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MIXED PSEUDO–ASSOCIATIVITIES OF BANDLER – KOHOUT COMPOSITIONS OF RELATIONS

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This paper considers compositions of relations based on the notion of the afterset and the foreset, i. e., the subproduct, the superproduct and the square product introduced by Bandler and Kohout with modification proposed by De Baets and Kerre. There are proven all possible mixed pseudo-associativity properties of Bandler – Kohout compositions of relations.

Keywords: Bandler - Kohout compositions of relation, mixed associativity

AMS Subject Classification: 04A05, 08A02

1. REVISION OF BANDLER – KOHOUT COMPOSITIONS OF RELATIONS

In 1980 Bandler and Kohout [1] introduced new compositions of relations. They extended these compositions to fuzzy relations and presented a list of applications of these compositions in medical diagnosis, generalized morphisms and information retrieval systems. They proved some properties of compositions among others three mixed pseudo-associativities. In [4, 5] we can find discussion about non-associative compositions. Twelve years later, in 1993, De Baets and Kerre [2] suggested little modifications of the original definitions, which will be used in this paper.

First, we will recall a few essential definitions. A relation between elements of two non empty sets X and Y is a subset of the Cartesian product of X and Y, i. e., $R \subseteq X \times Y$. The afterset xR of the element $x \in X$ and the foreset Ry of the element $y \in Y$ are defined, respectively, as

$$xR = \{y \in Y : (x, y) \in R\}, \tag{1}$$

$$Ry = \{x \in X : (x, y) \in R\}. \tag{2}$$

The domain $dom(R)$ and the range $rng(R)$ of a relation R are given as follows

$$dom(R) = \{x \in X : \exists_{y \in Y} (x, y) \in R\},$$

$$rng(R) = \{y \in Y : \exists_{x \in X} (x, y) \in R\}.$$

Denote by R^T the converse relation of a relation $R \subseteq X \times Y$ defined by

$$R^T = \{(y, x) \in Y \times X : (x, y) \in R\}.$$

It is easy to show, that for arbitrary relations $R, S \subseteq X \times Y$ we have

$$(R^T)^T = R, \qquad R \subseteq S \Leftrightarrow R^T \subseteq S^T.$$

One of the important relational calculus is the composition of relations. The classical composition of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is a relation $R \circ S \subseteq X \times Z$ given as follows

$$R \circ S = \{(x,z) \in X \times Z \colon \exists_{y \in Y}\ (x,y) \in R \text{ and } (y,z) \in S\}.$$

In [1], Bandler and Kohout have noticed, that the above definition can be written based on (1) and (2), as

$$R \circ S = \{(x,z) \in X \times Z \colon xR \cap Sz \neq \emptyset\}. \tag{3}$$

Inspired by this style of the notation, authors have introduced three new compositions called: the subcomposition, the supercomposition and the square composition, which are defined, respectively, as

$$\begin{aligned} R \lhd S &= \{(x,z) \in X \times Z \colon xR \subseteq Sz\}, \\ R \rhd S &= \{(x,z) \in X \times Z \colon Sz \subseteq xR\}, \\ R \diamond S &= \{(x,z) \in X \times Z \colon xR = Sz\}. \end{aligned}$$

In [2], De Baets and Kerre, have shown that the above definitions are not acceptable for empty foresets and aftersets, because $R \lhd S$, $R \rhd S$, $R \diamond S$ can contain a lot of unwanted couples. Therefore they have written them in the following ways

$$R \lhd S = \{(x,z) \in X \times Z \colon \emptyset \neq xR \subseteq Sz\}, \tag{4}$$

$$R \rhd S = \{(x,z) \in X \times Z \colon \emptyset \neq Sz \subseteq xR\}, \tag{5}$$

$$R \diamond S = \{(x,z) \in X \times Z \colon \emptyset \neq xR = Sz\}. \tag{6}$$

In this paper we will use the the above formulas. De Baets and Kerre have proved several properties of the above compositions and some relationships between them. Now we will recall basic properties, which will be essential for further discussions (see [2]). Let $R, R_1, R_2 \subseteq X \times Y$ and $S, S_1, S_2 \subseteq Y \times Z$.

- Containment:

$$R \diamond S \subseteq R \lhd S \subseteq R \circ S, \tag{7}$$

$$R \diamond S \subseteq R \rhd S \subseteq R \circ S. \tag{8}$$

- Convertibility:

$$(R \circ S)^T = S^T \circ R^T, \tag{9}$$

$$(R \lhd S)^T = S^T \rhd R^T, \tag{10}$$

$$(R \rhd S)^T = S^T \lhd R^T, \tag{11}$$

$$(R \diamond S)^T = S^T \diamond R^T. \tag{12}$$

- Monotonicity:

$$R_1 \subseteq R_2 \Rightarrow R_1 \circ S \subseteq R_2 \circ S, \tag{13}$$

$$(dom(R_1) = dom(R_2) \wedge R_1 \subseteq R_2) \Rightarrow R_2 \lhd S \subseteq R_1 \lhd S, \tag{14}$$

$$R_1 \subseteq R_2 \Rightarrow R_1 \rhd S \subseteq R_2 \rhd S, \tag{15}$$

$$S_1 \subseteq S_2 \Rightarrow R \circ S_1 \subseteq R \circ S_2, \tag{16}$$

$$S_1 \subseteq S_2 \Rightarrow R \lhd S_1 \subseteq R \lhd S_2, \tag{17}$$

$$(rng(S_1) = rng(S_2) \wedge S_1 \subseteq S_2) \Rightarrow R \rhd S_2 \subseteq R \rhd S_1. \tag{18}$$

- Associativity: It is well-known that the classical composition of relations is associative. In [2] authors have presented six mixed pseudo-associativity properties between the subcomposition, the supercomposition and the classical composition of relations. We will recall them bellow. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \circ (S \circ Q) = (R \circ S) \circ Q, \tag{19}$$

$$R \circ (S \rhd Q) \subseteq (R \circ S) \rhd Q, \tag{20}$$

$$R \lhd (S \circ Q) \supseteq (R \lhd S) \circ Q, \tag{21}$$

$$R \lhd (S \lhd Q) \subseteq (R \circ S) \lhd Q, \tag{22}$$

$$R \lhd (S \rhd Q) = (R \lhd S) \rhd Q, \tag{23}$$

$$R \rhd (S \circ Q) \supseteq (R \rhd S) \rhd Q. \tag{24}$$

A relation $R \subseteq X \times Y$ can be identified with its characteristic mapping, namely

$$\chi_R(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R, \\ 0, & \text{if } (x,y) \notin R. \end{cases}$$

