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*Kybernetika*, Vol. 43 (2007), No. 2, 209--220

Persistent URL: <http://dml.cz/dmlcz/135767>

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## COPULAS WITH GIVEN VALUES ON A HORIZONTAL AND A VERTICAL SECTION

FABRIZIO DURANTE, ANNA KOLESÁROVÁ, RADKO MESIAR AND CARLO SEMPI

In this paper we study the set of copulas for which both a horizontal section and a vertical section have been given. We give a general construction for copulas of this type and we provide the lower and upper copulas with these sections. Symmetric copulas with given horizontal section are also discussed, as well as copulas defined on a grid of the unit square. Several examples are presented.

*Keywords:* copula, horizontal section, vertical section, binary aggregation operator

*AMS Subject Classification:* 60E05, 62H05

### 1. INTRODUCTION

A two-dimensional *copula* (a copula, for short) is a function  $C$  from  $[0, 1]^2$  into  $[0, 1]$  that satisfies the following properties:

$$(C1) \quad C(x, 0) = C(0, x) = 0 \quad \text{for all } x \in [0, 1];$$

$$(C2) \quad C(x, 1) = C(1, x) = x \quad \text{for all } x \in [0, 1];$$

$$(C3) \quad \text{for all } x, x', y, y' \text{ in } [0, 1] \text{ with } x \leq x' \text{ and } y \leq y',$$

$$V_C([x, x'] \times [y, y']) = C(x', y') - C(x, y') - C(x', y) + C(x, y) \geq 0.$$

The conditions in (C1) and (C2) express the *boundary properties* of a copula  $C$ , (C3) is the *2-increasing property* of  $C$ , and  $V_C([x, x'] \times [y, y'])$  is called the *C-volume* of the rectangle  $[x, x'] \times [y, y']$  (see [20] for a thorough exposition). In other words, a copula is a binary aggregation operator with neutral element 1, and satisfying (C3) (see [1, 5]).

Copulas were used for the first time in a statistical context in order to join bivariate distribution functions (=d.f.'s) to their univariate marginal d.f.'s. In fact, according to Sklar's theorem (see [26, 27]), for each random vector  $(X, Y)$  there is a copula  $C_{X,Y}$  (uniquely defined, whenever  $X$  and  $Y$  are continuous) such that the joint distribution function  $F_{X,Y}$  of  $(X, Y)$  may be represented under the form

$$F_{X,Y}(x, y) = C_{X,Y}(F_X(x), F_Y(y)), \quad (1)$$

for all  $x, y \in \overline{\mathbb{R}}$ , where  $F_X$  and  $F_Y$  are the d.f.'s of  $X$  and  $Y$ , respectively. Conversely, given two univariate d.f.'s  $F_X$  and  $F_Y$ , and a copula  $C$ , the function  $F_{X,Y} : \overline{\mathbb{R}}^2 \rightarrow \overline{\mathbb{R}}$  defined by (1) is a bivariate distribution function. For a comprehensive introduction on the theoretical aspects of copulas, the reader is referred to the monographs by Joe [11] and Nelsen [20]. Relevant applications of copulas may be found, among others, in [3, 9, 17], for finance and actuarial science, and in [10, 24], for hydrology.

Since their introduction, copulas have played an important rôle not only in probability theory and statistics, but also in many other fields requiring the aggregation of incoming data, such as multi-criteria decision making (see [8]), probabilistic metric spaces (see [25]), fuzzy set theory (see [14, 22]).

In the literature, various methods are given in order to construct a copula when we have at disposal some information about the horizontal and the vertical sections of it: see, for example, [7, 12, 21, 23]. This paper aims at constructing new copulas with specified values on both a horizontal section and a vertical section of the unit square, i.e. on a cross-like set. Our construction, in particular, includes the construction of copulas with a fixed horizontal (or vertical) section considered in [13] (by using different techniques), and it is a natural extension and complement to Sections 3.2.5 and 3.2.6 of [20].

In Section 2 we construct an absolutely continuous copula with given values on a horizontal section and a vertical section. Such copulas are called *cross copulas* and the properties of the class of all cross copulas are considered in Section 3, where in particular some bounds are showed. Finally (Section 4), we apply the obtained results to some special constructions.

