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REMARKS ON TWO PRODUCT–LIKE CONSTRUCTIONS FOR COPULAS

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We investigate two constructions that, starting with two bivariate copulas, give rise to a new bivariate and trivariate copula, respectively. In particular, these constructions are generalizations of the $*$-product and the $\star$-product for copulas introduced by Darsow, Nguyen and Olsen in 1992. Some properties of these constructions are studied, especially their relationships with ordinal sums and shuffles of Min.

Keywords: copula, ordinal sum, shuffle of Min, concordance

AMS Subject Classification: 60E05

1. INTRODUCTION

Given $n \in \mathbb{N}$, $n \geq 2$, an *n-dimensional copula* (briefly, *n-copula*) is a function $C\colon [0,1]^n \to [0,1]$ such that

(C1) $C(u_1,\dots,u_n) = 0$ if $u_i = 0$ for at least one index $i \in \{1,2,\dots,n\}$;

(C2) $C(u_1,\dots,u_n) = u_i$ if $u_j = 1$ for each $j \neq i$;

(C3) C is *n-increasing*, i.e. for any n-box $B = \times_{i=1}^n [x_i, y_i] \subseteq [0,1]^n$, with $x_i \leq y_i$ for all $i = 1,2,\dots,n$, we have

$$V_C(B) := \sum_{\boldsymbol{z} \in \times_{i=1}^n \{x_i,y_i\}} (-1)^{N(\boldsymbol{z})} C(\boldsymbol{z}) \geq 0,$$

where $N(\boldsymbol{z}) = \operatorname{card}\{k \mid z_k = x_k\}$.

Two important examples of n-copulas are

$$\begin{aligned} M_n(u_1,\dots u_n) &= \min\{u_1,\dots,u_n\}, \\ \Pi_n(u_1,\dots,u_n) &= \prod_{i=1}^n u_i. \end{aligned}$$

Another important 2-copula is $W_2(u_1,u_2) = \max\{u_1 + u_2 - 1, 0\}$. By $\mathcal{C}_n$ we will denote the class of all n-copulas (see [4, 8] for more details).

In Statistics, multivariate copulas are mainly used in the construction of multivariate distribution functions (= d.f.'s) with given univariate marginals [4, 8], as asserted by the following Theorem due to A. Sklar [10].

Theorem 1. Let $X_1, X_2, \ldots, X_n$ be random variables with joint d.f. F and univariate marginals $F_1, F_2, \ldots, F_n$. Then there exists $C_n \in \mathcal{C}_n$ such that, for all $x_1, x_2, \ldots, x_n$ in $\mathbb{R}$,

$$F(x_1, x_2, \ldots, x_n) = C_n(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)). \tag{1}$$

If $F_1, F_2, \ldots, F_n$ are continuous, then C_n is unique; otherwise C_n is uniquely determined on $Ran\, F_1 \times Ran\, F_2 \times \cdots \times Ran\, F_n$.

Conversely, if C_n is an n-copula and $F_1, F_2, \ldots, F_n$ are univariate d.f.'s, then the function F defined by (1) is a multivariate d.f. with marginals $F_1, F_2, \ldots, F_n$.

One of the possible extensions of this problem is to construct n-dimensional d.f.'s with k given m-dimensional marginals, $1 \le m < n$ and $1 \le k \le \binom{n}{m}$. For example, given two bivariate d.f.'s F_{12} and F_{23}, one may wish to construct, if there exist, trivariate d.f.'s F such that F_{12} and F_{23} are, respectively, the d.f.'s of the first two and the last two components of the random triplet associated to F, viz.

$$F(x_1, x_2, 1) = F_{12}(x_1, x_2),$$
$$F(1, x_2, x_3) = F_{23}(x_2, x_3),$$

for all x_1, x_2, x_3 in $\mathbb{R}$.

The construction of trivariate d.f.'s with given bivariate marginals has been recently studied in [3]. Specifically, in order to characterize the class of all trivariate d.f.'s when their bivariate marginals are known, two new construction methods for copulas have been proposed.

The aim of this note is to present these methods and to study their relationships with other well known constructions. In Section 2 we define two operations on the set of copulas and we study their basic properties. In Section 3 we study how the constructions performed by these operations fit into the framework of ordinal sums and shuffles of Min.

