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# ON THE STRUCTURE OF CONTINUOUS UNINORMS

Paweł Drygaś

Uninorms were introduced by Yager and Rybalov [13] as a generalization of triangular norms and conorms. We ask about properties of increasing, associative, continuous binary operation $U$ in the unit interval with the neutral element $e \in [0,1]$. If operation $U$ is continuous, then $e = 0$ or $e = 1$. So, we consider operations which are continuous in the open unit square. As a result every associative, increasing binary operation with the neutral element $e \in (0,1)$, which is continuous in the open unit square may be given in $[0,1)^2$ or $(0,1]^2$ as an ordinal sum of a semigroup and a group. This group is isomorphic to the positive real numbers with multiplication. As a corollary we obtain the results of Hu, Li [7].

*Keywords:* uninorms, continuity, $t$-norms, $t$-conorms, ordinal sum of semigroups

*AMS Subject Classification:* 06F05, 03E72, 03B52

## 1. INTRODUCTION

Uninorms were introduced by Yager and Rybalov [13] as a generalization of triangular norms and conorms. However similar operations were considered in [3] and [4]. In [6] Fodor, Yager and Rybalov examined a general structure of uninorms. For example, the frame structure of uninorms and characterization of representable uninorms are presented.

In this paper we consider a more general class of operations than uninorms, i. e. operations from the class $\mathcal{U}(e) = \{U : [0,1]^2 \to [0,1] : U$ is an increasing, associative binary operation with the neutral element $e\}$ for $e \in [0,1]$, where we omit the assumption about the commutativity. We ask about properties of continuous operation $U$ in $\mathcal{U}(e)$ where $e \in [0,1]$. If operation $U$ is continuous then $e = 0$ or $e = 1$ (cf. [3]). So, we consider operations which are continuous in the open unit square. The structure of operations continuous on another subset of unit square we can find in [6, 11, 12].

First, in the Section 2 we present the notion of uninorms and the frame structure of uninorms. Next we present the construction of ordinal sum of semigroups. In Section 4 we present properties of the operation which is continuous in $(0,1)^2$.
As a result every operation in $\mathcal{U}(e)$ with $e \in (0,1)$, which is continuous in the open unit square may be given in $[0,1)^2$ or $(0,1]^2$ as an ordinal sum of a semigroup and a group. This group is isomorphic to the positive real numbers with multiplication.

Moreover this operation is commutative beyond from two points at the most. As a corollary we obtain results of Hu, Li [7] and Fodor, Yager, Rybalov [6].

## 2. NOTION OF UNINORMS

We discuss the structure of binary operations $U : [0,1]^2 \to [0,1]$.

**Definition 1.** (Yager and Rybalov [13]) An operation $U$ is called a uninorm if it is commutative, associative, increasing and has the neutral element $e \in [0,1]$.

Uninorms are generalizations of triangular norms (case $e = 1$) and triangular conorms (case $e = 0$). In the case $e \in (0,1)$ a uninorm $U$ is composed by using a triangular norm and a triangular conorm.

**Theorem 1.** (Fodor, Yager and Rybalov [6]) If a uninorm $U$ has the neutral element $e \in (0,1)$, then there exist a triangular norm $T$ and a triangular conorm $S$ such that

$$U = \begin{cases} T^* \text{ in } [0,e]^2, \\ S^* \text{ in } [e,1]^2, \end{cases} \tag{1}$$

where

$$\begin{cases} T^*(x,y) = \varphi^{-1}\left(T\left(\varphi(x),\varphi(y)\right)\right), \ \varphi(x) = x/e, & x,y \in [0,e], \\ S^*(x,y) = \psi^{-1}\left(S\left(\psi(x),\psi(y)\right)\right), \ \psi(x) = (x-e)/(1-e), & x,y \in [e,1]. \end{cases} \tag{2}$$

**Lemma 1.** (Fodor, Yager and Rybalov [6]) If $U$ is increasing and has the neutral element $e \in (0,1)$ then

$$\min \le U \le \max \ \text{ in } A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e). \tag{3}$$

Furthermore, if $U$ is associative, then $U(0,1), U(1,0) \in \{0,1\}$.

**Theorem 2.** (Li and Shi [10]) Let $e \in (0,1)$. If $T$ is an arbitrary triangular norm and $S$ is an arbitrary triangular conorm then formula (1) with $U = \min$ or $U = \max$ in $A(e)$ gives uninorms.

**Remark 1.** Uninorms from Theorem 2 are not continuous in some points such that one of the variables is equal to the neutral element.

**Example 1.** (Fodor, Yager and Rybalov [6]) Formula

$$U(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \frac{xy}{(1-x)(1-y)+xy}, & \text{if } x > 0 \text{ and } y > 0 \end{cases}$$

gives a uninorm with $e = \frac{1}{2}$, $T(x,y) = \frac{xy}{2-(x+y-xy)}$, $S(x,y) = \frac{x+y}{1+xy}$, $x,y \in [0,1]$. This uninorm is continuous apart from the points $(0,1)$ and $(1,0)$.