## 2. CROSS COPULAS

Let  $a$  and  $b$  be fixed in  $]0, 1[$ . We denote by  $\mathcal{H}_b$  the set of increasing and 1-Lipschitz functions  $h : [0, 1] \rightarrow [0, 1]$  such that  $h(0) = 0$  and  $h(1) = b$  with

$$\max\{0, t + b - 1\} \leq h(t) \leq \min\{t, b\};$$

and, analogously, we denote by  $\mathcal{V}_a$  the set of increasing and 1-Lipschitz functions  $v : [0, 1] \rightarrow [0, 1]$  such that  $v(0) = 0$  and  $v(1) = a$  with

$$\max\{0, t + a - 1\} \leq v(t) \leq \min\{a, t\}.$$

Let  $C$  be a copula. The *horizontal  $b$ -section* of  $C$  is the function  $h_{C,b} : [0, 1] \rightarrow [0, b]$  given by  $h_{C,b}(t) := C(t, b)$  and the *vertical  $a$ -section* of  $C$  is the function  $v_{C,a} : [0, 1] \rightarrow [0, a]$  given by  $v_{C,a}(t) := C(a, t)$ . Every copula  $C$  is 1-Lipschitz and satisfies, for all  $(x, y) \in [0, 1]^2$ ,

$$W(x, y) \leq C(x, y) \leq M(x, y),$$

where  $W(x, y) = \max\{x + y - 1, 0\}$  and  $M(x, y) = \min\{x, y\}$ . Thus it is immediate that  $h_{C,b}$  and  $v_{C,a}$  are in  $\mathcal{H}_b$  and  $\mathcal{V}_a$ , respectively.

Notice that horizontal and vertical sections have a statistical interpretation. In fact, if  $U$  and  $V$  are r.v.'s, uniformly distributed on  $[0, 1]$ , linked by the copula  $C$ , we have

$$h_{C,b}(t) = P(U \leq t, V \leq b) = bP(U \leq t \mid V \leq b),$$

and

$$v_{C,a}(t) = P(U \leq a, V \leq t) = aP(V \leq t \mid U \leq a).$$

In general, for any pair of continuous r.v.'s  $X$  and  $Y$  linked by a copula  $C$ , we have  $h_{C,b}(F_X(t)) = bF_{X|Y \leq Q_Y(b)}(t)$  and  $v_{C,a}(F_Y(t)) = aF_{Y|X \leq Q_X(a)}(t)$ , where  $Q_X$  and  $Q_Y$  are the quantile functions of  $X$  and  $Y$ , respectively. Thus, the horizontal and vertical sections of a copula express our knowledge about the conditional d.f.'s of  $X$  given  $Q_Y(b)$  and of  $Y$  given  $Q_X(a)$ .

The question arises whether, for each  $h \in \mathcal{H}_b$  and  $v \in \mathcal{V}_a$  such that  $h(a) = v(b)$ , there is a copula  $C$  such that  $h_{C,b} = h$  and  $v_{C,a} = v$ . Copulas of this type will be called *cross copulas* (or  $(h, v)$ -*cross copulas*) and their class will be denoted by  $\mathcal{C}_{h,v}$ .

The following result provides a copula in  $\mathcal{C}_{h,v}$ .

**Theorem 1.** Let  $h$  and  $v$  be given in  $\mathcal{H}_b$  and  $\mathcal{V}_a$ , respectively, with  $h(a) = v(b) = c$ , and  $\max\{a + b - 1, 0\} < c < \min\{a, b\}$ . Then the function  $\tilde{C}$  defined by

$$\tilde{C}(x, y) := \begin{cases} \frac{h(x)v(y)}{c}, & \text{on } [0, a] \times [0, b]; \\ \frac{(a - v(y))h(x) + (v(y) - c)x}{a - c}, & \text{on } [0, a] \times [b, 1]; \\ \frac{(b - h(x))v(y) + (h(x) - c)y}{b - c}, & \text{on } [a, 1] \times [0, b]; \\ x + y - 1 + \frac{(1 - x - b + h(x))(1 - y - a + v(y))}{1 - a - b + c}, & \text{on } [a, 1] \times [b, 1]; \end{cases}$$

is in  $\mathcal{C}_{h,v}$ .

**Proof.** It is immediate to prove that the function  $\tilde{C}$  is well defined and satisfies the boundary conditions (C1) and (C2). In order to prove that  $\tilde{C}$  is 2-increasing, notice that it is sufficient to prove that  $\tilde{C}$  is 2-increasing in each one of the four rectangles

$$[0, a] \times [0, b], [0, a] \times [b, 1], [a, 1] \times [0, b] \text{ and } [a, 1] \times [b, 1].$$

For  $0 \leq x \leq x' \leq a$  and  $0 \leq y \leq y' \leq b$ , we have

$$V_{\tilde{C}}([x, x'] \times [y, y']) = \frac{(h(x') - h(x))(v(y') - v(y))}{c},$$

which is positive, because  $h$  and  $v$  are increasing.