2. TWO CONSTRUCTIONS FOR COPULAS

In this section, we present the two constructions of copulas introduced in [3]. For a function $f\colon [0,1]^2 \to \mathbb{R}$, $\partial_1 f$ and $\partial_2 f$ will denote, respectively, the derivatives of f with respect to the first and the second variable.

Proposition 1. Let A and B be in $\mathcal{C}_2$ and let $\boldsymbol{C} = (C_t)_{t\in[0,1]}$ be a one-parameter family of 2-copulas. Then the mapping $A *_{\boldsymbol{C}} B\colon [0,1]^2 \to [0,1]$ defined by

$$\big(A *_{\boldsymbol{C}} B\big)(u_1, u_2) = \int_0^1 C_t\left(\partial_2 A(u_1, t), \partial_1 B(t, u_2)\right) \mathrm{d}t \tag{2}$$

is in $\mathcal{C}_2$.

Throughout this paper the boldface symbol $\boldsymbol{C}$ always denotes a family of 2-copulas (C_t) parametrized by $t \in [0,1]$. The copula $A *_{\boldsymbol{C}} B$ is called the $\boldsymbol{C}$*-product* of A and B. If $C_t = C$ for all $t \in [0,1]$, then we shall write $A *_C B$ instead of $A *_{\boldsymbol{C}} B$. In the special case when $C_t = \Pi_2$ for all $t \in [0,1]$, the $\boldsymbol{C}$-product coincides with the product of copulas studied in [1].

Just to give some examples, simple calculations show that, for every $A \in \mathcal{C}_2$ and every family $\boldsymbol{C}$

$$\begin{aligned} &A *_{\boldsymbol{C}} M_2 = A = M_2 *_{\boldsymbol{C}} A, \\ &\big(A *_{\boldsymbol{C}} W_2\big)(u_1, u_2) = u_1 - A(u_1, 1-u_2), \\ &\big(W_2 *_{\boldsymbol{C}} A\big)(u_1, u_2) = u_2 - A(1-u_1, u_2). \end{aligned}$$

Remark 1. For a fixed family $\boldsymbol{C}$, the $\boldsymbol{C}$-product induces a binary operation $*_{\boldsymbol{C}}$ on the set $\mathcal{C}_2$ given by

$$*_{\boldsymbol{C}} \colon \mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2, \qquad (A,B) \mapsto A *_{\boldsymbol{C}} B$$

This operation has neutral element M_2. However, while the operation $*_{\Pi_2}$ is associative [1], in general the operation $*_{\boldsymbol{C}}$ need not to be so. For example, given the 2-copula C, defined for every $u_1, u_2 \in [0,1]$, by

$$C(u_1, u_2) = u_1 u_2 + u_1 u_2 (1-u_1)(1-u_2),$$

simple (but really tedious) calculations yield $C *_C (C *_C C) \neq (C *_C C) *_C C$.

Example 1. Let $\boldsymbol{C} = (C_t)_{t\in[0,1]}$ be a family of 2-copulas. Then we have

$$\big(\Pi_2 *_{\boldsymbol{C}} \Pi_2\big)(u_1, u_2) = \int_0^1 C_t(u_1, u_2)\,\mathrm{d}t.$$

The function $\Pi_2 *_{\boldsymbol{C}} \Pi_2$ is a 2-copula, also known as the *convex sum* of the family $\boldsymbol{C}$ with respect to the Lebesgue measure in the unit interval [8, Section 3.2.4]. Moreover, the finite convex combination of the 2-copulas $(D_i)_{i=1,2,\ldots,n}$ with coefficients α_i, where $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$, can be obtained for a special choice of the family $\boldsymbol{C}$. Namely, put $\beta_0 = 0$ and $\beta_i = \sum_{j=1}^i \alpha_j$ for all $i = 1, 2, \ldots, n$. Now, it is sufficient to take $\boldsymbol{C} = (C_t)_{t\in[0,1]}$ such that $C_t = D_i$ whenever $t \in [\beta_{i-1}, \beta_i]$, in order to obtain that

$$\big(\Pi_2 *_{\boldsymbol{C}} \Pi_2\big)(u_1, u_2) = \sum_{i=1}^n \alpha_i D_i(u_1, u_2).$$