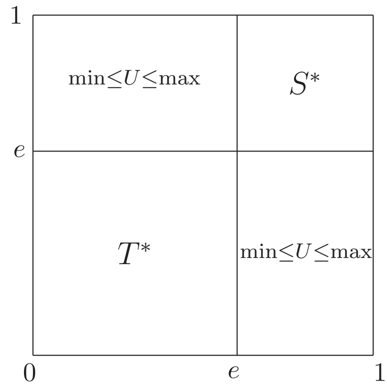


**Fig. 1.** Frame structure of uninorm $U$ with neutral element $e$.

**Theorem 3.** (Czogała and Drewniak [3]) If a uninorm is continuous then $e = 0$ or $e = 1$.

## 3. REMARK ABOUT THE ORDINAL SUM THEOREM

In this section we consider the ordinal sum and dual ordinal sum of semigroups. Next we present the characterization of continuous $t$-norms and $t$-conorms by using the ordinal sum theorem. Additional information about the ordinal sum of semigroups one may find in [1, 2, 5, 8, 9, 12].

**Theorem 4.** (Clifford [1], Climescu [2]) If $(X, F)$, $(Y, G)$ are disjoint semigroups then $(X \cup Y, H)$ is a semigroup, where $H$ is given by

$$H(x,y) = \begin{cases} F(x,y), & \text{if } \ x, y \in X, \\ G(x,y), & \text{if } \ x, y \in Y, \\ x, & \text{if } \ x \in X, \ y \in Y, \\ y, & \text{if } \ x \in Y, \ y \in X. \end{cases} \tag{4}$$

By duality we obtain

**Theorem 5.** (Drewniak and Drygaś [5]) If $(X, F)$, $(Y, G)$ are disjoint semigroups, then $(X \cup Y, H)$ is a semigroup, where $H$ is given by

$$H(x,y) = \begin{cases} F(x,y), & \text{if } \ x, y \in X, \\ G(x,y), & \text{if } \ x, y \in Y, \\ y, & \text{if } \ x \in X, \ y \in Y, \\ x, & \text{if } \ x \in Y, \ y \in X. \end{cases} \tag{5}$$

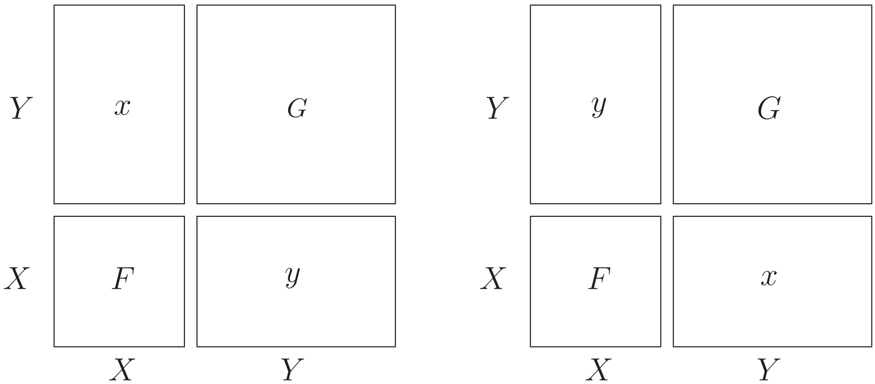


**Fig. 2.** Ordinal sum (left) and dual ordinal sum (right) of semigroups $(X, F)$ and $(Y, G)$.

For our consideration it will be useful to remember the characterization of continuous $t$-norms or $t$-conorms by using ordinal sum theorems.

**Theorem 6.** (Klement, Mesiar and Pap [9], p. 128, Sander [12]) Operation $T : [0,1]^2 \to [0,1]$ is continuous, associative, increasing, with the neutral element $e = 1$ iff there exists a family $\{(a_k, b_k)\}_{k\in A}$ (where $A \subset \mathbb{Q} \cap [0,1]$) of nonempty, pairwise disjoint, open subintervals of $[0,1]$ such that the operations $T_k = T|_{[a_k,b_k]^2}$ are continuous, increasing, associative with Archimedean property, neutral element $b_k$ and $T$ is given by

$$T(x,y) = \begin{cases} T_k(x,y), & \text{for } (x,y) \in (a_k,b_k]^2, \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{6}$$

Moreover, the operation $T$ is commutative.

**Theorem 7.** (Klement, Mesiar and Pap [9], p. 130) Operation $S : [0,1]^2 \to [0,1]$ is continuous, associative, increasing, with the neutral element $e = 0$ iff there exists a family $\{(a_k, b_k)\}_{k\in A}$ (where $A \subset \mathbb{Q} \cap [0,1]$) of nonempty, pairwise disjoint, open subintervals of $[0,1]$ such that the operations $S_k = S|_{[a_k,b_k]^2}$ are continuous, increasing, associative with Archimedean property, neutral element $a_k$ and $S$ is given by

$$S(x,y) = \begin{cases} S_k(x,y), & \text{for } (x,y) \in [a_k,b_k)^2, \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{7}$$

Moreover, the operation $S$ is commutative.