For  $0 \leq x \leq x' \leq a$  and  $b \leq y \leq y' \leq 1$ , we have

$$V_{\tilde{C}}([x, x'] \times [y, y']) = \frac{(x' - x - (h(x') - h(x)))(v(y') - v(y))}{a - c},$$

which is positive, because  $h$  is 1-Lipschitz and  $v$  is increasing.

For  $a \leq x \leq x' \leq 1$  and  $0 \leq y \leq y' \leq b$ , we have

$$V_{\tilde{C}}([x, x'] \times [y, y']) = \frac{(y' - y - (v(y') - v(y)))(h(x') - h(x))}{b - c},$$

which is positive, because  $v$  is 1-Lipschitz and  $h$  is increasing.

Finally, for  $a \leq x \leq x' \leq 1$  and  $b \leq y \leq y' \leq 1$ , we have

$$V_{\tilde{C}}([x, x'] \times [y, y']) = \frac{(x' - x - (h(x') - h(x)))(y' - y - (v(y') - v(y)))}{1 - a - b + c},$$

which is positive, because both  $h$  and  $v$  are 1-Lipschitz.

Hence,  $\tilde{C}$  is a copula and, since we have both  $h_{C,b} = h$  and  $v_{C,a} = v$ , it belongs to  $\mathcal{C}_{h,v}$ . □

Notice that the copula  $\tilde{C}$  is absolutely continuous (see [20]). In fact,  $\tilde{C}$  can be expressed in the form

$$\tilde{C}(x, y) = \int_0^x \int_0^y \tilde{c}(s, t) \, ds \, dt,$$

where  $\tilde{c}$  is the density of  $\tilde{C}$  given by

$$\tilde{c}(s, t) := \begin{cases} \frac{h'(s)v'(t)}{c}, & (s, t) \in [0, a] \times [0, b]; \\ \frac{(1 - h'(s))v'(t)}{a - c}, & (s, t) \in [0, a] \times [b, 1]; \\ \frac{(1 - v'(t))h'(s)}{b - c}, & (s, t) \in [a, 1] \times [0, b]; \\ \frac{(1 - h'(s))(1 - v'(t))}{1 - a - b + c}, & (s, t) \in [a, 1] \times [b, 1]. \end{cases}$$

Theorem 1 may be extended to the case in which either  $c = \max\{a + b - 1, 0\}$  or  $c = \min\{a, b\}$ . In this case, we can obtain also some copulas in  $\mathcal{C}_{h,v}$  that are connected with the above  $\tilde{C}$  in the following way:

- if  $c = 0$ , then

$$\tilde{C}_1(x, y) := \begin{cases} 0, & (x, y) \in [0, a] \times [0, b]; \\ \tilde{C}(x, y), & \text{otherwise;} \end{cases}$$

- if  $c = a + b - 1$ , then

$$\tilde{C}_2(x, y) := \begin{cases} W(x, y), & (x, y) \in [a, 1] \times [b, 1]; \\ \tilde{C}(x, y), & \text{otherwise;} \end{cases}$$

- if  $c = a$ , then

$$\tilde{C}_3(x, y) := \begin{cases} M(x, y), & (x, y) \in [0, a] \times [b, 1]; \\ \tilde{C}(x, y), & \text{otherwise;} \end{cases}$$

- if  $c = b$ , then

$$\tilde{C}_4(x, y) := \begin{cases} M(x, y), & (x, y) \in [a, 1] \times [0, b]; \\ \tilde{C}(x, y), & \text{otherwise.} \end{cases}$$

However, the copulas  $C_i$  ( $i \in \{1, 2, 3, 4\}$ ) are not absolutely continuous.

**Remark 1.** Notice that, at the beginning of this section, we considered  $a, b \in ]0, 1[$ . These limitations are justified by the fact that, if  $a, b \in \{0, 1\}$ , then the construction of cross copulas can be reduced to known cases. For example, If  $a = 0$ , then  $v_a = 0$  and, therefore, a copula in  $\mathcal{C}_{h,v}$  is simply a copula with given horizontal section at  $b$  and this case was considered in [13]. In particular, if also  $b = 0$ , then  $h_b = 0$  and any copula is a cross copula with these sections.

### 3. PROPERTIES OF THE CLASS OF CROSS COPULAS

Thanks to Theorem 1 and to the considerations that follow it, we know that  $\mathcal{C}_{h,v}$  is not empty; moreover, it is not difficult to verify that  $\mathcal{C}_{h,v}$  is also convex. Now, we are interested in considering the question of the existence of a pointwise supremum and a pointwise infimum in this class.