In all examples we consider relations $R, S \subseteq X \times X$, where $X = \{x_1, \dots, x_n\}$, so we use the matrix notation

$$r_{ij} = \chi_R(x_i, x_j), \qquad 1 \le i,j \le n,$$

$$s_{ij} = \chi_S(x_i, x_j), \qquad 1 \le i,j \le n.$$

It is easy to show that the compositions (3) – (6) of matrices are given by (see [3])

$$(R \circ S)_{ij} = \max_{1 \le k \le n} r_{ik} \wedge s_{kj}, \quad 1 \le i,j \le n,$$

$$(R \lhd S)_{ij} = \left(\min_{1 \le k \le n} r_{ik} \Rightarrow s_{kj} \right) \wedge \left(\max_{1 \le k \le n} r_{ik} \right), \quad 1 \le i,j \le n,$$

$$(R \rhd S)_{ij} = \left(\min_{1 \le k \le n} s_{kj} \Rightarrow r_{ik} \right) \wedge \left(\max_{1 \le k \le n} s_{kj} \right), \quad 1 \le i,j \le n,$$

$$(R \diamond S)_{ij} = \left(\min_{1 \le k \le n} r_{ik} \Leftrightarrow s_{kj} \right) \wedge \left(\max_{1 \le k \le n} r_{ik} \right), \quad 1 \le i,j \le n.$$

2. PARTIAL PSEUDO–ASSOCIATIVITIES OF MIXED COMPOSITIONS

The aim of this part is to find another mixed pseudo-associativity properties. We have 256 possibilities to put one of the four signs of compositions. The first series of properties is based on the associativity of the classical composition of relations.

Theorem 1. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \circ (S \circ Q) \supseteq (R \circ S) \star_1 Q, \tag{25}$$
$$R \star_1 (S \circ Q) \subseteq (R \circ S) \circ Q, \tag{26}$$
$$R \circ (S \circ Q) \supseteq (R \star_1 S) \star_2 Q, \tag{27}$$
$$R \star_2 (S \star_1 Q) \subseteq (R \circ S) \circ Q, \tag{28}$$

where $\star_1 \in \{\lhd, \rhd, \diamond\}$ and $\star_2 \in \{\circ, \lhd, \rhd, \diamond\}$.

P r o o f. According to (19), (7) and (8) we have

$$R \circ (S \circ Q) = (R \circ S) \circ Q \supseteq (R \circ S) \rhd Q \supseteq (R \circ S) \diamond Q,$$
$$R \circ (S \circ Q) = (R \circ S) \circ Q \supseteq (R \circ S) \lhd Q,$$

which proves (25) for $\star_1 \in \{\lhd, \rhd, \diamond\}$. Now we will investigate (27) when $\star_1$ is $\lhd$. By properties (7), (13) and (19) we obtain

$$(R \lhd S) \circ Q \subseteq (R \circ S) \circ Q = R \circ (S \circ Q).$$

By the above inclusion and the containment conditions once more we can deduce that

$$(R \lhd S) \diamond Q \subseteq (R \lhd S) \lhd Q \subseteq (R \lhd S) \circ Q \subseteq (R \circ S) \circ Q = R \circ (S \circ Q)$$

and

$$(R \lhd S) \rhd Q \subseteq (R \lhd S) \circ Q \subseteq (R \circ S) \circ Q = R \circ (S \circ Q),$$

so we obtain (27) for $\star_1$ equal $\lhd$ and $\star_2 \in \{\circ, \lhd, \rhd, \diamond\}$. When $\star_1$ is equal $\rhd$ or $\diamond$ the proofs are dual. The inclusions (26) and (28) we can deduce from inclusions (25) and (27), respectively, and the convertibility properties putting $R = Q^T$, $Q = R^T$, $S = S^T$. □

If in (25) we put $\star_1 = \circ$, then we obtain the classical associativity condition (19). So far we intensively have used the associativity of the classical composition. Now we will apply another properties from the mixed pseudo-associativity conditions.

Theorem 2. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \star (S \rhd Q) \subseteq (R \circ S) \rhd Q, \tag{29}$$
$$R \lhd (S \circ Q) \supseteq (R \lhd S) \star Q, \tag{30}$$

where $\star \in \{\circ, \lhd, \rhd, \diamond\}$.

P r o o f. For $\star = \circ$ the inclusion (29) has been proved in [2] (see also (20)). Using this inclusion and the containment properties we obtain

$$R \diamond (S \rhd Q) \subseteq R \lhd (S \rhd Q) \subseteq R \circ (S \rhd Q) \subseteq (R \circ S) \rhd Q,$$
$$R \rhd (S \rhd Q) \subseteq R \circ (S \rhd Q) \subseteq (R \circ S) \rhd Q.$$

The next inclusion we can deduce from (21) or it follows immediately from (29) using the convertibility condition for $R = Q^T, S = S^T, Q = R^T$. For $\star = \lhd$ in (29)

$$\begin{aligned} Q^T \lhd (S^T \rhd R^T) \subseteq (Q^T \circ S^T) \rhd R^T \;&\Leftrightarrow\; (Q^T \lhd (S^T \rhd R^T))^T \subseteq ((Q^T \circ S^T) \rhd R^T)^T \\ &\Leftrightarrow\; (S^T \rhd R^T)^T \rhd Q \subseteq R \lhd (Q^T \circ S^T)^T \\ &\Leftrightarrow\; (R \lhd S) \rhd Q \subseteq R \lhd (S \circ Q). \end{aligned}$$

The proofs of the other inclusions are dual. □

Theorem 3. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \lhd (S \star Q) \supseteq (R \diamond S) \star Q, \tag{31}$$

$$R * (S \diamond Q) \subseteq (R * S) \rhd Q, \tag{32}$$

where $\star \in \{\rhd, \circ\}$ and $* \in \{\lhd, \circ\}$.