## 4. MAIN RESULTS

In Theorems 6 and 7 a characterization of continuous operations in the class $\mathcal{U}(1)$ and $\mathcal{U}(0)$ respectively is given. Moreover, if operation in the class $\mathcal{U}(e)$ is continuous, then $e = 0$ or $e = 1$ (see Theorem 3). Thus, we ask about the structure of operations in the class $\mathcal{U}(e)$ which are continuous in the open unit square for $e \in (0,1)$.

**Lemma 2.** Let $e \in (0,1)$. If operation $U \in \mathcal{U}(e)$ is continuous in $(0,1)^2$ then operation $U|_{[0,e]^2}$ is isomorphic to a continuous $t$-norm and $U|_{[e,1]^2}$ is isomorphic to a continuous $t$-conorm.

P r o o f. First we prove that operation $U|_{[e,1]^2}$ is continuous. The operator $U$ is continuous in $(0,1)^2$. From this we obtain the continuity of the operation $U|_{[e,1]^2}$ in $[e,1)^2$. Moreover $U(x,y) \geq \max(x,y)$ for $x, y \in [e,1]$ and $U(x,1) = U(1,x) = 1$ for $x \in [e,1]$. Let $x, y \in [e,1]$, then $1 \geq U(x,y) \geq \max(x,y)$, $\lim_{x\to 1} \max(x,y) = 1$ and $\lim_{y\to 1} \max(x,y) = 1$. It means that $\lim_{x\to 1} U(x,y) = 1$ and $\lim_{y\to 1} U(x,y) = 1$, i. e. functions $U(x,t)$ and $U(t,y)$, $t \in [e,1]$ are continuous for all $x, y \in [e,1]$. This implies continuity of the operation $U|_{[e,1]^2}$. It means, that $U|_{[e,1]^2}$ is a continuous, associative, increasing operation with neutral element $e$, then it is isomorphic to a continuous $t$-conorm.

In similar way we obtain that the operation $U|_{[0,e]^2}$ is isomorphic to a continuous $t$-norm. □

**Lemma 3.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$. If there exists $a \in [0,e)$ such that $U(x,y) = x$ for $x \in (a,e)$, $y \in (e,1)$ or $U(x,y) = y$ for $x \in (e,1)$, $y \in (a,e)$ then $U$ is not continuous in $(0,1)^2$.

P r o o f. Let $U(x,y) = x$ for $x \in (a,e)$, $y \in (e,1)$. Take $s \in (e,1)$ and let $f(t) = U(t,s)$, $t \in [0,1]$. We have $f(t) = U(t,s) = t < e$ for $t \in (a,e)$ and $f(e) = s > e$. It means, that the function $f$ is not continuous at the point $e$. This implies, that $U$ is not continuous in $(0,1)^2$.

In similar way as above we obtain the second part of Lemma. □

In the next part of this paper we need the following lemmas

**Lemma 4.** (Klement, Mesiar and Pap [9]) Let $J = [a,b]$ and $F : J^2 \to J$ be associative, increasing operation with the neutral element $b$. If $x \in J$ is an idempotent element of operation $F$ and functions $f(t) = F(x,t)$, $h(t) = F(t,x)$, $t \in J$ are continuous in $J$ then $F(x,y) = F(y,x) = \min(x,y)$ for $y \in J$.

**Lemma 5.** Let $J = [a,b]$ and $F : J^2 \to J$ be associative, increasing operation with the neutral element $a$. If $x \in J$ is an idempotent element of operation $F$ and functions $f(t) = F(x,t)$, $h(t) = F(t,x)$, $t \in J$ are continuous in $J$ then $F(x,y) = F(y,x) = \max(x,y)$ for $y \in J$.

**Lemma 6.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists $b \in (0,e)$ such that $U(b,y) = b$ for $y \in (b,e)$ or $U(x,b) = b$ for $x \in (b,e)$ then $U(x,y) = U(y,x) = \min(x,y)$ for $x \in [0,b]$ and $y \in [b,1)$.

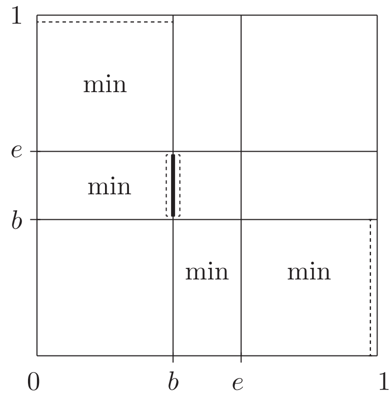


**Fig. 3.** The operation $U$ from the Lemma 6.

Proof. Let $x \in [0, b]$ and $y \in (e, 1)$. For all $t \in (b, e)$ we have $U(b, t) = b$. By the continuity of the operation $U$ we have $U(b, b) = b$. This means that $b$ is an idempotent element of the continuous operation $U|_{[0,e]^2}$ and by Lemma 4 we have $U(b, t) = U(t, b) = \min(t, b)$ for $t \in [0, e]$. Hence, by monotonicity of $U$ we have $U(s, t) = \min(s, t)$ for $s \in [0, b]$, $t \in [b, e]$.