**Theorem 2.** Let  $h$  be in  $\mathcal{H}_b$  and  $v$  be in  $\mathcal{V}_a$  with  $h(a) = v(b) = c$ . Then the functions  $C_{h,v} : [0, 1]^2 \rightarrow [0, 1]$  and  $C^{h,v} : [0, 1]^2 \rightarrow [0, 1]$  given, respectively, by

$$C_{h,v}(x, y) := \begin{cases} \max\{0, h(x) + v(y) - c\}, & (x, y) \in [0, a] \times [0, b]; \\ \max\{h(x), x + v(y) - a\}, & (x, y) \in [0, a] \times [b, 1]; \\ \max\{v(y), h(x) + y - b\}, & (x, y) \in [a, 1] \times [0, b]; \\ \max\{h(x) + v(y) - c, x + y - 1\}, & (x, y) \in [a, 1] \times [b, 1]; \end{cases}$$

and

$$C^{h,v}(x, y) := \begin{cases} \min\{h(x), v(y)\}, & (x, y) \in [0, a] \times [0, b]; \\ \min\{h(x) + v(y) - c, y\}, & (x, y) \in [0, a] \times [b, 1]; \\ \min\{x, h(x) + v(y) - c\}, & (x, y) \in [a, 1] \times [0, b]; \\ \min\{x + v(y) - a, h(x) + y - b\}, & (x, y) \in [a, 1] \times [b, 1]; \end{cases}$$

are in  $\mathcal{C}_{h,v}$ .

*Proof.* It is immediate to check that  $C_{h,v}$  and  $C^{h,v}$  are well defined and satisfy the boundary conditions (C1) and (C2). In order to prove that they are 2-increasing, it is sufficient to prove that they are 2-increasing in each of the four rectangles

$$[0, a] \times [0, b], [0, a] \times [b, 1], [a, 1] \times [0, b] \text{ and } [a, 1] \times [b, 1]. \tag{2}$$

Notice that, if  $A : [0, 1]^2 \rightarrow \mathbb{R}$  is 2-increasing, then  $A(f(x), g(y))$  is 2-increasing for all increasing functions  $f, g : [0, 1] \rightarrow [0, 1]$  (see [5, Proposition 4.2]); moreover, the sum of 2-increasing functions is also 2-increasing. Now, notice that, in each of the four rectangles in (2), the expressions of  $C_{h,v}$  and  $C^{h,v}$  can be reduced to the forms

$$\begin{aligned} C_{h,v}(x, y) &= f_1(x) + f_2(y) + W(f_3(x), f_4(y)), \\ C^{h,v}(x, y) &= g_1(x) + g_2(y) + M(g_3(x), g_4(y)), \end{aligned}$$

for suitable increasing functions  $f_i$  and  $g_i$  ( $i = 1, 2, 3, 4$ ) and, hence,  $C_{h,v}$  and  $C^{h,v}$  are 2-increasing. Therefore, they are copulas and it is immediately seen that they belong to  $\mathcal{C}_{h,v}$ .  $\square$

**Theorem 3.** Let  $h$  be in  $\mathcal{H}_b$  and  $v$  be in  $\mathcal{V}_a$  with  $h(a) = v(b) = c$ . Then, for every copula  $C \in \mathcal{C}_{h,v}$  and for all  $x, y \in [0, 1]$ , we have

$$C_{h,v}(x, y) \leq C(x, y) \leq C^{h,v}(x, y). \tag{3}$$

*Proof.* Let  $C$  be in  $\mathcal{C}_{h,v}$  and let  $(x, y)$  be in  $[0, a] \times [0, b]$ . Since  $C$  is 2-increasing, we have

$$\begin{aligned} C(x, y) + c &\geq h(x) + v(y), \\ h(x) - C(x, y) &\geq 0, \quad v(y) - C(x, y) \geq 0, \end{aligned}$$

viz.  $C$  satisfies (3).

In the other cases, the proof proceeds in an analogous manner.  $\square$

Notice that both  $C_{h,v}$  and  $C^{h,v}$  are singular copulas. Moreover, since  $C_{h,v} \neq C^{h,v}$ , an infinite number of copulas belong to the set  $\mathcal{C}_{h,v}$ ; in fact, for each  $\lambda \in [0, 1]$ , the copula  $\lambda C_{h,v} + (1 - \lambda)C^{h,v}$  is in  $\mathcal{C}_{h,v}$ .