For any $A, B, C_1, C_2 \in \mathcal{C}_2$, note that $C_1 \leq C_2$ implies $A *_{C_1} B \leq A *_{C_2} B$. This fact suggests to use the operation $*_{\boldsymbol{C}}$ as a method for the construction of one-parameter families of copulas $(C_\alpha)_{\alpha\in J}$, where J is an interval in $\mathbb{R}$, which are monotone with respect to the parameter, i.e. $C_\alpha \leq C_{\alpha'}$ whenever $\alpha \leq \alpha'$. Indeed, it suffices to fix $A, B \in \mathcal{C}_2$ and consider a positively ordered family $(C_\alpha)_{\alpha\in J}$ of 2-copulas. Then $(A *_{C_\alpha} B)_{\alpha\in J}$ is also a positively ordered family of 2-copulas.

Example 2. Let $(C_\alpha)_{\alpha\in[-1,1]}$ be the Farlie–Gumbel–Morgenstern family of copulas defined by the expression

$$C_\alpha(u_1,u_2) = u_1u_2 + \alpha u_1u_2(1-u_1)(1-u_2). \tag{3}$$

Denote $\widetilde{C}_\alpha = C_0 *_{C_\alpha} C_1$. From Proposition 1 we have that $(\widetilde{C}_\alpha)_{\alpha\in[-1,1]}$ is a one-parameter family of 2-copulas given by

$$\widetilde{C}_\alpha(u_1,u_2) = \frac{u_1u_2}{3}(3+\alpha(u_1-1)(u_1(u_1-1)+3)(u_2-1))$$

for every $\alpha \in [-1,1]$. Moreover, $\widetilde{C}_{\alpha_1} \le \widetilde{C}_{\alpha_2}$ whenever $\alpha_1 \le \alpha_2$.

The $\boldsymbol{C}$-product can be extended to obtain 3-copulas in the following way [3].

Proposition 2. Let A and B be in $\mathcal{C}_2$ and let $\boldsymbol{C}$ be a family of 2-copulas. Then the mapping $A \star_{\boldsymbol{C}} B\colon [0,1]^3 \to [0,1]$ defined by

$$\big(A \star_{\boldsymbol{C}} B\big)(u_1,u_2,u_3) = \int_0^{u_2} C_t\left(\partial_2 A(u_1,t), \partial_1 B(t,u_3)\right) \mathrm{d}t$$

is in $\mathcal{C}_3$.

The 3-copula $A \star_{\boldsymbol{C}} B$ is called the $\boldsymbol{C}$*-lifting* of A and B. If $C_t = C$ for every $t \in [0,1]$, then we shall write $A \star_C B$ instead of $A \star_{\boldsymbol{C}} B$. In the special case when $C_t = \Pi_2$ for all t in $[0,1]$, the $\boldsymbol{C}$-lifting coincides with the $\star$-operation considered in [1, 5].

Just to give some examples, simple calculations show that, for every $A \in \mathcal{C}_2$ and every family of 2-copulas $\boldsymbol{C}$

$$\begin{aligned}
\big(A \star_{\boldsymbol{C}} M_2\big)(u_1,u_2,u_3) &= A(u_1, u_2 \wedge u_3),\\
\big(M_2 \star_{\boldsymbol{C}} A\big)(u_1,u_2,u_3) &= A(u_1 \wedge u_2, u_3),\\
\big(W_2 \star_{\boldsymbol{C}} W_2\big)(u_1,u_2,u_3) &= W_2(u_1 \wedge u_3, u_2).
\end{aligned}$$

Remark 2. Notice that, given $A, B \in \mathcal{C}_2$ and a family of 2-copulas $\boldsymbol{C}$, the 2-marginals of $A \star_{\boldsymbol{C}} B$ are

$$\begin{aligned}
\big(A \star_{\boldsymbol{C}} B\big)(u_1,u_2,1) &= A(u_1,u_2),\\
\big(A \star_{\boldsymbol{C}} B\big)(u_1,1,u_3) &= \big(A *_{\boldsymbol{C}} B\big)(u_1,u_3),\\
\big(A \star_{\boldsymbol{C}} B\big)(1,u_2,u_3) &= B(u_2,u_3).
\end{aligned}$$