Suppose that there exists $z \in (e, 1)$ such that $U(b, z) \geq e$. By continuity of the operation $U$ and condition $U(b, e) = b$ there exists $w \in (e, z]$ such that $U(b, w) = e$. Then

$$b = U(b, e) = U(b, U(b, w)) = U(U(b, b), w) = U(b, w) = e,$$

which is a contradiction. Therefore $U(b, y) < e$ for all $y \in (e, 1)$. By continuity of the operation $U$ and condition $U(e, y) = y$ there exists $v \in (b, e)$ such that $U(v, y) = e$. Therefore for all $x \leq b$ we have

$$U(x, y) = U(\min(x, v), y) = U(U(x, v), y) = U(x, U(v, y)) = U(x, e) = x.$$

By commutativity of the operation $U|_{[0,e]^2}$ we obtain $U(y, x) = x$ for $x \in [0, b]$ and $y \in [b, e]$. In similar way as above we obtain $U(y, x) = \min(x, y)$ for $x \in [0, b]$, $y \in [b, 1)$. If we assume that $U(x, b) = b$ for $x \in (b, e)$ then the proof is analogous.□

By duality we obtain

**Lemma 7.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. If there exists $a \in (e, 1)$, such that $U(a, y) = a$ for $y \in (e, a)$ or $U(x, a) = a$ for $x \in (e, a)$ then $U(x, y) = U(y, x) = \max(x, y)$ for $x \in [a, 1]$ and $y \in (0, a]$.

**Lemma 8.** (cf. Hu and Li [7]) Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. Then there exist idempotent elements $a \in [0, e)$ and $b \in (e, 1]$ such that operations $U|_{(a,e]^2}$ and $U|_{[e,b)^2}$ are strictly increasing. Moreover $a = 0$ or $b = 1$.

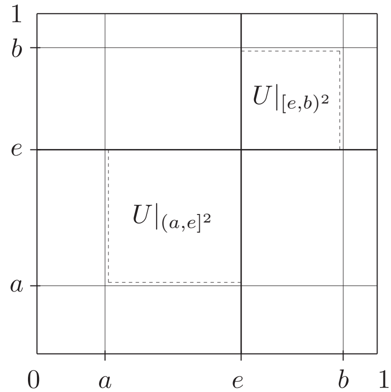


**Fig. 4.** The operation $U \in \mathcal{U}(e)$ from Lemma 8.

P r o o f . By Lemma 2 operation $U|_{[0,e]^2}$ is isomorphic to a continuous $t$-norm. By Theorem 6 there exists a countably family of intervals $(a_k, b_k) \subset [0, e]$ such that $U|_{[0,e]^2}$ is an ordinal sum of semigroups $T_k = U|_{[a_k,b_k]^2}$ with Archimedean property or $T_k = \min$.

Suppose that there does not exist such $a \in [0, e)$ that $U|_{[a,e]^2}$ is a semigroup with Archimedean property. Then there exists $r \in [0, e)$ such that $U|_{[r,e]^2} = \min$ or for every neighborhood of the point $e$ there exists $k$ such that interval $(a_k, b_k)$ is included in that neighborhood, i. e. there exists an increasing subsequence $\{b_{k_n}\}$ of sequence $\{b_k\}$ convergent to $e$. So, we construct the sequence of idempotent elements $\{c_n\}$, e. g. $c_n = e - \frac{1}{n+\left[\frac{1}{e-r}\right]} \in [r, e)$ in the first case, and $c_n = b_{k_n}$ in the second case. According to (6) we have $U(c_n, y) = c_n$ for all $y \in (c_n, e)$. By Lemma 6, $U(x, y) = x$ for $x \in [0, c_n]$ and $y \in (e, 1)$. It implies that $U(x, y) = x$ for $x \in [0, e) = \bigcup_{n=1}^{\infty}[0, c_n]$ and $y \in (e, 1)$. Now, by Lemma 3, operation $U$ is not continuous in $(0, 1)^2$, which is a contradiction. So, there exists $a \in [0, e)$ such that $U|_{[a,e]^2}$ is isomorphic to a continuous Archimedean $t$-norm. Moreover $a$ is an idempotent element of operation $U$ and the zero element of operation $U|_{[a,e]^2}$.