Proof. Using properties (7), (13), (15), (21) and (23) we obtain

$$\begin{aligned} (R \diamond S) \circ Q \subseteq (R \lhd S) \circ Q \subseteq R \lhd (S \circ Q), \\ (R \diamond S) \rhd Q \subseteq (R \lhd S) \rhd Q = R \lhd (S \rhd Q). \end{aligned}$$

Inclusions (32) follow immediately from inclusions (31) and convertibility properties as was shown in the previous proof. □

Theorem 4. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \lhd (S \circ Q) \supseteq (R \diamond S) \star Q, \tag{33}$$

where $\star \in \{\lhd, \rhd, \diamond\}$,

$$R * (S \diamond Q) \subseteq (R \circ S) \rhd Q, \tag{34}$$

where $* \in \{\rhd, \lhd, \diamond\}$,

$$(R \diamond S) \star_1 Q \subseteq R \star_2 (S \rhd Q), \tag{35}$$

where $\star_1 \in \{\rhd, \diamond\}$ and $\star_2 \in \{\lhd, \circ\}$,

$$R \star_1 (S \diamond Q) \subseteq (R \lhd S) \star_2 Q, \tag{36}$$

where $\star_1 \in \{\lhd, \diamond\}$ and $\star_2 \in \{\rhd, \circ\}$.

Proof. To prove (33) we will use (31) with $\star = \circ$ and the containment properties, so

$$\begin{aligned} (R \diamond S) \diamond Q \subseteq (R \diamond S) \lhd Q \subseteq (R \diamond S) \circ Q \subseteq R \lhd (S \circ Q), \\ (R \diamond S) \rhd Q \subseteq (R \diamond S) \circ Q \subseteq R \lhd (S \circ Q). \end{aligned}$$

Now we will consider (35). Using again (31) with $\star = \rhd$ and the containment properties we obtain

$$(R \diamond S) \diamond Q \subseteq (R \diamond S) \rhd Q \subseteq R \lhd (S \rhd Q) \subseteq R \circ (S \rhd Q).$$

By comparing the first and the last, the first and the third and finally the second and the last part in the above inclusions we obtain (35). Inclusions (34) and (36) follow immediately from (33) and (35), respectively, and convertibilities properties as was shown in the previous proof. □

Theorem 5. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \diamond (S \lhd Q) \subseteq (R \circ S) \lhd Q, \tag{37}$$

$$R \rhd (S \circ Q) \supseteq (R \rhd S) \diamond Q. \tag{38}$$

Proof. By the containment properties and according to (22), (24) we can conclude that

$$R \diamond (S \lhd Q) \subseteq R \lhd (S \lhd Q) \subseteq (R \circ S) \lhd Q,$$

$$R \rhd (S \circ Q) \supseteq (R \rhd S) \rhd Q \supseteq (R \rhd S) \diamond Q,$$

so we proved (37) and (38). □

Next theorem is based on (23).

Theorem 6. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \diamond (S \rhd Q) \subseteq (R \lhd S) \star_1 Q, \tag{39}$$

$$R \star_2 (S \rhd Q) \supseteq (R \lhd S) \diamond Q, \tag{40}$$

where $\star_1 \in \{\circ, \rhd\}$, $\star_2 \in \{\circ, \lhd\}$, and

$$R \lhd (S \rhd Q) \subseteq (R \lhd S) \circ Q, \tag{41}$$

$$R \circ (S \rhd Q) \supseteq (R \lhd S) \rhd Q. \tag{42}$$

Proof. According to (7), (8) and (23) we obtain

$$R \diamond (S \rhd Q) \subseteq R \lhd (S \rhd Q) = (R \lhd S) \rhd Q \subseteq (R \lhd S) \circ Q,$$

so the proof of this theorem is complete because if we compare the first and the last part of the above inclusions we have (39) for $\star_1 = \circ$. By comparing the first and the third part we obtain (39) for $\star_1 = \rhd$. If we take into consideration the second and the last part we get (41). In the aftermath of (39) and (41) using the convertibility properties we obtain (40) and (42), respectively. □

Theorem 7. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \rhd (S \circ Q) \supseteq (R \diamond S) * Q, \tag{43}$$

where $* \in \{\rhd, \diamond\}$, and

$$R * (S \diamond Q) \subseteq (R \circ S) \lhd Q, \tag{44}$$

where $* \in \{\lhd, \diamond\}$.

Proof. By properties (8), (15) and (24) we obtain

$$(R \diamond S) \rhd Q \subseteq (R \rhd S) \rhd Q \subseteq R \rhd (S \circ Q),$$

so we have (43) for $\star = \rhd$. Using the containment conditions once more we get

$$(R \diamond S) \diamond Q \subseteq (R \diamond S) \rhd Q \subseteq R \rhd (S \circ Q),$$

which finishes the proof of (43). If we put $R = Q^T, S = S^T$ and $Q = R^T$ in (43) and we will use the convertibility properties we can conclude last inclusions. □

Proofs of the next theorems are based on the basic definitions.

Theorem 8. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$(R \circ S) * Q \subseteq R \circ (S \lhd Q), \tag{45}$$

where $* \in \{\lhd, \diamond\}$, and

$$R * (S \circ Q) \subseteq (R \rhd S) \circ Q, \tag{46}$$

where $* \in \{\rhd, \diamond\}$.

