata and Yazaki [7]. This is contrast result to the following: For any $m \geq 1$, $a(\cdot)$ and $\mathcal{P}^0$ if the anisotropy is symmetric, then the solution curve keeps admissibility. See Giga and Giga [5].

We can also observe that no swallow tail occurs because of the adapted time step control. Then we can continue the scheme until the maximal existence time. Moreover several convergence results are known until the maximal existence time. See, e. g., [5, 18] and references therein. At this stage, before the maximal existence time, it is open whether $\mathcal{P}^t$ becomes convex or not. For instance, the following non-convexified phenomenon is known: In the case where $\beta(k,\nu) = a(\nu)|k|^{m-1}k$, there exist $m \in (0,1)$, $a(\cdot)$ and $\mathcal{P}^0$ such that nonconvex solution curve shrinks homothetically, i. e., there exists a nonconvex self-similar solution polygonal curve. See Ishiwata, Ushijima, Yagisita and Yazaki [9].

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