Now we show that $U|_{(a,e]^2}$ is strictly increasing. Suppose that it is not. It means that $U|_{[a,e]^2}$ is isomorphic to the Łukasiewicz $t$-norm $T_L$. By continuity of $U$ there exist $p \in (a, e)$ and $w \in (e, 1)$ such that $U(p, w) = e$. By the fact that $U|_{[a,e]^2}$ is isomorphic to $T_L$ (all elements from $(a, e)$ are zero divisors, where zero element is equal to $a$) it follows that $U(p, q) = U(q, p) = a$ for some $q \in (a, e)$ and by monotonicity of operation $U$ and because $U(a, a) = a$ we have $U(t, p) = a$ for all $t \in [a, q]$. Therefore $U(t, U(p, w)) = U(t, e) = t$ and $U(U(t, p), w) = U(a, w)$. By associativity of $U$ we have $U(a, w) = t$ for all $t \in [a, q]$, which leads to a contradiction. Thus $U|_{(a,e]^2}$ is strictly increasing.

In similar way we prove that there exists idempotent element $b \in (e, 1]$, which is the zero element of $U|_{[e,b]^2}$, such that $U|_{[e,b)^2}$ is strictly increasing.

Suppose that $a > 0$ and $b < 1$. Since $U(a, y) = a$ for all $y \in (a, e)$, Lemma 6 implies that $U(x, y) = \min(x, y)$ for $x \in [0, a]$ and $y \in (e, 1)$. Similarly, since $b$ is the zero element of $U|_{[e,b]^2}$, Lemma 7 implies that $U(x,y) = \max(x,y)$ for $x \in (0,e)$ and $y \in [b,1]$. Therefore $U(x,y) = x$ and $U(x,y) = y$ for $x \in (0,a]$ and $y \in [b,1)$, which is a contradiction.

Accordingly $a = 0$ or $b = 1$. □

**Lemma 9.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists $a \in [0,e)$ such that operations $U|_{(a,e]^2}$ and $U|_{[e,1)^2}$ are strictly increasing then the operation $U|_{(a,1)^2}$ is strictly increasing.

Proof. To show, that $U|_{(a,1)^2}$ is strictly increasing we must show that $U$ is strictly increasing on the set $(a,e] \times [e,1) \cup [e,1) \times (a,e]$. By Lemma 2 operations $U|_{[0,e]^2}$ and $U|_{[e,1]^2}$ are commutative. Let $x, y \in (a,e]$, $x < y$ and $z \in [e,1)$. Suppose that $U(x,z) = U(y,z)$. Then $z > e$ because $U(x,e) = x < y = U(y,e)$.

If $U(x,z) = U(y,z) < e$ then by continuity of $U$ and inequality $U(e,z) = z > e$ there exists $s \in (x,e)$ such that $U(s,z) = e$. Then

$$\begin{aligned} x &= U(x,e) = U(x,U(s,z)) = U(U(x,s),z) = U(U(s,x),z) = U(s,U(x,z)) \\ &= U(s,U(y,z)) = U(U(s,y),z) = U(U(y,s),z) = U(y,U(s,z)) = U(y,e) = y, \end{aligned}$$

which is a contradiction.

If $U(x,z) = U(y,z) \geq e$ then, by continuity of $U$ and condition $U(x,e) = x$, $x < y \leq e$, there exists $c \in (e,z]$ such that $U(x,c) = y$. From $U(y,e) = y \leq e \leq U(y,z)$, there exists $d \in [e,z]$ such that $U(y,d) = e$. Thus $U(e,z) = z$ and

$$\begin{aligned} z &= U(e,z) = U(U(y,d),z) = U(y,U(d,z)) = U(y,U(z,d)) \\ &= U(U(x,c),U(z,d)) = U(x,U(c,U(z,d))) = U(x,U(U(c,z),d)) \\ &= U(x,U(U(z,c),d)) = U(x,U(z,U(c,d))) = U(x,U(z,U(d,c))) \\ &= U(U(x,z),U(d,c)) = U(U(y,z),U(d,c)) = U(y,U(z,U(d,c))) \\ &= U(y,U(U(z,d),c)) = U(y,U(U(d,z),c)) = U(y,U(d,U(z,c))) \\ &= U(U(y,d),U(z,c)) = U(e,U(z,c)) = U(z,c). \end{aligned}$$

Moreover operation $U|_{[e,1)^2}$ is strictly increasing and $z, c \in (e,1)$. This leads to a contradiction. Therefore $U$ is strictly increasing with respect to the first variable in the $(a,e] \times [e,1)$.

Now let $x, y \in [e,1)$, $x < y$ and $z \in (a,e]$. Suppose that $U(z,x) = U(z,y)$. Then $z < e$ because $U(e,x) = x < y = U(e,y)$.
If $U(z,x) = U(z,y) > e$ then, by continuity of $U$ and inequality $U(z,e) = z < e$, there exists $s \in (e,x)$ such that $U(z,s) = e$. Therefore

$$\begin{aligned} x &= U(e,x) = U(U(z,s),x) = U(z,U(s,x)) = U(z,U(x,s)) = U(U(z,x),s) \\ &= U(U(z,y),s) = U(z,U(y,s)) = U(z,U(s,y)) = U(U(z,s),y) = U(e,y) = y, \end{aligned}$$

which is a contradiction.