Proof. We will prove the inclusion (45) for $* = \lhd$. First note that

$$\begin{aligned}
(x,u) \in{} & R \circ (S \lhd Q) \\
& \Leftrightarrow \exists_y \ [(x,y) \in R \wedge (y,u) \in S \lhd Q] \\
& \Leftrightarrow \exists_y \ [(x,y) \in R \wedge \emptyset \neq yS \subseteq Qu] \\
& \Leftrightarrow \exists_y \ \{(x,y) \in R \wedge \emptyset \neq yS \wedge \forall_z \ [(y,z) \in S \Rightarrow (z,u) \in Q]\} \\
& \Leftrightarrow \exists_y \ \{(x,y) \in R \wedge \exists_z \ (y,z) \in S \wedge \forall_z \ [(y,z) \in S \Rightarrow (z,u) \in Q]\} \\
& \Leftrightarrow \exists_y \ \{(x,y) \in R \wedge \exists_z \ (y,z) \in S \wedge \\
& \qquad \forall_z \ (x,y) \in R \wedge [(y,z) \in S \Rightarrow (z,u) \in Q]\} \\
& \Leftrightarrow \exists_y \ \{(x,y) \in R \wedge \exists_z \ (y,z) \in S \wedge \\
& \qquad \forall_z \ \{(x,y) \in R \wedge [((x,y) \in R \wedge (y,z) \in S) \Rightarrow (z,u) \in Q]\}\} \\
& \Leftrightarrow \exists_y \ \{(x,y) \in R \wedge \exists_z \ (y,z) \in S \wedge \\
& \qquad \forall_z \ [((x,y) \in R \wedge (y,z) \in S) \Rightarrow (z,u) \in Q]\} \\
& \Leftrightarrow \exists_y \ \{(x,y) \in R \wedge \exists_z \ (y,z) \in S \wedge \\
& \qquad \forall_z \ [((x,y) \in R \wedge (y,z) \in S) \Rightarrow (z,u) \in Q]\} \\
& \Leftrightarrow \exists_y \ \{\exists_z \ [(x,y) \in R \wedge (y,z) \in S] \wedge \\
& \qquad \forall_z \ [((x,y) \in R \wedge (y,z) \in S) \Rightarrow (z,u) \in Q]\}\,.
\end{aligned} \tag{A}$$

Further,

$$\begin{aligned}
(x,u) \in{} & (R \circ S) \lhd Q \\
& \Leftrightarrow \emptyset \neq x(R \circ S) \subseteq Qu \\
& \Leftrightarrow \emptyset \neq x(R \circ S) \wedge x(R \circ S) \subseteq Qu \\
& \Leftrightarrow \exists_z \ [(x,z) \in R \circ S] \wedge \forall_z \ [(x,z) \in R \circ S \Rightarrow (z,u) \in Q] \\
& \Leftrightarrow \exists_z \ \{\exists_y \ [(x,y) \in R \wedge (y,z) \in S]\} \wedge \\
& \qquad \forall_z \ \{\exists_y \ [(x,y) \in R \wedge (y,z) \in S] \Rightarrow (z,u) \in Q\} \\
& \Leftrightarrow \exists_z \ \exists_y \ [(x,y) \in R \wedge (y,z) \in S] \wedge \\
& \qquad \forall_z \ \forall_y \ \{[(x,y) \in R \wedge (y,z) \in S] \Rightarrow (z,u) \in Q\} \\
& \Leftrightarrow \exists_y \ \exists_z \ [(x,y) \in R \wedge (y,z) \in S] \wedge \\
& \qquad \forall_y \ \forall_z \ \{[(x,y) \in R \wedge (y,z) \in S] \Rightarrow (z,u) \in Q\}\,.
\end{aligned} \tag{B}$$

By comparing (A) and (B) we see that (A) can be deduced from (B). Using the monotonicity property we have

$$R \diamond (S \circ Q) \subseteq R \lhd (S \circ Q) \subseteq (R \rhd S) \circ Q.$$

The inclusion (46) follows immediately from foregoing property using the convertibility properties. □

Theorem 9. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$R \lhd (S \diamond Q) \subseteq (R \circ S) \diamond Q, \tag{47}$$

$$R \lhd (S \diamond Q) \subseteq (R \lhd S) * Q, \tag{48}$$

where $* = \{\lhd, \diamond\}$, and

$$(R \diamond S) \rhd Q \subseteq R \diamond (S \circ Q), \tag{49}$$

$$(R \diamond S) \rhd Q \subseteq R * (S \rhd Q), \tag{50}$$

where $* \in \{\rhd, \diamond\}$.

P r o o f. First we will prove (47) and (48). In the both cases in the left side of inequalities we obtain

$$\begin{aligned}
(x,u) \in & R \lhd (S \diamond Q) \\
&\Leftrightarrow \emptyset \neq xR \subseteq (S \diamond Q)u \\
&\Leftrightarrow \emptyset \neq xR \wedge (S \diamond Q)u \neq \emptyset \wedge \forall_{y \in xR}\, y \in (S \diamond Q)u \\
&\Leftrightarrow \emptyset \neq xR \wedge \forall_y\ [y \in xR \Rightarrow (y,u) \in S \diamond Q] \\
&\Leftrightarrow \emptyset \neq xR \wedge \forall_y\ [(x,y) \in R \Rightarrow (yS = Qu \neq \emptyset)] \\
&\Leftrightarrow \emptyset \neq xR \wedge Qu \neq \emptyset \wedge yS \neq \emptyset \wedge \forall_y\ \{(x,y) \in R \Rightarrow [\forall_z\ (z \in yS \Leftrightarrow z \in Qu)]\} \\
&\Leftrightarrow \emptyset \neq xR \wedge Qu \neq \emptyset \wedge yS \neq \emptyset \wedge \\
&\qquad \forall_z\ \forall_y\ \{(x,y) \in R \Rightarrow [(y,z) \in S \Leftrightarrow (z,u) \in Q]\}\,,
\end{aligned} \tag{C}$$

while for the right side of (48), for $* = \diamond$, we get

$$\begin{aligned}
(x,u) \in & (R \lhd S) \diamond Q \\
&\Leftrightarrow x(R \lhd S) = Qu \neq \emptyset \\
&\Leftrightarrow Qu \neq \emptyset \wedge x(R \lhd S) \neq \emptyset \wedge \forall_z\ [z \in x(R \lhd S) \Leftrightarrow z \in Qu] \\
&\Leftrightarrow Qu \neq \emptyset \wedge x(R \lhd S) \neq \emptyset \wedge \forall_z\ [(x,z) \in R \lhd S \Leftrightarrow (z,u) \in Q] \\
&\Leftrightarrow Qu \neq \emptyset \wedge x(R \lhd S) \neq \emptyset \wedge \forall_z\ [\emptyset \neq xR \subseteq Sz \Leftrightarrow (z,u) \in Q] \\
&\Leftrightarrow Qu \neq \emptyset \wedge xR \neq \emptyset \wedge Sz \neq \emptyset \wedge \\
&\qquad \forall_z\ [\forall_y\ ((x,y) \in R \Rightarrow (y,z) \in S) \Leftrightarrow (z,u) \in Q]
\end{aligned} \tag{D}$$