Actually, the $\boldsymbol{C}$-lifting is exactly the tool that allows us to construct all the 3-copulas having A and B as respective marginals [3]. In particular, A, B and $A *_{\boldsymbol{C}} B$ are compatible, viz. there exist three random variables U_1, U_2 and U_3, U_i uniformly distributed on $[0,1]$ for $i \in \{1,2,3\}$, such that A is the distribution function of (U_1,U_2), $A *_{\boldsymbol{C}} B$ is the d.f. of (U_1,U_3) and B is the d.f. of (U_2,U_3).

The $\boldsymbol{C}$-lifting also provides a general method to construct families of 3-copulas that are *positively ordered* with respect to the *concordance order*, which is a condition stronger than pointwise order between functions [8]. In fact, we recall that given $C_1, C_2 \in \mathcal{C}_3$, the copula C_2 is said to be greater than C_1 in the concordance order, and we write $C_1 \preceq C_2$, if $C_1 \leq C_2$ and $\bar{C}_1 \leq \bar{C}_2$, where $\bar{C}\colon [0,1]^3 \to [0,1]$ is the survival function of C, defined by

$$\begin{aligned}\bar{C}&(u_1,u_2,u_3)\\ &= 1-u_1-u_2-u_3+C(u_1,u_2,1)+C(u_1,1,u_3)+C(1,u_2,u_3)-C(u_1,u_2,u_3).\end{aligned}$$

The relationship between the $\boldsymbol{C}$-lifting and the concordance order is summarized in the following proposition [3].

Proposition 3. Let C, C' be in $\mathcal{C}_2$. If $C \leq C'$, then $A \star_C B \preceq A \star_{C'} B$ for all $A, B \in \mathcal{C}_2$.

Therefore, if we fix $A, B \in \mathcal{C}_2$ and consider a positively ordered family $(C_\alpha)_{\alpha \in J}$ of 2-copulas, where J is an interval of $\mathbb{R}$, then $(A \star_{C_\alpha} B)_{\alpha \in J}$ is a family of 3-copulas positively ordered with respect to the concordance order.

Example 3. Let $(C_\alpha)_{\alpha \in [-1,1]}$ be again the Farlie–Gumbel–Morgenstern family of copulas (3). Then $(\Pi_2 \star_{C_\alpha} \Pi_2)_{\alpha \in [-1,1]}$ is a one-parameter family of 3-copulas given by the expression

$$\begin{aligned}\big(\Pi_2 \star_{C_\alpha} \Pi_2\big)(u_1,u_2,u_3) &= u_1u_2u_3(1+\alpha(1-u_1)(1-u_3))\\ &= u_2C_\alpha(u_1,u_3).\end{aligned}$$

This family is increasing in α with respect to the concordance order.

Remark 3. Given $C \in \mathcal{C}_n$, it can be easily derived that, for each permutation π of $\{1,2,\dots,n\}$, the function $C^\pi\colon [0,1]^n \to [0,1]$ given by

$$C^\pi(u_1,u_2,\dots,u_n) = C(u_{\pi(1)},u_{\pi(2)},\dots,u_{\pi(n)})$$

is also in $\mathcal{C}_n$. In particular, by applying this fact to the $\boldsymbol{C}$-lifting, we obtain that, for all 2-copulas A, B, for any family of 2-copulas $\boldsymbol{C}$, and for each permutation π of $\{1,2,3\}$, the operation $(A \star_{\boldsymbol{C}} B)^\pi$ is also in $\mathcal{C}_3$, but in general $(A \star_{\boldsymbol{C}} B)^\pi \neq A \star_{\boldsymbol{C}} B$.

3. RELATIONSHIPS WITH OTHER CONSTRUCTIONS

In this section we show how the $\boldsymbol{C}$-product is related to two known constructions of 2-copulas: the ordinal sums and the shuffles of Min. In both cases we will discuss the possible extensions to the trivariate case.