**Example 1.** Let  $h(t) = bt$  and  $v(t) = at$  be the horizontal and vertical sections at  $a$  and  $b$ , respectively. Then we have that  $C_{h,v}$  is given by

$$C_{h,v}(x, y) = \begin{cases} \max \{0, bx + ay - ab\}, & (x, y) \in [0, a] \times [0, b]; \\ \max \{ax, x + ay - a\}, & (x, y) \in [0, a] \times [b, 1]; \\ \max \{ay, bx + y - b\}, & (x, y) \in [a, 1] \times [0, b]; \\ \max \{bx + ay - ab, x + y - 1\}, & (x, y) \in [a, 1] \times [b, 1]. \end{cases}$$

On the other hand  $C^{h,v}$  is given by

$$C^{h,v}(x, y) = \begin{cases} \min \{bx, ay\}, & (x, y) \in [0, a] \times [0, b]; \\ \min \{bx + ay - ab, y\}, & (x, y) \in [0, a] \times [b, 1]; \\ \min \{bx + ay - ab, x\}, & (x, y) \in [a, 1] \times [0, b]; \\ \min \{x + ay - a, bx + y - b\}, & (x, y) \in [a, 1] \times [b, 1]. \end{cases}$$

### 4. SPECIAL CONSTRUCTIONS

In this section, we apply our methods to some special construction in the class of copulas.

#### 4.1. Symmetric horizontal copulas

As already noted, in [13] the authors considered a class of copulas with a given horizontal section, constructed an absolutely continuous copula with this section, and provided the smallest and greatest copulas in this class. All the copulas thus obtained are, in general, non-symmetric. Here, on the contrary, we apply the results of the previous sections in order to construct symmetric copulas having a given horizontal (or vertical) section. We recall that symmetry is a relevant statistical property for a copula  $C$ , as it corresponds to the case of a pair of exchangeable random variables having  $C$  as their copula.

Let  $a \in ]0, 1[$  be fixed. For each  $h \in \mathcal{H}_a$  we aim at constructing a symmetric copula  $C$  such that  $h_{C,a} = h$ . In this case, symmetry entails  $v_{C,a} = h$ ; as a consequence, we can apply the above methods to the construction of a copula  $C$  having a horizontal section and a vertical section at  $a$  which are both equal to  $h$ .

**Theorem 4.** Let  $h$  be in  $\mathcal{H}_a$ ,  $h(a)=c$ ,  $\max\{2a - 1, 0\} < c < a$ . Then the function  $\tilde{C}$  given by

$$\tilde{C}(x, y) := \begin{cases} \frac{h(x)h(y)}{c}, & \text{on } [0, a] \times [0, a]; \\ \frac{(a - h(y))h(x) + (h(y) - c)x}{a - c}, & \text{on } [0, a] \times [a, 1]; \\ \frac{(a - h(x))h(y) + (h(x) - c)y}{a - c}, & \text{on } [a, 1] \times [0, a]; \\ x + y - 1 + \frac{(1 - x - a + h(x))(1 - y - a + h(y))}{1 - 2a + c}, & \text{on } [a, 1] \times [a, 1]; \end{cases}$$

is a symmetric copula with horizontal section at  $a$  equal to  $h$ .

Moreover, in the class of such symmetric copulas, the lower and upper bound are given, respectively, by:

$$C_h(x, y) := \begin{cases} \max\{0, h(x) + h(y) - c\}, & (x, y) \in [0, a] \times [0, a]; \\ \max\{h(x), x + h(y) - a\}, & (x, y) \in [0, a] \times [a, 1]; \\ \max\{h(y), h(x) + y - a\}, & (x, y) \in [a, 1] \times [0, a]; \\ \max\{h(x) + h(y) - c, x + y - 1\}, & (x, y) \in [a, 1] \times [a, 1]; \end{cases}$$

and by

$$C^h(x, y) := \begin{cases} \min\{h(x), h(y)\}, & (x, y) \in [0, a] \times [0, a]; \\ \min\{h(x) + hy - c, y\}, & (x, y) \in [0, a] \times [a, 1]; \\ \min\{x, h(x) + h(y) - c\}, & (x, y) \in [a, 1] \times [0, a]; \\ \min\{x + h(y) - a, h(x) + y - a\}, & (x, y) \in [a, 1] \times [a, 1]. \end{cases}$$



As was done in Section 2 the above theorem may be extended to the case when either one of the strict inequalities for  $c$  is an equality, namely when either  $c = \max\{0, 2a - 1\}$  or  $c = a$ . In this case, we can obtain the following copulas:

- if  $c = 0$ , then

$$\tilde{C}_1(x, y) := \begin{cases} 0, & (x, y) \in [0, a] \times [0, a]; \\ \tilde{C}(x, y), & \text{otherwise;} \end{cases}$$