If $U(z,x) = U(z,y) \le e$ then, by continuity of $U$ and condition $U(e,y) = y$, $e \le x < y$, there exists $c \in (z,e)$ such that $U(c,y) = x$. From $U(e,x) = x > e \ge U(z,x)$ there exists $d \in [z,e]$ such that $U(d,x) = e$. Therefore

$$\begin{aligned} z &= U(z,e) = U(z,U(d,x)) = U(U(z,d),x) = U(U(d,z),x) \\ &= U(U(d,z),U(c,y)) = U(d,U(z,U(c,y))) = U(d,U(U(z,c),y)) \\ &= U(d,U(U(c,z),y)) = U(d,U(c,U(z,y))) = U(U(d,c),U(z,y)) \\ &= U(U(c,d),U(z,x)) = U(U(U(c,d),z),x) = U(U(c,U(d,z)),x) \\ &= U(U(c,U(z,d)),x) = U(U(U(c,z),d),x) = U(U(c,z),U(d,x)) \\ &= U(U(c,z),e) = U(c,z). \end{aligned}$$

Moreover, operation $U|_{(a,e]^2}$ is strictly increasing and $z, c \in (a,e)$. This leads to a contradiction. Thus $U$ is strictly increasing with respect to second variable on $(a,e] \times [e,1)$.

In a similar way we prove that $U$ is strictly increasing on $[e,1) \times (a,e]$. □

**Theorem 8.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists an idempotent element $a \in [0,e)$ of $U$ such that operations $U|_{(a,e]^2}$ and $U|_{[e,1)^2}$ are strictly increasing, then operation $U|_{[0,1)^2}$ is an ordinal sum of continuous semigroup $U|_{[0,a]^2}$ with the neutral element $a$ and continuous group $U|_{(a,1)^2}$ with Archimedean property and the neutral element $e$.

Proof. By Lemma 2, the operation $U|_{[0,e]^2}$ is isomorphic to a continuous $t$-norm and, since $a$ is an idempotent element of this operation, $U|_{[0,a]^2}$ is also isomorphic to a continuous $t$-norm. By Lemma 9, operation $U|_{(a,1)^2}$ is strictly increasing and therefore it is isomorphic to the real numbers with addition. Now, taking into account Lemma 6 we have that $U|_{[0,1)^2}$ is an ordinal sum of the semigroup $U|_{[0,a]^2}$ and the group $U|_{(a,1)^2}$. □

Similarly, we obtain the following results:

**Lemma 10.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists $b \in (e,1]$ such that operations $U|_{(0,e]^2}$ and $U|_{[e,b)^2}$ are strictly increasing then the operation $U|_{(0,b)^2}$ is strictly increasing.

**Theorem 9.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists an idempotent element $b \in (e,1]$ of $U$ such that operations $U|_{(0,e]^2}$ and $U|_{[e,b)^2}$ are strictly increasing then operation $U|_{(0,1]^2}$ is a dual ordinal sum of continuous group $U|_{(0,b)^2}$ with Archimedean property and the neutral element $e$ and continuous semigroup $U|_{[b,1]^2}$ with the neutral element $b$.

So, we have the characterization of this operation in the open unit square. Now we ask about it's structure on the boundary.

**Lemma 11.** Let $e \in (0,1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists an idempotent element $a \in [0,e)$ of $U$ such that operations $U|_{(a,e]^2}$ and $U|_{[e,1)^2}$ are strictly increasing then there exist idempotent elements $c, d \in [0,a]$ of operation $U$ such that

$$U(x,1) = \begin{cases} x, & \text{if } x \in [0,c), \\ 1, & \text{if } x \in (c,1], \\ x \text{ or } 1, & \text{if } x = c, \end{cases} \tag{8}$$

$$U(1,x) = \begin{cases} x, & \text{if } x \in [0,d), \\ 1, & \text{if } x \in (d,1], \\ x \text{ or } 1, & \text{if } x = d. \end{cases} \tag{9}$$

Moreover $c = d$.

Proof. By the Lemma 1, $U(0,1) = 0$ or $U(0,1) = 1$. If $U(0,1) = 1$ then by monotonicity of $U$ we have $U(x,1) = 1$ for $x \in [0,1]$. Therefore we obtain (8) for $c = 0$. Moreover 0 is an idempotent element of the operation $U$.
If $U(0,1) = 0$ then by Theorem 9 the semigroup $U|_{(a,1)^2}$ is isomorphic to the real numbers with addition. Thus we have $\lim_{y\to 1} U(x,y) = 1$ for $x \in (a,1)$ and by monotonicity of the operation $U$ we obtain $U(x,1) = 1$ for $x \in (a,1]$. Let $x \in (0,a]$. First we will prove that $U(x,1) = x$ or $U(x,1) = 1$. Suppose that there exists $z \in (0,a]$ such that $z < U(z,1) < 1$ and let $w = U(z,1)$.
If $w \in (a,1)$ then for $y \in (e,1)$, by associativity of $U$ and strictly monotonicity of $U|_{(a,1)^2}$, we obtain

$$\begin{aligned} w &= U(z,1) = U(z,U(y,1)) = U(z,U(1,y)) \\ &= U(U(z,1),y) = U(w,y) > U(w,e) = w, \end{aligned}$$

which is a contradiction.