3.1. Ordinal sums

Let $(C_i)_{i\in\mathcal{I}}$ be a family of 2-copulas indexed by the (at most) countable set $\mathcal{I}$. Let $(]a_i, b_i[)_{i\in\mathcal{I}}$ be a family of pairwise disjoint subintervals of $[0,1]$ indexed by the same set $\mathcal{I}$. The *ordinal sum* of $(C_i)_{i\in\mathcal{I}}$ with respect to $(]a_i, b_i[)_{i\in\mathcal{I}}$ is the copula $C\colon [0,1]^2 \to [0,1]$ given by

$$C(u_1,u_2) = \begin{cases} a_i + (b_i - a_i)C_i\left(\frac{u_1-a_i}{b_i-a_i}, \frac{u_2-a_i}{b_i-a_i}\right) & \text{if } u_1,u_2 \in\,]a_i,b_i[\,, \\ M_2(u_1,u_2) & \text{otherwise.} \end{cases}$$

Usually, we make use of the notation $C = (\langle a_i, b_i, C_i\rangle)_{i\in\mathcal{I}}$ (see, for example, [8, 9] for more details).

The relationship between the ordinal sum construction and the $\boldsymbol{C}$-product is summarized in the next result.

Proposition 4. Let $C = (\langle a_i, b_i, C_i\rangle)_{i\in\mathcal{I}}$ be an ordinal sum. Then

$$C = C_\Pi *_{\boldsymbol{C}} C_\Pi,$$

where $C_\Pi = (\langle a_i, b_i, \Pi_2\rangle)_{i\in\mathcal{I}}$, and $\boldsymbol{C} = (C_t)_{t\in[0,1]}$ is any family of 2-copulas such that $C_t = C_i$ whenever $t \in\,]a_i, b_i[$ for some $i \in \mathcal{I}$.

Proof. Given $[a,b] \subseteq [0,1]$, define $U_{[a,b]}\colon [0,1] \to [0,1]$ by the expression

$$U_{[a,b]}(x) = \begin{cases} 0 & \text{if } x \le a, \\ \frac{x-a}{b-a} & \text{if } a < x \le b, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that

$$\partial_2 C_\Pi(u_1,t) = \begin{cases} U_{[a_i,b_i]}(u_1) & \text{if } t \in\,]a_i,b_i[\,, \\ \mathbf{1}_{[t,1]}(u_1) & \text{otherwise,} \end{cases}$$

and, since C_Π is symmetric, $\partial_1 C_\Pi(t,u_2)$ has an analogous expression.

Now, let $u_1, u_2 \in\,]a_i, b_i[$ for some $i \in \mathcal{I}$. In this case, we have

$$\begin{aligned}(C_\Pi *_{\boldsymbol{C}} C_\Pi)(u_1,u_2) &= \int_0^{a_i} C_t(1,1)\,\mathrm{d}t + \int_{a_i}^{b_i} C_t(U_{[a_i,b_i]}(u_1), U_{[a_i,b_i]}(u_2))\,\mathrm{d}t \\ &= a_i + (b_i - a_i)C_i\left(\frac{u_1-a_i}{b_i-a_i}, \frac{u_2-a_i}{b_i-a_i}\right) \\ &= C(u_1,u_2).\end{aligned}$$

Otherwise, if $(u_1,u_2) \notin\,]a_i,b_i[^2$ for all $i \in \mathcal{I}$, we have

$$(C_\Pi *_{\boldsymbol{C}} C_\Pi)(u_1,u_2) = \int_0^{u_1\wedge u_2} C_t(1,1)\,\mathrm{d}t = M_2(u_1,u_2),$$

which concludes the proof. □

Now, we will consider the above result in order to obtain a new 3-copula by means of the $\boldsymbol{C}$-lifting.