- if  $c = 2a - 1$ , then

$$\tilde{C}_2(x, y) := \begin{cases} W(x, y), & (x, y) \in [a, 1] \times [a, 1]; \\ \tilde{C}(x, y), & \text{otherwise;} \end{cases}$$

- if  $c = a$ , then

$$\tilde{C}_3(x, y) := \begin{cases} M(x, y), & (x, y) \in ([0, a] \times [a, 1]) \cup ([a, 1] \times [0, a]); \\ \tilde{C}(x, y), & \text{otherwise.} \end{cases}$$

**Example 2.** Let  $h(t) = \max\{0, t - 1/4\}$  and  $v(t) = \max\{0, t - 1/4\}$  be the horizontal and vertical sections at  $a = b = 3/4$ , respectively. Then,  $C_{h,v} = W$  and

$$C^{h,v}(x, y) = \begin{cases} \min\{x, y - \frac{3}{4}\}, & (x, y) \in [0, \frac{1}{4}] \times [\frac{3}{4}, 1]; \\ \min\{x - \frac{1}{4}, y - \frac{1}{4}\}, & (x, y) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]; \\ \min\{x - \frac{3}{4}, y\}, & (x, y) \in [\frac{3}{4}, 1] \times [0, \frac{1}{4}]; \\ W(x, y), & \text{otherwise;} \end{cases}$$

viz.  $C^{h,v}$  is a special kind of  $W$ -ordinal sum (see [4, 18]).

### 4.2. Transformations of cross copulas

Let  $C$  be a copula and let  $\varphi : [0, 1] \rightarrow [0, 1]$  be an increasing concave bijection of  $[0, 1]$ . It is known (see [6, 15, 19]) that the mapping  $\mathcal{C}_\varphi : [0, 1]^2 \rightarrow [0, 1]$  defined by

$$C_\varphi(x, y) := \varphi^{-1}(C(\varphi(x) + \varphi(y)))$$

is also a copula, called the  $\varphi$ -transformation of  $C$ .

For all  $h \in \mathcal{H}_b$  and  $v \in \mathcal{V}_a$ , it is not difficult (but tedious) to show that, if  $C$  is in  $C_{h,v}$ , then  $C_\varphi$  is in  $C_{h^*,v^*}$ , where

$$h^* := \varphi^{-1} \circ h \circ \varphi \quad \text{and} \quad v^* := \varphi^{-1} \circ v \circ \varphi$$

are in  $\mathcal{H}_{\varphi^{-1}(b)}$  and in  $\mathcal{V}_{\varphi^{-1}(a)}$ , respectively.

Now, let  $\varphi : [0, 1] \rightarrow [0, 1]$  be an increasing and concave bijection and denote by  $(C_{h,v})_\varphi$  the class of all copulas  $C_\varphi$  where  $C \in C_{h,v}$ . If  $C_{h^*,v^*}$  and  $C^{h^*,v^*}$  are the bounds in  $C_{h^*,v^*}$  given by Theorem 3, then, for all  $x$  and  $y$  in  $[0, 1]$ ,

$$C_{h^*,v^*}(x, y) \leq (C_{h,v})_\varphi(x, y) \quad \text{and} \quad C^{h^*,v^*}(x, y) \geq (C^{h,v})_\varphi(x, y).$$

**Example 3.** Let  $h(t) = v(t) = t/2$  be the horizontal and vertical sections at  $a = b = 1/2$ . Given  $\varphi(t) = \sqrt{t}$ , we have  $h^*(t) = v^*(t) = t/4$ , and, for every  $(x, y) \in [0, 1/4]^2$ ,

$$C_{h^*,v^*}(x, y) = \frac{1}{4} \max \left\{ 0, x + y - \frac{1}{4} \right\}, \quad C^{h^*,v^*}(x, y) = \frac{1}{4} \min\{x, y\},$$

and

$$(C_{h,v})_\varphi(x, y) = \frac{1}{4} \left( \max \left\{ 0, \sqrt{x} + \sqrt{y} - \frac{1}{2} \right\} \right)^2, \quad (C^{h,v})_\varphi(x, y) = \frac{1}{4} \min\{x, y\}.$$

Therefore,  $C^{h^*,v^*} = (C^{h,v})_\varphi$  and  $C_{h^*,v^*} < (C_{h,v})_\varphi$ .

### 4.3. Copulas with a given value at a point

Now, we use our method for the construction of an absolutely continuous copula such that the value of the copula is specified at a single interior point of  $[0, 1]^2$ .