If $w \in (z,a]$ then by the conditions $U(0,w) = 0$, $U(e,w) = w$ and continuity of $U|_{[0,e]^2}$ there exists $v \in (0,e)$ such that $U(v,w) = z$ and by associativity of $U$, we obtain

$$\begin{aligned} w &= U(z,1) = U(U(v,w),1) = U(U(v,U(z,1)),1) \\ &= U(U(v,z),U(1,1)) = U(U(v,z),1) = U(v,U(z,1)) = U(v,w) = z, \end{aligned}$$

which is a contradiction. Therefore $U(x,1) = x$ or $U(x,1) = 1$ for $x \in [0,1]$.

Thus, for $c = \inf\{x \in [0,a] : \ U(x,1) = 1\}$ we obtain (8), moreover $c \in [0,a]$.

Let $x \in (0,c)$, $y \in (c,e]$ then we have

$$\begin{aligned} U(x,y) &= U(y,x) = U(y,U(x,1)) = U(U(y,x),1) \\ &= (U(x,y),1) = U(x,U(y,1)) = U(x,1) = x = \min(x,y). \end{aligned}$$

By monotonicity of $U$ and inequality $U|_{[0,e]^2} \leq \min$ we obtain $U(c,y) = c$ for $y \in (c,e)$. By above and continuity of $U$ we have $U(c,c) = c$, i. e. $c$ is an idempotent element of operation $U$. Similarly we prove (9).

To prove that $c = d$ suppose that $d < c$. Then there exists $y \in (d, c)$ such that $U(1, y) = 1$ and $U(y, 1) = y$. Taking $z \in (d, y)$ we have $U(1, z) = 1$ and

$$y = U(y, 1) = U(y, U(1, z)) = U(U(y, 1), z) = U(y, z) \leq U(e, z) = z < y,$$

which is a contradiction, thus $d \geq c$.

If we suppose that $d > c$ then there exists $y \in (c, d)$ such that $U(1, y) = y$ and $U(y, 1) = 1$. Taking $z \in (y, d)$ we have

$$z = U(1, z) = U(U(y, 1), z) = U(y, U(1, z)) = U(y, z) \leq U(y, e) = y < z,$$

which is a contradiction. Thus $c = d$. □

**Lemma 12.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. If there exists an idempotent element $b \in (e, 1]$ of $U$ such that operations $U|_{(0,e]^2}$ and $U|_{[e,b)^2}$ are strictly increasing then there exist idempotent elements $p, q \in [b, 1]$ of operation $U$ such that

$$U(x, 0) = \begin{cases} 0, & \text{if } x \in [0, p), \\ x, & \text{if } x \in (p, 1], \\ 0 \text{ or } x, & \text{if } x = p, \end{cases} \tag{10}$$

$$U(0, x) = \begin{cases} 0, & \text{if } x \in [0, q), \\ x, & \text{if } x \in (q, 1], \\ 0 \text{ or } x, & \text{if } x = q. \end{cases} \tag{11}$$

Moreover $p = q$.

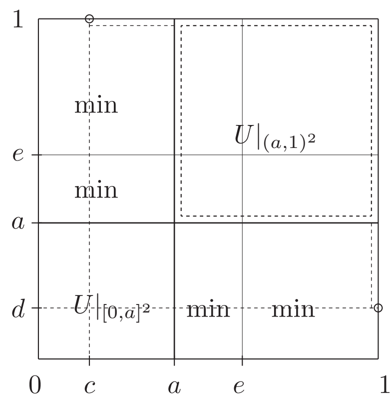


**Fig. 5.** Operation $U \in \mathcal{U}(e)$ continuous in the open unit square with $a > 0$.

As a results of our considerations we obtain

**Theorem 10.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. Then one of the following two cases holds:

(i) There exist idempotent elements $a \in [0, e)$ and $c \in [0, a]$ of operation $U$ such that $U|_{[0,1)^2}$ is an ordinal sum of continuous semigroup $U|_{[0,a]^2}$ with the neutral element $a$ and continuous group $U|_{(a,1)^2}$ with Archimedean property and the neutral element $e$ and conditions (8) and (9) hold.

(ii) There exist idempotent elements $b \in (e, 1]$ and $p \in [b, 1]$ of operation $U$, such that $U|_{(0,1]^2}$ is a dual ordinal sum of continuous semigroup $U|_{[b,1]^2}$ with the neutral element $b$ and continuous group $U|_{(0,b)^2}$ with Archimedean property and the neutral element $e$ and conditions (10) and (11) hold.