Comparing these formulas we see that (D) follows from (C), because

$$\{\exists_y\ \varphi(y) \wedge \forall_y\ [\varphi(y) \Rightarrow (\psi(y) \Leftrightarrow \lambda)]\} \Rightarrow \{\forall_y\ [\varphi(y) \Rightarrow \psi(y)] \Leftrightarrow \lambda\}\,.$$

Now we consider the right side of (47).

$$\begin{aligned}
(x,u) \in & (R \circ S) \diamond Q \\
&\Leftrightarrow x(R \circ S) = Qu \neq \emptyset \\
&\Leftrightarrow Qu \neq \emptyset \wedge x(R \circ S) \neq \emptyset \wedge \forall_z\ [z \in x(R \circ S) \Leftrightarrow z \in Qu] \\
&\Leftrightarrow Qu \neq \emptyset \wedge xR \neq \emptyset \wedge Sz \neq \emptyset \wedge \forall_z\ [(x,z) \in R \circ S \Leftrightarrow (z,u) \in Q] \\
&\Leftrightarrow Qu \neq \emptyset \wedge xR \neq \emptyset \wedge yS \neq \emptyset \wedge \\
&\qquad \forall_z\ [\exists_y\ ((x,y) \in R \wedge (y,z) \in S) \Leftrightarrow (z,u) \in Q]
\end{aligned} \tag{E}$$

By the following property

$$[\exists_y\ \varphi(y) \wedge \forall_y\ (\varphi(y) \Rightarrow \psi(y))] \Rightarrow \exists_y\ (\varphi(y) \wedge \psi(y)) \tag{$*$}$$

we see, that (E) follows from (D), so (E) can be deduced from (C). The inequality (48) for $* = \triangleleft$ we obtain using the monotonicity property

$$R \triangleleft (S \diamond Q) \subseteq (R \triangleleft S) \diamond Q \subseteq (R \triangleleft S) \triangleleft Q.$$

Inequalities (49), (50) can be deduced from the above and the convertibility property. □

Theorem 10. If $R \subseteq X \times Y$, $S \subseteq Y \times Z$ and $Q \subseteq Z \times U$, then

$$(R \diamond S) \diamond Q \subseteq R \diamond (S \circ Q), \tag{51}$$
$$(R \diamond S) \diamond Q \subseteq R * (S \triangleright Q), \tag{52}$$

where $* \in \{\triangleright, \diamond\}$, and

$$R \diamond (S \diamond Q) \subseteq (R \circ S) \diamond Q, \tag{53}$$
$$R \diamond (S \diamond Q) \subseteq (R \triangleleft S) * Q, \tag{54}$$

where $* \in \{\triangleleft, \diamond\}$.

P r o o f. We will prove (51) and (52). First we will consider left side of the above inequalities.

$$\begin{aligned}(x,u) \in & (R \diamond S) \diamond Q \\ & \Leftrightarrow x(R \diamond S) = Qu \neq \emptyset \\ & \Leftrightarrow Qu \neq \emptyset \wedge x(R \diamond S) \neq \emptyset \wedge \forall_z\ [(x,z) \in R \diamond S \Leftrightarrow (z,u) \in Q] \\ & \Leftrightarrow Qu \neq \emptyset \wedge xR \neq \emptyset \wedge Sz \neq \emptyset \wedge \forall_z\ [xR = Sz \neq \emptyset \Leftrightarrow (z,u) \in Q] \\ & \Leftrightarrow Qu \neq \emptyset \wedge xR \neq \emptyset \wedge Sz \neq \emptyset \wedge \\ & \qquad \forall_z\ [\forall_y\ ((x,y) \in R \Leftrightarrow (y,z) \in S) \Leftrightarrow (z,u) \in Q]\,. \end{aligned} \tag{F}$$

The right side of inequality (52) for $* = \diamond$ has the form

$$\begin{aligned}(x,u) \in & R \diamond (S \triangleright Q) \\ & \Leftrightarrow xR = (S \triangleright Q)u \neq \emptyset \\ & \Leftrightarrow xR \neq \emptyset \wedge (S \triangleright Q)u \neq \emptyset \wedge \forall_y\ [(x,y) \in R \Leftrightarrow (y,u) \in S \triangleright Q] \\ & \Leftrightarrow xR \neq \emptyset \wedge yS \neq \emptyset \wedge Qu \neq \emptyset \wedge \\ & \qquad \forall_y\ \{(x,y) \in R \Leftrightarrow \forall_z\ [(z,u) \in Q \Rightarrow (y,z) \in S]\} \end{aligned} \tag{G}$$

The expression (G) follows from (F) because

$$\{\exists_z\ \lambda(z) \wedge \forall_z\ \{\forall_y\ [\varphi(y) \Leftrightarrow \psi(y,z)] \Leftrightarrow \lambda(z)\}\} \Rightarrow \forall_y\ \{\varphi(y) \Leftrightarrow \forall_z\ [\lambda(z) \Rightarrow \psi(y,z)]\}\,.$$

For the right side of the inequality (51) we can write

$$\begin{aligned}(x,u)\in & R\diamond(S\circ Q)\\ &\Leftrightarrow xR=(S\circ Q)u\neq\emptyset\\ &\Leftrightarrow xR\neq\emptyset\wedge(S\circ Q)u\neq\emptyset\wedge\forall_y\ [(x,y)\in R\Leftrightarrow(y,u)\in S\circ Q]\\ &\Leftrightarrow xR\neq\emptyset\wedge yS\neq\emptyset\wedge Qu\neq\emptyset\wedge\\ &\qquad\qquad \forall_y\ \{(x,y)\in R\Leftrightarrow\exists_z\ [(y,z)\in S\wedge(z,u)\in Q]\}\,,\end{aligned}\tag{H}$$

so (H) follows from (G) by ($*$) and we see that (H) is the consequence of (F). The inequality (52) for $*=\triangleright$ we can obtain from following, using monotonicity,

$$(R\diamond S)\diamond Q\subseteq R\diamond(S\triangleright Q)\subseteq R\triangleright(S\triangleright Q).$$