Specifically, given  $(a, b)$  in  $]0, 1]^2$  and  $\theta$  in  $[0, 1]$  such that  $\max\{a + b - 1, 0\} < \theta < \min\{a, b\}$ , we want to construct a copula  $\tilde{C}$  such that  $\tilde{C}(a, b) = \theta$ . To this end, consider

$$h(t) = \begin{cases} \frac{\theta t}{a}, & t \in [0, a]; \\ \frac{b - \theta}{1 - a}(t - a) + \theta, & t \in [a, 1]; \end{cases}$$

and

$$v(t) = \begin{cases} \frac{\theta t}{b}, & t \in [0, b]; \\ \frac{a - \theta}{1 - b}(t - b) + \theta, & t \in [b, 1]. \end{cases}$$

It is easily proved that  $h$  and  $v$  are in  $\mathcal{H}_b$  and  $\mathcal{V}_a$ , respectively, and the required copula  $\tilde{C}$  that satisfies  $\tilde{C}(a, b) = \theta$  is given by Theorem 1. Notice that the smallest and greatest copulas with given value at a point of  $[0, 1]^2$  are given in [20, Theorem 3.2.3].

**Remark 2.** The above copula  $\tilde{C}$  has a mass  $\theta$  uniformly distributed over the rectangle  $[0, a] \times [0, b]$ , a mass  $a - \theta$  uniformly distributed over the rectangle  $[0, a] \times [b, 1]$ , a mass  $b - \theta$  uniformly distributed over the rectangle  $[a, 1] \times [0, b]$ , and a mass  $1 - a - b + \theta$  uniformly distributed over the rectangle  $[a, 1] \times [b, 1]$  (see [20] for more details).

### 4.4. Copulas defined on a grid

Let  $C$  be a copula and let  $a_i, b_j$  be in  $[0, 1]$ ,  $i \in \{1, 2, \dots, n\}$  and  $j \in \{1, 2, \dots, m\}$ , such that  $a_i < a_{i+1}$  and  $b_j < b_{j+1}$ . Let  $\{h_j\}_j$  be the horizontal sections of  $C$  at the point  $b_j$ ,  $h_j = h_{C,b_j}$ , and, analogously, let  $\{v_i\}_i$  be the vertical sections of  $C$  at the point  $a_i$ ,  $v_i = v_{C,a_i}$ . Intuitively, we construct a grid in the unit square in the

following way: divide  $[0, 1]^2$  into vertical strips by means of vertical straight lines through the points  $(a_i, 0)$ , and into horizontal strips by means of horizontal straight lines going through the points  $(0, b_j)$ .

The question is whether there exists another copula  $D$  that takes the same values as  $C$  on this grid and which is different from  $C$ . By using the same arguments of the proof of Theorem 1 (and by considering several cases), we can construct an absolutely continuous copula  $D$  having the properties just specified. Moreover, in a similar way, it is also possible to determine the smallest and the greatest copula with given values on a grid (see, also, [2] for a deep discussion on this topic).

In particular, let  $\{(a_i, b_i)\}$  be a finite set of points in  $[0, 1]^2$ , let  $C$  be a copula, and suppose that  $C(a_i, b_i) = \theta_i$ . Then, we can always construct another copula  $D$  that coincides with  $C$  at the points  $(a_i, b_i)$  but which differs from  $C$ . In fact, it is sufficient to construct a suitable system of horizontal and vertical sections by means of the triple  $(a_i, b_i, \theta_i)$  (for example, by using piecewise linear sections), and, then, associate to it a copula  $D$ . As a consequence, the extension of a discrete copula to a copula is not, in general, unique (see [16]).

## 5. CONCLUSIONS

We have studied the class of copulas with given values on a horizontal and a vertical section, called cross copulas. In this class, we have provided absolutely continuous copulas and the lower and upper (pointwise) bounds. The methods have been, hence, applied in some special constructions: symmetric copulas with given horizontal sections, transformations of copulas, copulas with given values on a point or on a grid of the unit square.

Motivated by the statistical interpretation of the (horizontal and vertical) sections of a copulas, it could be also of interest to consider the construction of  $n$ -dimensional copulas, with  $n \geq 3$ , with given values on some sections of  $[0, 1]^n$  or, in general, with known values on special  $(n - 1)$ -dimensional subdomains of  $[0, 1]^2$ . This topic will be the object of future investigations.

## ACKNOWLEDGEMENTS

The second author was supported by the grant VEGA 1/3012/06, the third by the grant VEGA 1/4209/07. Both these authors also acknowledge the support of Science and Technology Assistance Agency under the contract No. APVT-20-003204. The fourth author gratefully acknowledges the support of the project “Metodi Stocastici in Finanza Matematica” of the Italian M.I.U.R.