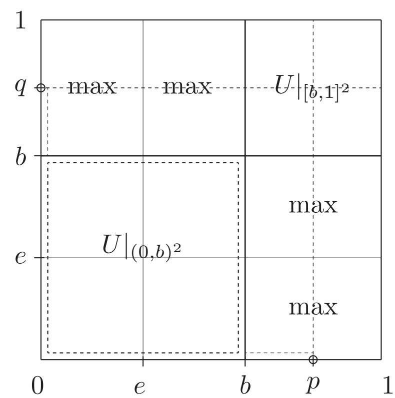


**Fig. 6.** Operation $U \in \mathcal{U}(e)$ continuous in the open unit square with $b < 1$.

P r o o f. By Lemma 8 there exist $a \in [0, e)$ and $b \in (e, 1]$ $(a = 0$ or $b = 1)$ such that $U|_{(a,b)^2}$ is strictly increasing (Lemma 9 and 10).

If $b = 1$ then by Theorem 8 and Lemma 11 we obtain (i).

If $a = 0$ then by Theorem 9 and Lemma 9 we obtain (ii). □

**Remark 2.** Operation $U$ in the previous theorem is commutative in the set

(i) $[0, 1]^2 \setminus \{(c, 1), (1, c)\}$,

(ii) $[0, 1]^2 \setminus \{(0, p), (p, 0)\}$.

## 5. CONCLUSION

By the above consideration we obtain the following results known from the papers [6] and [7]

**Theorem 11.** (Hu and Li [7], Theorem 4.5) Let $e \in (0,1)$ and $U$ be a uninorm which is continuous in $(0,1)^2$. Then $U$ can be represented as follows:

$$\text{(i)}\ \ U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}), & \text{if } x,y \in [0,a], \\ h^{-1}(h(x)+h(y)), & \text{if } x,y \in (a,1), \\ x, & \text{if } x \in [0,a],\ y \in (a,1) \text{ or } x \in [0,c),\ y = 1, \\ y, & \text{if } x \in (a,1), y \in [0,a] \text{ or } x = 1,\ y \in [0,c), \\ 1, & \text{if } x \in (c,1], y = 1 \text{ or } x = 1,\ y \in (c,1], \\ x \text{ or } y, & \text{if } x = c, y = 1 \text{ or } x = 1, y = c, \end{cases}$$

where $a \in [0,e)$, $c \in [0,a]$, $U(c,c) = c$, function $h : [a,1] \to [-\infty,+\infty]$ is strict and $h(a) = -\infty$, $h(e) = 0$, $h(1) = +\infty$;

$$\text{(ii)}\ \ U(x,y) = \begin{cases} e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}), & \text{if } x,y \in [b,1], \\ h^{-1}(h(x)+h(y)), & \text{if } x,y \in (0,b), \\ y, & \text{if } x \in (0,b),\ y \in [b,1] \text{ or } x = 0,\ y \in (p,1], \\ x, & \text{if } x \in [b,1],\ y \in (0,b) \text{ or } x \in (p,1],\ y = 0, \\ 0, & \text{if } x = 0, y \in [0,p) \text{ or } x \in [0,p),\ y = 0, \\ x \text{ or } y, & \text{if } x = p, y = 0, \text{ or } x = 0,\ y = p, \end{cases}$$

where $b \in (e,1]$, $p \in [b,1]$, $U(p,p) = p$, function $h : [0,b] \to [-\infty,+\infty]$ is strict and $h(0) = -\infty$, $h(e) = 0$, $h(b) = +\infty$.

**Theorem 12.** (Fodor, Yager and Rybalkov [6]) Let $e \in (0,1)$ and $U$ be a uninorm continuous without the points $(0,1)$ and $(1,0)$. Then operations $U|_{(0,e]^2}$ and $U|_{[e,1)^2}$ are strictly increasing and

$$U(x,y) = \begin{cases} h^{-1}(h(x)+h(y)), & \text{for } (x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}, \\ 0 \text{ or } 1, & \text{elsewhere}, \end{cases} \tag{12}$$

where $h : [0,1] \to [-\infty,+\infty]$ is an increasing bijection such that $h(e) = 0$.

P r o o f. Operation $U|_{(0,1)^2}$ is continuous. Suppose that in Theorem 10 the condition $(i)$ holds, i.e. there exists $a \in [0,e)$, such that operation $U|_{(a,1)^2}$ is strictly increasing. By Lemma 11 there exists $c \in [0,a]$ such that (8) holds.

Suppose that $c < a$, then for $x \in (c,a)$ and $y \in (e,1)$ we have $U(x,y) = \min(x,y) = x$ and $U(x,1) = 1$. It means that $U$ is not continuous at the points $(x,1)$, $x \in (c,a)$. Therefore $c = a$.

Suppose now, that $a > 0$. By Lemma 11 we have $U(x,1) = x$ for $x \in [0,a)$ and $U(x,1) = 1$ for $x \in (a,1]$. It means that the point $(a,1)$ is a point of discontinuity of the operation $U$, which leads to a contradiction. Thus $a = 0$. Now, directly by the above theorem, we obtain (12). □

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