Inequalities (53) and (54) can be deduced from the above and the convertibility properties. □

Now we will present examples, which show that the equalities do not hold in general. The matrix R will be the same for all examples. Matrices S and Q will be taken from the following

$$A_1=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\ A_2=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\ A_3=\begin{pmatrix}1&0&0\\0&0&0\\1&0&0\end{pmatrix},\ A_4=\begin{pmatrix}1&0&0\\1&0&0\\0&0&0\end{pmatrix},$$

$$A_5=\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix},\ A_6=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix},\ A_7=\begin{pmatrix}0&0&1\\0&0&1\\1&0&0\end{pmatrix},\ A_8=\begin{pmatrix}1&0&1\\0&0&1\\1&0&0\end{pmatrix},$$

$$A_9=\begin{pmatrix}0&1&1\\0&0&1\\1&0&0\end{pmatrix},\ A_{10}=\begin{pmatrix}0&0&0\\0&1&0\\1&0&0\end{pmatrix},\ A_{11}=\begin{pmatrix}1&0&0\\1&0&1\\1&0&0\end{pmatrix},\ A_{12}=\begin{pmatrix}0&1&0\\1&0&1\\1&0&0\end{pmatrix},$$

$$A_{13}=\begin{pmatrix}1&1&0\\0&1&0\\1&0&0\end{pmatrix},\ A_{14}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\ A_{15}=\begin{pmatrix}0&1&0\\0&1&0\\0&1&0\end{pmatrix},\ A_{16}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$A_{17}=\begin{pmatrix}1&0&1\\1&0&1\\1&0&0\end{pmatrix},\ R=\begin{pmatrix}1&1&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Example 1. We will show that all inclusions in (25)–(54) cannot be replaced by the equality. If $S=A_8$, $Q=A_{16}$, then

$$R\circ(S\circ Q)=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}>\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{cases}(R\circ S)\triangleleft Q,\\(R\circ S)\diamond Q,\\(R\star_1 S)\star_2 Q,\end{cases}$$

where $\star_1 \in \{\rhd, \lhd, \diamond\}$, $\star_2 \in \{\circ, \rhd, \lhd, \diamond\}$. We have shown strong inequalities in (25) for $\star \in \{\lhd, \diamond\}$ and in (27). Putting $S = A_7$ and $Q = A_3$ we obtain

$$R \circ (S \circ Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (R \circ S) \rhd Q,$$

$$\left.\begin{array}{l} R \diamond (S \lhd Q) \\ R \lhd (S \rhd Q) \end{array}\right\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} < \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left\{\begin{array}{l} (R \circ S) \lhd Q \\ (R \rhd S) \circ Q \end{array}\right. ,$$

which show strong inequalities in (25) for $\star = \rhd$, (37) and (41). Moreover

$$(R \diamond S) \star Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} < \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R \lhd (S \star Q),$$

for $\star \in \{\rhd, \circ\}$ so we have the strong inequalities from (31). Taking $S = A_{12}$ and $Q = A_{18}$ we calculate that

$$(R \circ S) \rhd Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R \star (S \rhd Q),$$

where $\star \in \{\circ, \rhd, \lhd, \diamond, \}$, which is from (29). For $S = A_{11}$, $Q = R$ we have

$$(R \lhd S) \star Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R \diamond (S \rhd Q),$$

where $\star \in \{\rhd, \circ\}$. We have shown the strong inequalities for (39). The same results we have for (33) and (35).

Taking $S = A_7$ and $Q = A_5$ we show strong inequalities in (43):

$$R \rhd (S \circ Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (R \diamond S) \star Q,$$

where $\star \in \{\rhd, \diamond\}$. Putting $S = A_8$ and $Q = A_2$ we obtain

$$R \circ (S \lhd Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left\{\begin{array}{l} (R \circ S) \lhd Q \\ (R \circ S) \diamond Q \end{array}\right. ,$$

which show strong inequality in (45). For R, S as above and $Q = A_3$ we get

$$R \lhd (S \diamond Q) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} < \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \{ \, (R \circ S) \diamond Q \ \%116 (R \lhd S) \lhd Q \ \%118 (R \lhd S) \diamond Q \ ,$$

so we obtain strong inequality in (47) and (48). Using the same matrices we examine (52). Taking $S = A_7$, $Q = A_5$ we obtain

$$R \diamond (S \circ Q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (R \diamond S) \diamond Q,$$

i. e., we have the strong inequality in (51). The rest strong inequalities we obtain from the above using convertibility properties.

3. VERIFICATION OF NON–ASSOCIATIVITY

Other mixed pseudo-associativity properties are not possible to obtain. This results we put into the table. The first column contains examined properties, while in the last column there are dual properties to these presented in the first column. They are obtained by putting $R = Q^T, S = S^T, Q = R^T$. The second column contains, respectively, matrices S and Q for which we have $R \star_1 (S \star_2 Q) < (R \star_3 S) \star_4 Q$ where $\star_i \in \{\circ, \triangleright, \triangleleft, \diamond\}$ for $i = 1, \ldots, 4$. In the third column there are, respectively, matrices S and Q fulfilling the opposite inequalities.