(Received November 11, 2006.)

## REFERENCES

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- [1] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar: Aggregation operators: Properties, classes and construction methods. In: Aggregation Operators (T. Calvo, G. Mayor, and R. Mesiar, eds.), Physica-Verlag, Heidelberg 2002, pp. 3–107.

- [2] H. Carley: Maximum and minimum extensions of finite subcopulas. *Comm. Statist. Theory Methods* 31 (2002), 2151–2166.
- [3] U. Cherubini, E. Luciano, and W. Vecchiato: *Copula Methods in Finance*. Wiley, New York 2004.
- [4] B. De Baets and H. De Meyer: Orthogonal grid construction for copulas. *IEEE Trans. Fuzzy Systems* (2007), to appear.
- [5] F. Durante, R. Mesiar, P. L. Papini, and C. Sempi: 2-increasing binary aggregation operators. *Inform. Sci.* 177 (2007), 111–129.
- [6] F. Durante and C. Sempi: Copula and semicopula transforms. *Internat. J. Math. Sci.* 2005 (2005), 645–655.
- [7] A. Erdelyi and J. M. González-Barrios: On the construction of families of absolutely continuous copulas with given restrictions. *Comm. Statist. Theory Methods* 35 (2006), 649–659.
- [8] J. C. Fodor and M. Roubens: *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht 1994.
- [9] E. W. Frees and E. A. Valdez: Understanding relationships using copulas. *North Amer. Act. J.* 2 (1998), 1–25.
- [10] C. Genest and A.-C. Favre: Everything you always wanted to know about copula modeling but were afraid to ask. *J. Hydrologic Engrg.* 12 (2007), to appear.
- [11] H. Joe: *Multivariate Models and Dependence Concepts*. Chapman & Hall, London 1997.
- [12] E. P. Klement and A. Kolesárová: Extensions to copulas and quasi-copulas as special 1-Lipschitz aggregation operators. *Kybernetika* 41 (2005), 329–348.
- [13] E. P. Klement, A. Kolesárová, R. Mesiar, and C. Sempi: Copulas constructed from the horizontal section. *Comm. Statist. Theory Methods*, to appear.
- [14] E. P. Klement, R. Mesiar, and E. Pap: *Triangular Norms*. Kluwer Academic Publishers, Dordrecht 2000.
- [15] E. P. Klement, R. Mesiar, and E. Pap: Transformations of copulas. *Kybernetika* 41 (2005), 425–434.
- [16] A. Kolesárová, R. Mesiar, J. Mordelová, and C. Sempi: Discrete copulas. *IEEE Trans. Fuzzy Systems* 14 (2006), 698–705.
- [17] A. J. McNeil, R. Frey, and P. Embrechts: *Quantitative Risk Management. Concepts, Techniques and Tools*. Princeton University Press, Princeton, N.J. 2005.
- [18] R. Mesiar and J. Szolgay: W-ordinals sum of copulas and quasi-copulas. In: *Proc. MAGIA 2004 Conference, Kočovce 2004*, pp. 78–83.
- [19] P. M. Morillas: A method to obtain new copulas from a given one. *Metrika* 61 (2005), 169–184.
- [20] R. B. Nelsen: *An Introduction to Copulas*. Springer, New York 2006.
- [21] R. B. Nelsen, J. J. Quesada-Molina, and J. A. Rodríguez-Lallena: Bivariate copulas with cubic sections. *J. Nonparametr. Statist.* 7 (1997), 205–220.
- [22] H. T. Nguyen and E. A. Walker: *A First Course in Fuzzy Logic*. Chapman & Hall/CRC, Boca Raton 2006.
- [23] J. J. Quesada-Molina and J. A. Rodríguez-Lallena: Bivariate copulas with quadratic sections. *J. Nonparametr. Statist.* 5 (1995), 323–337.
- [24] G. Salvadori, C. De Michele, N. T. Kottegoda, and R. Rosso: *Extremes in Nature. An Approach Using Copulas*. (WTS Library Series, Vol. 56.) Springer-Verlag, Berlin 2007.
- [25] B. Schweizer and A. Sklar: *Probabilistic Metric Spaces*. Elsevier, New York 1983.
- [26] A. Sklar: Fonctions de répartition à  $n$  dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* 8 (1959), 229–231.
- [27] A. Sklar: Random variables, joint distribution functions, and copulas. *Kybernetika* 9 (1973), 449–460.

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