Let $(C_i)_{i\in\mathcal{I}}$ be a family of 2-copulas and let $(]a_i, b_i[)_{i\in\mathcal{I}}$ be a family of pairwise disjoint subintervals of $[0, 1]$. Then we call *lifting ordinal sum* of $(C_i)_{i\in\mathcal{I}}$ with respect to $(]a_i, b_i[)_{i\in\mathcal{I}}$ the 3-copula $\widetilde{C}$ defined by

$$\widetilde{C} = C_\Pi \star_{\boldsymbol{C}} C_\Pi,$$

where $C_\Pi = (\langle a_i, b_i, \Pi_2\rangle)_{i\in\mathcal{I}}$, and $\boldsymbol{C} = (C_t)_{t\in[0,1]}$ is any family such that $C_t = C_i$ whenever $t \in]a_i, b_i[$ for some $i \in \mathcal{I}$.

Example 4. The *lifting ordinal sum* of (Π_2, Π_2) with respect to the family of sub-intervals $(]0, \frac{1}{2}[,]\frac{1}{2}, 1[)$ is the 3-copula $\widetilde{C}$ given by the expression

$$\widetilde{C}(u_1, u_2, u_3) = \begin{cases} u_2 C_1(u_1, u_3) & \text{if } u_2 \le \frac{1}{2}, \\ \frac{1}{2}C_1(u_1, u_3) + (u_2 - \frac{1}{2})C_2(u_1, u_3) & \text{otherwise}, \end{cases}$$

where $C_1 = (\langle 0, \frac{1}{2}, \Pi_2\rangle)$ and $C_2 = (\langle \frac{1}{2}, 1, \Pi_2\rangle)$.

Remark 4. The above result can also be applied to a related construction for copulas, called W-ordinal sum [2, 6]. Specifically, let $C = (\langle a_i, b_i, C_i\rangle)^W_{i\in\mathcal{I}}$ be the W-ordinal sum of $(C_i)_{i\in\mathcal{I}}$ with respect to $(]a_i, b_i[)_{i\in\mathcal{I}}$. Then

$$C = C_\Pi *_{\boldsymbol{C}} C^W_\Pi,$$

where $C_\Pi = (\langle a_i, b_i, \Pi_2\rangle)_{i\in\mathcal{I}}$ is the (usual) ordinal sum, $C^W_\Pi = (\langle a_i, b_i, \Pi_2\rangle)^W_{i\in\mathcal{I}}$ is the W-ordinal sum, and $\boldsymbol{C} = (C_t)_{t\in[0,1]}$ is any family of 2-copulas such that $C_t = C_i$ whenever $t \in]a_i, b_i[$ for some $i \in \mathcal{I}$.

3.2. Shuffles of Min

It is well known that every 2-copula C induces a probability measure λ_C on $[0, 1]^2$ given, for every x_1, y_1, x_2, y_2 in $[0, 1]$, $x_1 \le y_1$ and $x_2 \le y_2$, by

$$\lambda_C([x_1, y_1] \times [x_2, y_2]) = V_C([x_1, y_1] \times [x_2, y_2]).$$

In particular, the *support* of every copula C is the complement of the union of all open subsets of $[0, 1]^2$ with λ_C-measure equal to 0.

By using the probability measure induced by a copula, in [7] the authors introduced a new method for obtaining copulas, called *shuffles of Min*, by means of an intuitive geometrical "manipulation" of the measure induced by M_2 (see [7, 8] for more details).

The support of any shuffle of Min is the graph of a bijection f on $[0, 1]$ such that:

(a) f is piecewise linear;

(b) each component of the graph of f has slope 1 or -1;

(c) f has at most finitely many discontinuity points.

Conversely, from every bijection $f\colon [0,1] \to [0,1]$ satisfying (a), (b) and (c), we can construct a shuffle of Min M_f whose support is given by the graph of f.

From [7] (see also [5, 8]) it follows that, if M_f is the shuffle of Min corresponding to f, then there exist two random variables U_1 and U_2, both uniformly distributed on $[0,1]$, such that M_f is the d.f. of (U_1, U_2) and $U_2 = f(U_1)$.

Remark 5. By using the results of [5], we also have that, if M_f is the shuffle of Min associated with a bijection f, then, because f is a measure-preserving transformation,

$$M_f(u_1, u_2) = \lambda\big(f([0, u_1]) \cap [0, u_2]\big),$$

where λ is the Lebesgue measure on $[0,1]$.

In the next result, we compose two shuffles of Min by means of the $\boldsymbol{C}$-product.