Properties	Matrices	Matrices	Dual properties
$R \circ (S \triangleleft Q) \nparallel (R \circ S) \triangleright Q$ $R \circ (S \triangleleft Q) \nparallel (R \triangleleft S) \star Q$ $\star \in \{\circ, \triangleleft, \triangleright, \diamond\}$	$S = A_{13}$ $Q = R$	$S = A_{10}$ $Q = A_3$	$R \triangleleft (S \circ Q) \nparallel (R \triangleright S) \circ Q$ $R \star (S \triangleright Q) \nparallel (R \triangleright S) \circ Q$ $\star \in \{\triangleright, \triangleleft, \diamond\}$
$R \circ (S \triangleleft Q) \nparallel (R \triangleright S) \star Q$ $\star \in \{\circ, \triangleleft, \triangleright, \diamond\}$	$S = A_{10}$ $Q = A_2$	$S = A_{11}$ $Q = R$	$R \star (S \triangleleft Q) \nparallel (R \triangleright S) \circ Q$ $\star \in \{\circ, \triangleright, \triangleleft, \diamond\}$
$R \triangleright (S \circ Q) \nparallel (R \circ S) \star Q$ $\star \in \{\triangleleft, \diamond\}$	$S = A_{17}$ $Q = A_3$	$S = A_{10}$ $Q = A_3$	$R \diamond (S \circ Q) \nparallel (R \circ S) \triangleleft Q$
$R \circ (S \triangleright Q) \nparallel (R \circ S) \star Q$ $\star \in \{\triangleleft, \diamond\}$	$S = A_{12}$ $Q = A_{17}$	$S = A_8$ $Q = R$	$R \star (S \circ Q) \nparallel (R \triangleleft S) \circ Q$ $\star \in \{\triangleright, \diamond\}$
$R \circ (S \triangleleft Q) \nparallel (R \diamond S) \star Q$ $R \circ (S \diamond Q) \nparallel (R \diamond S) \star Q$ $\star \in \{\circ, \triangleleft, \triangleright, \diamond\}$ $R \triangleright (S \diamond Q) \nparallel (R \diamond S) \star Q$ $R \triangleright (S \triangleleft Q) \nparallel (R \diamond S) \star Q$ $\star \in \{\triangleleft, \diamond\}$	$S = A_{17}$ $Q = A_2$	$S = A_{12}$ $Q = A_1$	$R \star (S \diamond Q) \nparallel (R \triangleright S) \circ Q$ $R \star (S \diamond Q) \nparallel (R \diamond S) \circ Q$ $\star \in \{\circ, \triangleright, \triangleleft, \diamond\}$ $R \star (S \diamond Q) \nparallel (R \diamond S) \triangleleft Q$ $R \star (S \diamond Q) \nparallel (R \triangleright S) \triangleleft Q$ $\star \in \{\triangleright, \diamond\}$
$R \circ (S \triangleright Q) \nparallel (R \triangleleft S) \star Q$ $\star \in \{\circ, \triangleleft, \}$	$S = A_7$ $Q = A_3$	$S = A_8$ $Q = R$	$R \triangleright (S \triangleright Q) \nparallel (R \triangleleft S) \circ Q$
$R \circ (S \triangleright Q) \nparallel (R \triangleright S) \star Q$ $\star \in \{\triangleright, \triangleleft, \diamond\}$	$S = A_{12}$ $Q = A_{14}$	$S = A_8$ $Q = R$	$R \star (S \triangleleft Q) \nparallel (R \triangleleft S) \circ Q$ $\star \in \{\triangleleft, \triangleright, \diamond\}$
$R \circ (S \triangleright Q) \nparallel (R \diamond S) \star Q$ $\star \in \{\circ, \triangleleft\}$	$S = A_7$ $Q = A_3$	$S = A_8$ $Q = R$	$R \star (S \diamond Q) \nparallel (R \triangleleft S) \circ Q$ $\star \in \{\circ, \triangleright, \}$
$R \circ (S \diamond Q) \nparallel (R \circ S) \star Q$ $\star \in \{\triangleleft, \diamond\}$ $R \triangleright (S \diamond Q) \nparallel (R \circ S) \diamond Q$	$S = A_{12}$ $Q = A_6$	$S = A_8$ $Q = A_2$	$R \star (S \circ Q) \nparallel (R \diamond S) \circ Q$ $\star \in \{\triangleright, \diamond\}$ $R \diamond (S \circ Q) \nparallel (R \diamond S) \triangleleft Q$
$R \circ (S \diamond Q) \nparallel (R \triangleleft S) \star Q$ $\star \in \{\triangleleft, \triangleright, \diamond\}$	$S = A_{13}$ $Q = R$	$S = A_{10}$ $Q = A_3$	$R \star (S \triangleright Q) \nparallel (R \diamond S) \circ Q$ $\star \in \{\triangleright, \triangleleft, \diamond\}$

$R \circ (S \diamond Q) \nparallel (R \rhd S) \star Q$ $R \lhd (S \circ Q) \nparallel (R \rhd S) \star Q$ $\star \in \{\lhd, \rhd, \diamond\}$	$S = A_{10}$ $Q = A_2$	$S = A_{11}$ $Q = R$	$R \star (S \lhd Q) \nparallel (R \diamond S) \circ Q$ $R \star (S \lhd Q) \nparallel (R \circ S) \rhd Q$ $\star \in \{\rhd, \lhd, \diamond\}$
$R \lhd (S \circ Q) \nparallel (R \circ S) \star Q$ $\star \in \{\lhd, \rhd, \diamond\}$	$S = A_{10}$ $Q = A_3$	$S = A_8$ $Q = A_5$	$R \star (S \circ Q) \nparallel (R \circ S) \rhd Q$ $\star \in \{\rhd, \diamond\}$
$R \lhd (S \lhd Q) \nparallel (R \circ S) \diamond Q$	$S = A_{10}$ $Q = A_3$	$S = A_7$ $Q = A_3$	$R \diamond (S \circ Q) \nparallel (R \rhd S) \rhd Q$
$R \lhd (S \lhd Q) \nparallel (R \lhd S) \star Q$ $R \lhd (S \lhd Q) \nparallel (R \diamond S) \star Q$ $\star \in \{\lhd, \rhd, \diamond\}$	$S = A_8$ $Q = A_2$	$S = A_{12}$ $Q = A_6$	$R \star (S \rhd Q) \nparallel (R \rhd S) \rhd Q$ $R \star (S \diamond Q) \nparallel (R \rhd S) \rhd Q$ $\star \in \{\rhd, \lhd, \diamond\}$
$R \lhd (S \lhd Q) \nparallel (R \rhd S) \star Q$ $R \lhd (S \diamond Q) \nparallel (R \diamond S) \star Q$ $\star \in \{\lhd, \rhd, \diamond\}$ $R \lhd (S \diamond Q) \nparallel (R \rhd S) \star Q$ $\star \in \{\lhd, \diamond\}$	$S = A_8$ $Q = A_2$	$S = A_{15}$ $Q = A_1$	$R \star (S \lhd Q) \nparallel (R \rhd S) \rhd Q$ $R \star (S \diamond Q) \nparallel (R \diamond S) \rhd Q$ $\star \in \{\rhd, \diamond\}$ $R \star (S \lhd Q) \nparallel (R \diamond S) \rhd Q$ $\star \in \{\rhd, \diamond\}$
$R \lhd (S \rhd Q) \nparallel (R \circ S) \star Q$ $\star \in \{\lhd, \diamond\}$ $R \lhd (S \rhd Q) \nparallel (R \lhd S) \lhd Q$	$S = A_7$ $Q = A_3$	$S = A_{17}$ $Q = R$	$R \star (S \circ Q) \nparallel (R \lhd S) \rhd Q$ $\star \in \{\rhd, \diamond\}$ $R \rhd (S \rhd Q) \nparallel (R \lhd S) \rhd Q$
$R \lhd (S \rhd Q) \nparallel (R \rhd S) \star Q$ $\star \in \{\lhd, \diamond\}$ $R \lhd (S \rhd Q) \nparallel (R \diamond S) \lhd Q$	$S = A_9$ $Q = A_5$	$S = A_{11}$ $Q = R$	$R \star (S \lhd Q) \nparallel (R \lhd S) \rhd Q$ $\star \in \{\rhd, \diamond\}$ $R \rhd (S \diamond Q) \nparallel (R \lhd S) \rhd Q$
$R \rhd (S \circ Q) \nparallel (R \lhd S) \star Q$ $R \rhd (S \lhd Q) \nparallel (R \lhd S) \star Q$ $\star \in \{\lhd, \diamond\}$ $R \rhd (S \diamond Q) \nparallel (R \lhd S) \diamond Q$	$S = A_{11}$ $Q = R$	$S = A_{12}$ $Q = A_1$	$R \star (S \rhd Q) \nparallel (R \circ S) \lhd Q$ $R \star (S \rhd Q) \nparallel (R \rhd S) \lhd Q$ $\star \in \{\rhd, \diamond\}$ $R \diamond (S \rhd Q) \nparallel (R \diamond S) \lhd Q$
$R \rhd (S \circ Q) \nparallel (R \rhd S) \lhd Q$ $R \rhd (S \circ Q) \nparallel (R \diamond S) \lhd Q$ $R \rhd (S \lhd Q) \nparallel (R \circ S) \diamond Q$	$S = A_8$ $Q = A_3$	$S = A_{12}$ $Q = A_1$	$R \rhd (S \lhd Q) \nparallel (R \circ S) \lhd Q$ $R \rhd (S \diamond Q) \nparallel (R \circ S) \lhd Q$ $R \diamond (S \circ Q) \nparallel (R \rhd S) \lhd Q$
$R \rhd (S \lhd Q) \nparallel (R \rhd S) \star Q$ $\star \in \{\lhd, \diamond\}$	$S = A_{10}$ $Q = A_2$	$S = A_9$ $Q = A_2$	$R \diamond (S \lhd Q) \nparallel (R \rhd S) \lhd Q$
$R \rhd (S \rhd Q) \nparallel (R \circ S) \diamond Q$	$S = A_{12}$, $Q = A_6$	$S = A_8$ $Q = A_2$	$R \diamond (S \circ Q) \nparallel (R \lhd S) \lhd Q$
$R \rhd (S \rhd Q) \nparallel (R \lhd S) \lhd Q$	$S = A_{11}$ $Q = R$	$S = A_9$ $Q = A_1$	$R \diamond (S \rhd Q) \nparallel (R \lhd S) \lhd Q$
$R \rhd (S \rhd Q) \nparallel (R \rhd S) \diamond Q$	$S = A_{12}$ $Q = A_5$	$S = A_8$ $Q = A_3$	$R \diamond (S \lhd Q) \nparallel (R \lhd S) \lhd Q$
$R \rhd (S \rhd Q) \nparallel (R \diamond S) \lhd Q$	$S = A_7$ $Q = A_3$	$S = A_{14}$ $Q = A_1$	$R \rhd (S \diamond Q) \nparallel (R \lhd S) \lhd Q$
$R \rhd (S \diamond Q) \nparallel (R \rhd S) \diamond Q$	$S = A_{12}$ $Q = A_5$	$S = A_8$ $Q = A_3$	$R \diamond (S \lhd Q) \nparallel (R \diamond S) \lhd Q$
$R \diamond (S \circ Q) \nparallel (R \circ S) \diamond Q$	$S = A_8$ $Q = A_3$	$S = A_7$ $Q = A_5$	

$R \diamond (S \circ Q) \nparallel (R \triangleleft S) \diamond Q$	$S = A_{11}$ $Q = R$	$S = A_7$ $Q = A_5$	$R \diamond (S \triangleright Q) \nparallel (R \circ S) \diamond Q$
$R \diamond (S \circ Q) \nparallel (R \triangleright S) \diamond Q$	$S = A_{10}$ $Q = A_2$	$S = A_7$ $Q = A_5$	$R \diamond (S \triangleleft Q) \nparallel (R \circ S) \diamond Q$
$R \diamond (S \triangleleft Q) \nparallel (R \star S) \diamond Q$ $\star \in \{\triangleleft, \triangleright, \diamond\}$	$S = A_8$ $Q = A_2$	$S = A_7$ $Q = A_5$	$R \diamond (S \star Q) \nparallel (R \triangleright S) \diamond Q$ $\star \in \{\triangleright, \diamond\}$
$R \diamond (S \triangleright Q) \nparallel (R \triangleleft S) \diamond Q$	$S = A_{11}$ $Q = R$	$S = A_{17}$ $Q = A_2$	
$R \diamond (S \diamond Q) \nparallel (R \diamond S) \diamond Q$	$S = A_8$ $Q = A_2$	$S = A_{17}$ $Q = A_4$	

4. CONCLUSION

In this paper we proved mixed pseudo-associativity properties of modified Bandler – Kohout composition of relations. We shows also examples showing, that presented inclusions are the best possible results. The obtained facts could not be valid for the original definition of Bandler – Kohout compositions of relations.

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