Proposition 5. Let M_f and M_g be shuffles of Min corresponding to f and g, respectively. Then, for any family of 2-copulas $\boldsymbol{C}$, the 2-copula $M_f *_{\boldsymbol{C}} M_g$ is the shuffle of Min corresponding to $g \circ f$.

P r o o f. It is known from [3] that, for every family $\boldsymbol{C}$, $M_f *_{\boldsymbol{C}} M_g$ is always the same copula, and M_f, $M_f *_{\boldsymbol{C}} M_g$ and M_g are compatible. Then there exist three random variables U_1, U_2 and U_3, U_i uniformly distributed on $[0,1]$ for every $i \in \{1,2,3\}$, such that M_f is the d.f. of (U_1, U_2), $M_f *_{\boldsymbol{C}} M_g$ is the d.f. of (U_1, U_3), and M_g is the d.f. of (U_2, U_3). But, because M_f and M_g are shuffles of Min, we have $U_2 = f(U_1)$ and $U_3 = g(U_2)$. Therefore, $U_3 = (g \circ f)(U_1)$ and, hence, $M_f *_{\boldsymbol{C}} M_g$ is the shuffle of Min $M_{g\circ f}$ corresponding to the bijection $g \circ f$. □

Example 5. Consider the two shuffles of Min, M_f and M_g, generated, respectively, by the bijections f and g defined by

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \frac{3}{2} - x & \text{otherwise,} \end{cases} \qquad g(x) = 1 - x.$$

Then, for every family $\boldsymbol{C}$, $M_f *_{\boldsymbol{C}} M_g$ is the shuffle of Min generated by

$$(g \circ f)(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{2}\right], \\ x - \frac{1}{2} & \text{otherwise,} \end{cases}$$

and, hence, its support is given by the segment connecting $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, and by the segment connecting $(\frac{1}{2}, 0)$ and $(1, \frac{1}{2})$.

Now, we will consider the above result in order to obtain a new 3-copula by means of the $\boldsymbol{C}$-lifting.

Proposition 6. Let M_f and M_g be shuffles of Min corresponding to f and g, respectively, Then $M_f \star_{\boldsymbol{C}} M_g$ is the 3-copula $\widetilde{C}$ given by

$$\widetilde{C}(u_1, u_2, u_3) = \lambda\big(f([0, u_1]) \cap [0, u_2] \cap g^{-1}([0, u_3])\big). \tag{4}$$

Proof. It is known from [3] that, for every family $\boldsymbol{C}$, $M_f \star_{\boldsymbol{C}} M_g$ is the only 3-copula $\widetilde{C}$ with 2-marginals M_f, $M_f *_{\boldsymbol{C}} M_g$ and M_g. Then there exist three random variables U_1, U_2 and U_3, U_i uniformly distributed on $[0, 1]$ for every $i \in \{1, 2, 3\}$, such that M_f is the d.f. of (U_1, U_2), $M_f *_{\boldsymbol{C}} M_g$ is the d.f. of (U_1, U_3), M_g is the d.f. of (U_2, U_3), and $\widetilde{C}$ is the d.f. of (U_1, U_2, U_3). Moreover, because M_f and M_g are shuffles of Min, we have that

$$U_2 = f(U_1),$$
$$U_3 = g(U_2) = (g \circ f)(U_1).$$

As a consequence, we obtain that, for every $u_1, u_2, u_3 \in [0, 1]$,

$$\begin{aligned} \widetilde{C}(u_1, u_2, u_3) &= P(U_1 \le u_1, U_2 \le u_2, U_3 \le u_3) \\ &= P(U_1 \le u_1, f(U_1) \le u_2, (g \circ f)(U_1) \le u_3) \\ &= \lambda\big([0, u_1] \cap f^{-1}([0, u_2]) \cap (g \circ f)^{-1}([0, u_3])\big), \end{aligned}$$

and, since f and g are measure-preserving transformations,

$$\widetilde{C}(u_1, u_2, u_3) = \lambda\big(f([0, u_1]) \cap [0, u_2] \cap g^{-1}([0, u_3])\big),$$

which is the desired assertion. □

Notice that the function given by (4) is a copula in view of a more general result on the representation of copulas by means of measure-preserving transformations [5].

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